Fractional Hedonic Games: Individual and Group Stability

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Introduction

Coalition formation provides a versatile framework for analyzing cooperative behavior in multi-agent systems. In particular, hedonic coalition formation has gained considerable attention in the literature. An interesting class of hedonic games recently introduced by Aziz et al. are *fractional hedonic games*. In these games, the utility an agent assigns to a coalition is his average valuation for the members of his coalition. We study whether popular solution concepts, i.e., stability notions, admit a solution for various subclasses of fractional hedonic games. Furthermore, we examine the computational complexity of checking whether a solution exists.

Fractional Hedonic Games

A **hedonic game** is a pair $(N, \succsim)$, where

- $N = \{1, \dots, n\}$ is a set of *agents* and
- $\succsim = (\succsim_1, \dots, \succsim_n)$ is a tuple of complete, reflexive, and transitive *preference relations* over coalitions.

Outcomes of hedonic games are *partitions* of the agents (or *coalition structures*). The preferences of an agent over partitions only depend on this preferences over coalitions (the hedonic aspect).

A hedonic game is a **fractional hedonic game (FHG)** if, for every agent i, there exists a *valuation function* $v_i \colon N \to \mathbb{R}$ such that

$$S \succsim_i T \quad \text{if and only if} \quad \sum_{j \in S} \frac{v_i(j)}{|S|} \geq \sum_{j \in T} \frac{v_i(j)}{|T|},$$

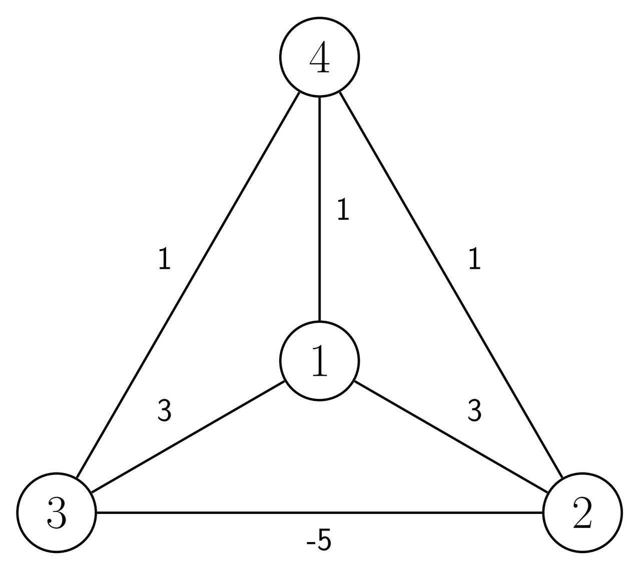
for all coalitions S, T that contain i.

Apart from unrestricted FHGs, we consider two classes of FHGs. An FHG is

- *symmetric* if $v_i(j) = v_j(i)$ for all $i, j \in N$ and
- *simple* if $v_i(j) \in \{0, 1\}$ for all $i, j \in N$.

Every FHG can be represented as weighted digraph $G = (N, N \times N, v)$, i.e., the weight of the edge (i, j) is the valuation of agent i for agent j.

Example



An example of a symmetric FHG with four agents. The valuation of agent i for agent j is the weight of the edge (i, j), e.g., $v_1(2) = v_2(1) = 3$. The preferences of agent 1 over the coalitions he is a member of are as follows:

$$\{1,2,3,4\} \succ_1 \{1,2,3\} \succ_1 \{1,2\} \sim_1 \{1,3\} \succ_1 \{1,2,4\} \sim_1 \{1,3,4\} \succ_1 \{1,4\} \succ_1 \{1\}$$

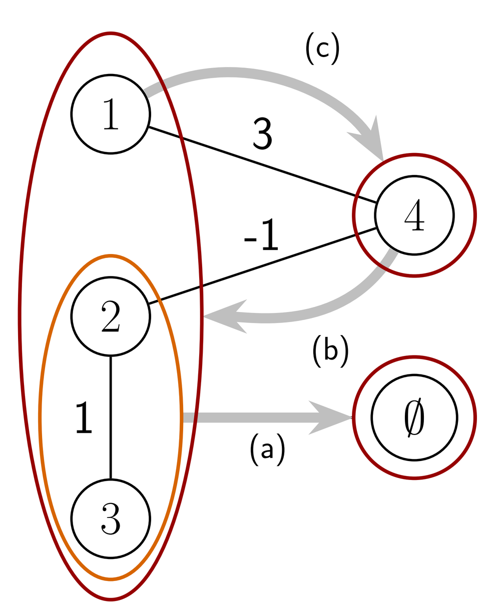
Stability Notions

Hedonic games are analyzed using stability notions, which formalize desirable or optimal ways in which the agents can be partitioned (based on the agents preferences over the coalitions). We consider three notions of stability:

(a) A coalition S blocks a partition π if every agent in S prefers S to his coalition in π. A partition that is not blocked by any coalition is **core stable**.

(b) A partition π is **Nash stable** if no agent can benefit from leaving his coalition in π and joining another existing (possibly empty) coalition.

(c) A partition π is **individually stable** if no agent can benefit from leaving his coalition in π and joining another existing (possibly empty) coalition without making some member of the coalition he joins worse off.

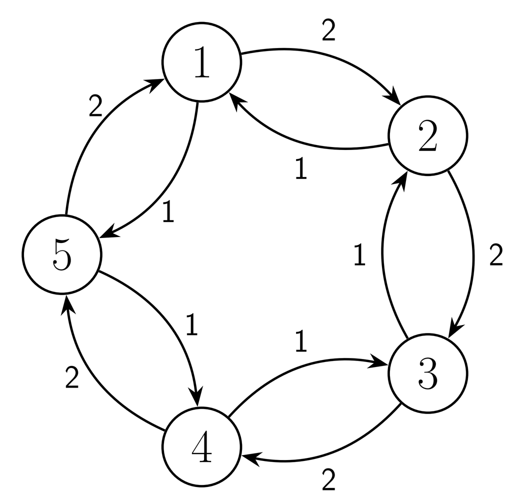


The FHG above illustrates the definitions of the stability notions. All omitted edges have weight 0.

Existence of Stable Partitions

Stable partitions may fail to exist due to cyclic deviations.

Theorem In *unrestricted* FHGs, individually stable partitions may not exist.



An FHG that does not admit an individually stable partition. All omitted edges have weight -4.

Open Problem Do symmetric FHGs always admit an individually stable partition?

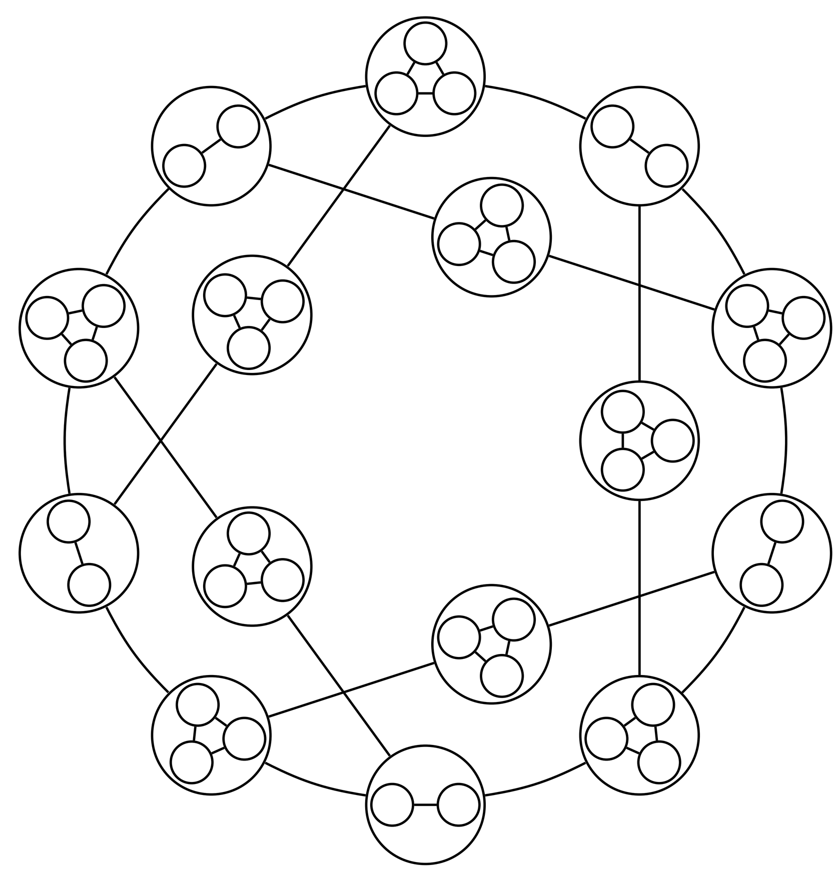
If all valuations are non-negative, the grand coalition consisting of all agents is always Nash (and hence individually) stable (Bilò et al.).

Theorem In *symmetric* FHGs, Nash stable partitions may not exist.

The example with four agents in the left column shows a symmetric FHG that does not admit a Nash stable partition.

Aziz et al. proved that core stable partitions may not exist in unrestricted FHGs and always exist in FHGs represented by certain types of graphs, e.g., trees and complete k-partite graphs. They left open the important problem whether every simple (and symmetric) FHG admits a core stable partition.

Theorem In *simple and symmetric FHGs*, core stable partitions may not exist.



A simple and symmetric FHG (with 40 agents) that does not admit a core stable partition. All depicted edges have weight 1. All missing edges have weight 0.

Computational Complexity

We show that various decision problems associated with FHGs are computationally hard by providing a generic reduction proof from *Exact Cover by* 3*-sets*.

Theorem Deciding whether there exists

- an individually stable partition in unrestricted FHGs is NP-complete,
- a Nash stable partition in symmetric FHGs is NP-complete, and
- a core stable partition in symmetric FHGs is NP-hard.

Open Problem Is deciding whether a core stable partition exists in simple and symmetric FHGs NP-hard?

Computational hardness of any of these decision problems implies hardness of computing a stable partition for the corresponding stability notion and class of FHGs.

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References

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