Minimal Extending Sets in Tournaments

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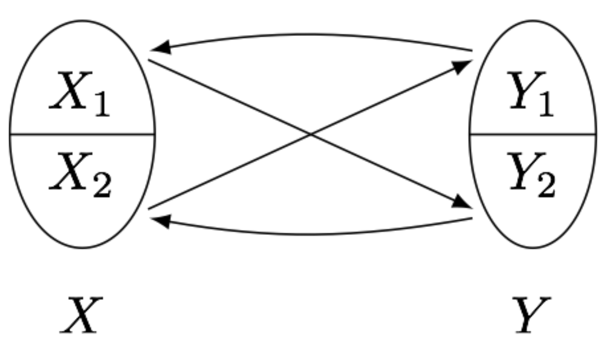
Motivation

Tournament solutions map a tournament to a nonempty subset of its alternatives and play an important role in social choice theory. The tournament solution **ME** was defined in 2011 by Brandt using the notion of **extending sets**.

Theorem (Brandt, 2011)
If every tournament admits a **unique** inclusion-minimal extending set, then ME satisfies many desirable properties.

Theorem (Brandt et al., 2013)
There exists a tournament with two **disjoint** extending sets that has about 10^{136} vertices.



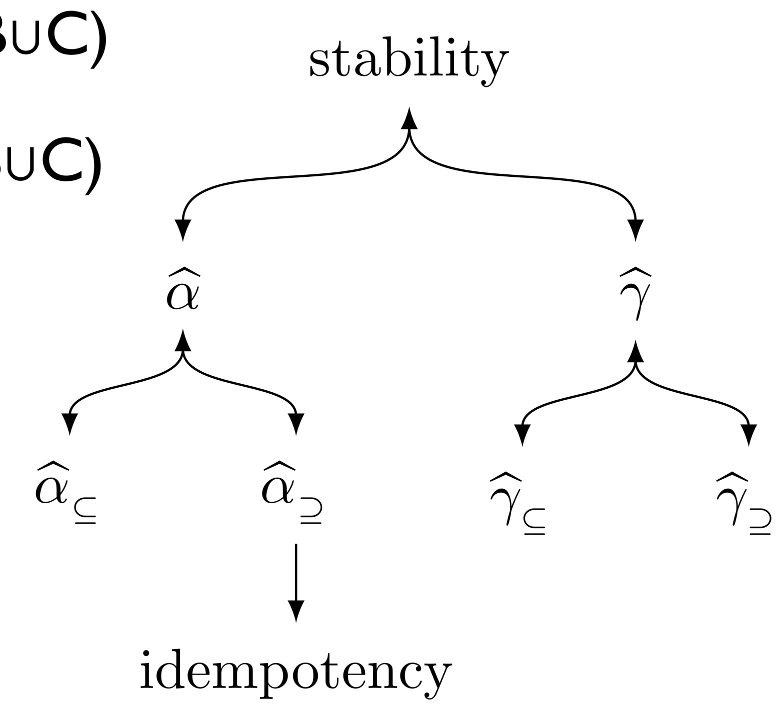
Open
Which of the properties are actually satisfied by ME?

Preliminaries

- A **tournament** T = (A, >) consists of a set of alternatives A and an asymmetric and complete relation >.
- The **Banks set** BA(T) comprises all alternatives that are maximal elements in inclusion-maximal transitive subtournaments of T.
- A set of alternatives B⊆A is an **extending set** in T if there exists no a in A∖B s.t. a is the maximal element in an inclusion-maximal transitive subtournament in B∪{a}, i.e., it should not be the case that a is in BA($T|_{B\cup\{a\}}$).
- The tournament solution **ME** is defined as the union of all inclusion-minimal extending sets of a tournament.

Desirable Properties

- **Dominance-based properties**
  - **monotonicity**
    chosen alternatives are still chosen when reinforced
  - **independence of unchosen alternatives**
    choice set unaffected by changes among unchosen alternatives
- **Choice-theoretic properties**
  - **stability**
    set X is chosen from two different sets B and C if and only if it is chosen from the union of these sets
  - can be factorized into
    - $\widehat{\alpha}$: X = S(B) = S(C) ⇐ X = S(B∪C)
    - $\widehat{\gamma}$: X = S(B) = S(C) ⇒ X = S(B∪C)
  - can again be factorized into
    - $\widehat{\alpha} \Leftrightarrow \widehat{\alpha}_{\subseteq}$ and $\widehat{\alpha}_{\supseteq}$
    - $\widehat{\gamma} \Leftrightarrow \widehat{\gamma}_{\subseteq}$ and $\widehat{\gamma}_{\supseteq}$
  - $\widehat{\alpha}_{\supseteq}$ implies **idempotency**



Results

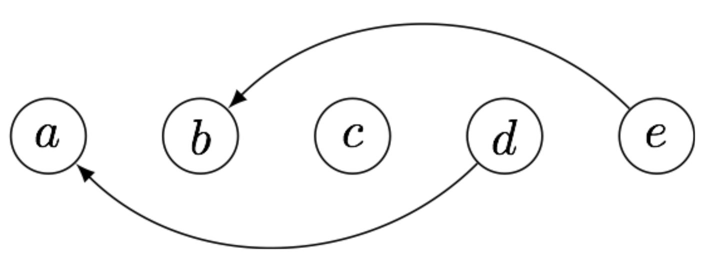
	min. ext. sets	∪ min. ext. sets = ME
monotonicity	✔	✘
independence of unchosen alternatives	✔	✘
$\widehat{\alpha}$	✔	✘
$\widehat{\gamma}$	✔	✘
stability	✔	✘
$\widehat{\alpha}_{\supseteq}$ and idempotency	✔	✔
contained in the Banks set	✔	✔
irregularity	✔	✔
composition-consistency	✔	✔
efficiently computable	✘	✘

Consequences

- In principle, ME is **severely flawed**. But, whenever ME violates a property, it is due to **rare counterexamples**!
- ME is a very appealing solution concept for all tournaments with a **unique** minimal extending set.
- No concrete example of a tournament with **multiple** minimal extending sets is known despite significant efforts to find one.

Example

- In this tournament, {a, b, c} is the unique minimal extending set
- ME(T) = { a, b, c }
- In contrast, BA(T) = { a, b, c, d }



Omitted edges point rightwards.

Criticism of the axiomatic method

ME satisfies the considered properties for all practical purposes

What does it mean if a tournament solution (or any other mathematical object) **in principle** violates some desirable properties, but **no concrete example** of a violation is known and will perhaps ever be known?

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THIS MATERIAL IS BASED UPON WORK SUPPORTED BY THE DEUTSCHE FORSCHUNGSGEMEINSCHAFT (DFG) UNDER GRANTS BR 2312/7-1.
PAUL HARRENSTEIN WAS ALSO PARTLY SUPPORTED BY THE ERC UNDER ADVANCED GRANT 291528 ("RACE").

MAY 2014