It Only Takes a Few: On the Hardness of Voting With a Constant Number of Agents

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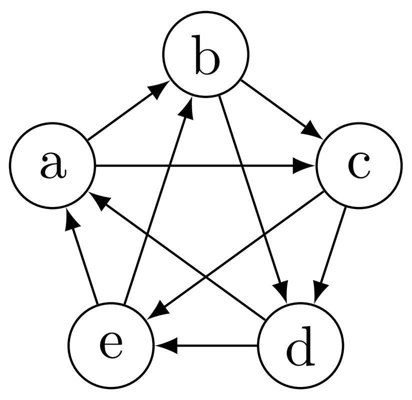
Motivation

Many social choice functions select outcomes on the basis of the pairwise majority relation.

1	1	1
a	d	c
b	e	e
c	a	b
d	b	d
e	c	a

preference profile



pairwise majority relation

Many reductions used for proving computational hardness results in social choice theory involve the construction of majority graphs abstracting away from the preference profiles. This is warranted by McGarvey's theorem, which also provides an upper bound on the number of voters required to obtain these graphs.

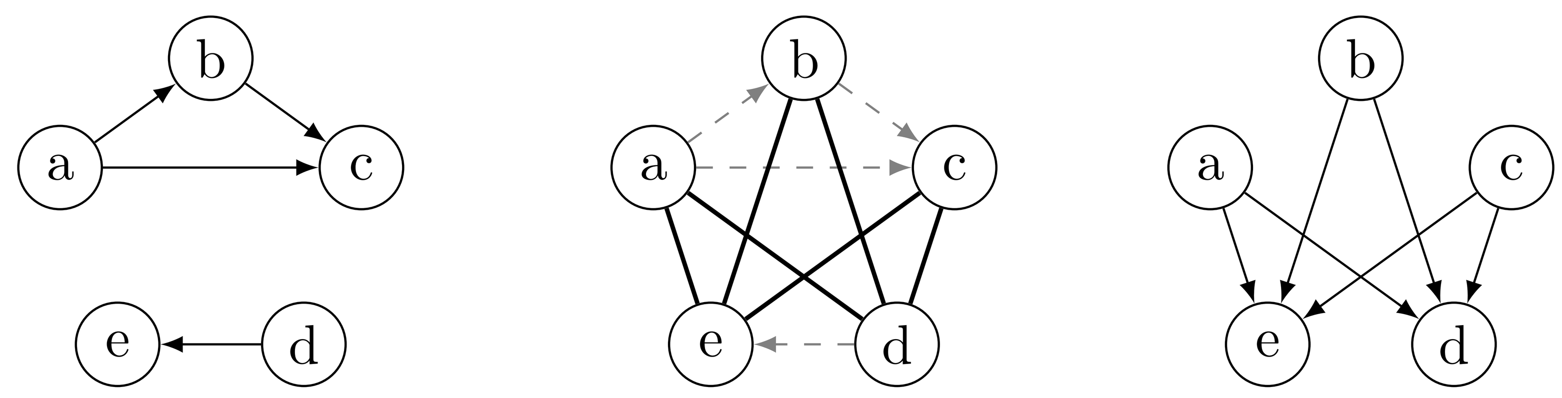
McGarvey (1953): Every directed asymmetric graph on *n* alternatives is induced by the pairwise majority relation on *n(n-1)* voters.

Sophisticated upper and lower bounds were established by Erdős and Moser, who showed that $\Theta(n/\log n)$ voters are required to induce every such graph. Therefore, it is not known whether the hardness results still hold if the number of voters is fixed. This is the issue we address for a number of voting rules, namely the Banks set, the tournament equilibrium set and ranked pairs.

> **"In particular, it would be interesting to know whether some of the problems [...] remain NP-hard if [the number of voters] is a given constant."**
> **(Hudry, 2008)**

Two Voters

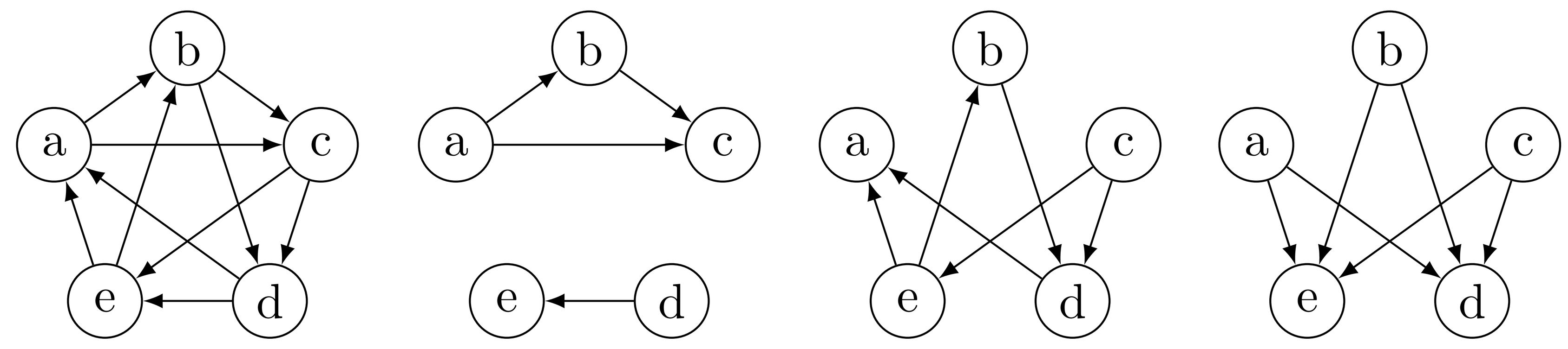
Dushnik and Miller (1941): A majority graph is induced by a 2-voter profile if and only if it is transitive and the indifference part is transitively orientable.



transitive majority graph — indifference part — transitive orientation of indifference part

Three Voters

A majority graph is induced by a 3-voter profile if and only if its edge set can be partitioned in a transitive set E_1 and an acyclic and transitively reorientable set E_2.



majority graph — E_1 transitive — E_2 acyclic and transitively reorientable

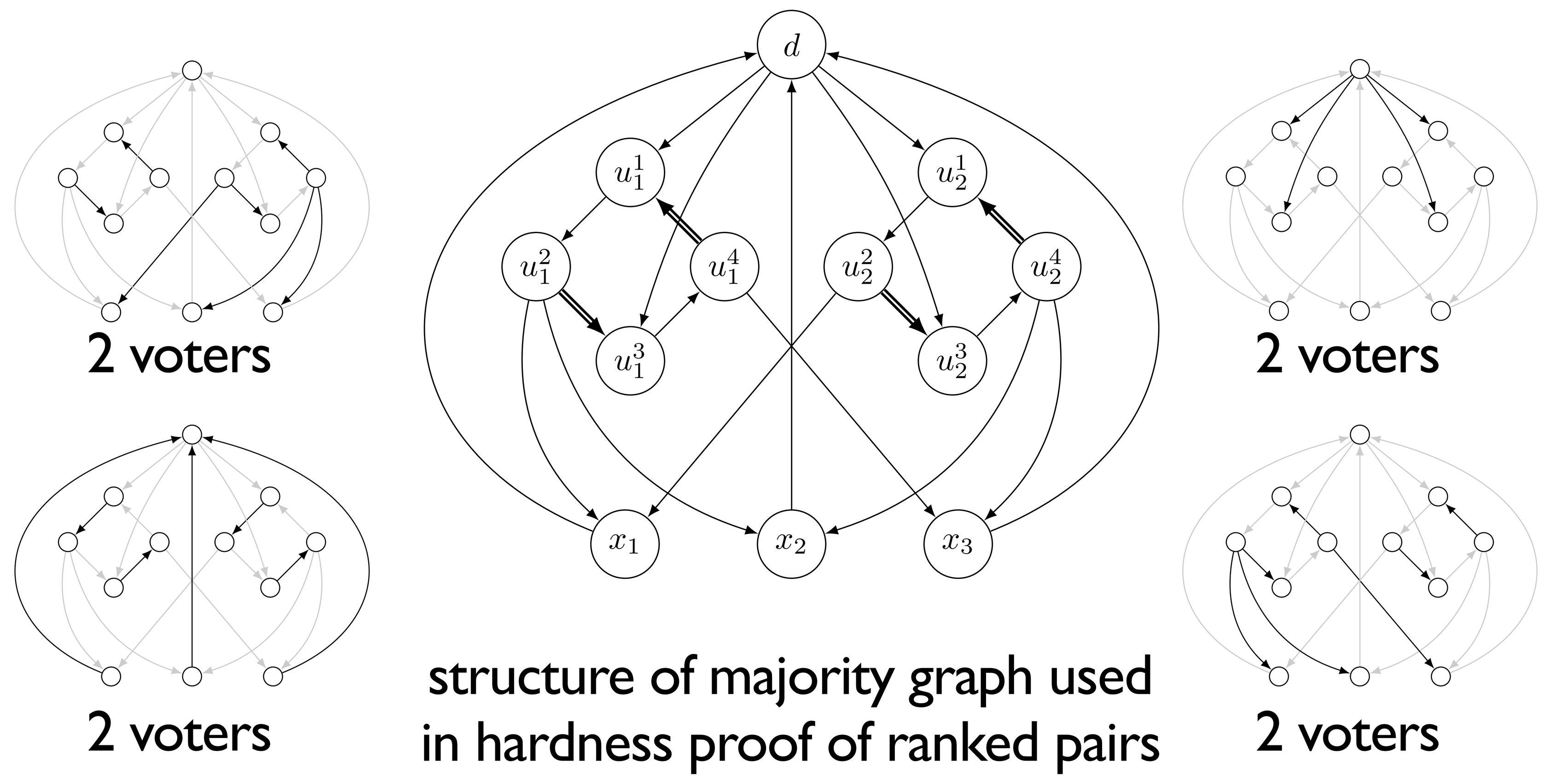
Sufficient Conditions For Multiple Voters

Let (V,E) be a tournament and $(V,E_1),\ldots,(V,E_k)$ subgraphs of (V,E) induced by 2-voter profiles. Let, moreover, $E_{k+1} = E \setminus (E_1 \cup \ldots \cup E_k)$.

- **If E_{k+1} is empty, (V,E) is induced by a $2k$-voter profile.**
- **If E_{k+1} is acyclic, (V,E) is induced by a $2k+1$-voter profile.**

Similar results can be obtained for weighted tournaments and general directed graphs as well.

Refining Hardness Proofs



structure of majority graph used in hardness proof of ranked pairs

We examined the hardness proofs for common, hard-to-compute social choice functions. It turns out that they hold even for a small constant number of voters.

Voting rule	NP-hard for n ≥
Banks set	**7**
tournament equilibrium set	**7**
ranked pairs	**8** (even), **11** (odd)

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