Extending Tournament Solutions

Felix Brandt
TU München
Germany

Markus Brill
Duke University
USA

Paul Harrenstein
University of Oxford
United Kingdom

Introduction

- Majoritarian (or C1) social choice functions only take into account the pairwise majority relation between alternatives
- It is well known that the majority relation can have cycles (Condorcet's paradox)
- When there are no ties, the majority relation is a tournament
- Most majoritarian functions have only been defined for tournaments
- We propose a generic way to extend tournament solutions to weak tournaments

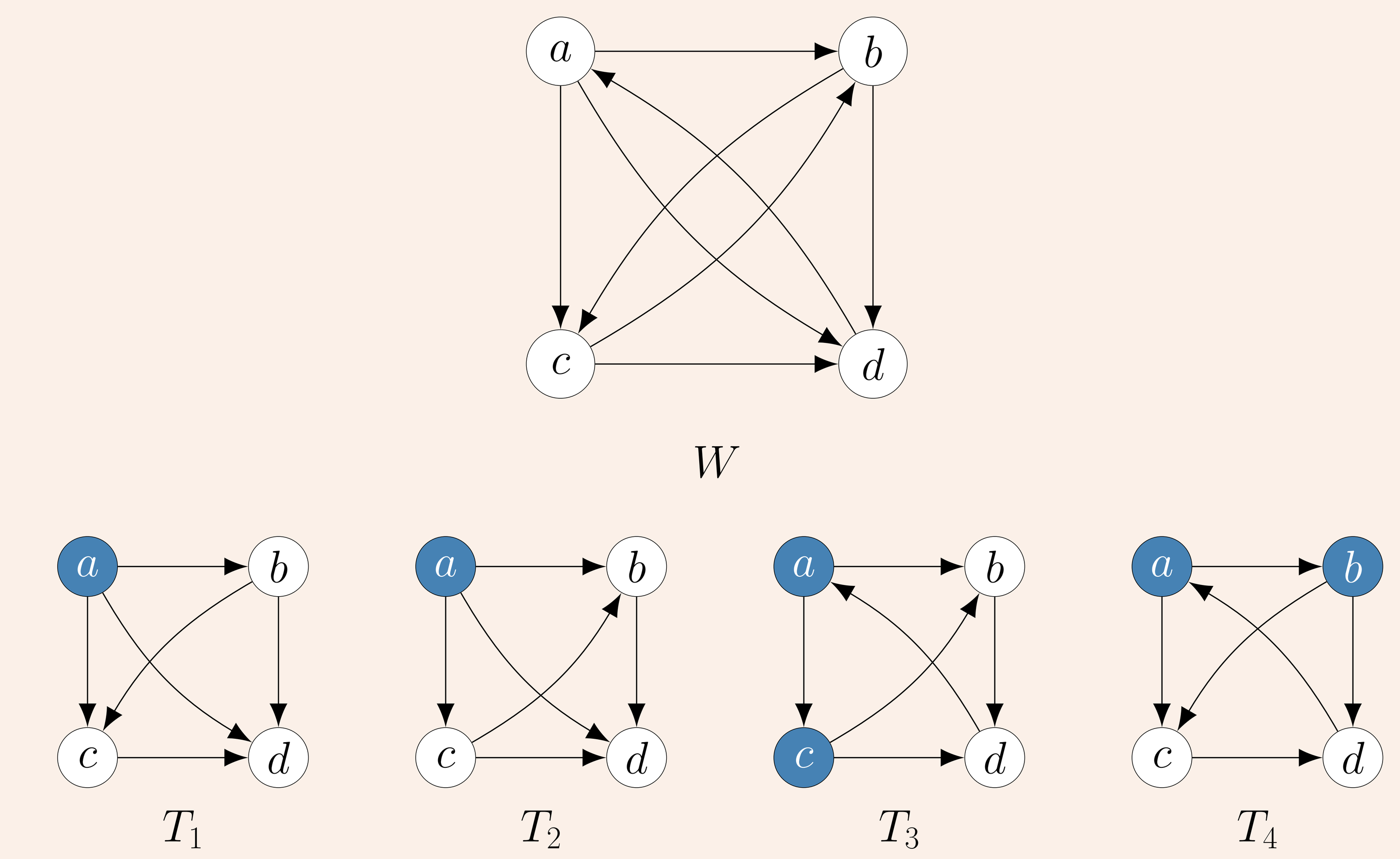
Tournaments and Weak Tournaments

- **Weak tournament**: a complete directed graph $W = (A, \succsim)$
- **Tournament**: a complete and antisymmetric directed graph $T = (A, \succsim)$
- A tournament $T = (A, \succsim)$ is an **orientation** of a weak tournament $W = (A, \succsim')$ if $T \subseteq W$
- A **tournament solution** S maps each tournament $T = (A, \succsim)$ to a nonempty subset $S(T) \subseteq A$ of alternatives

The Conservative Extension

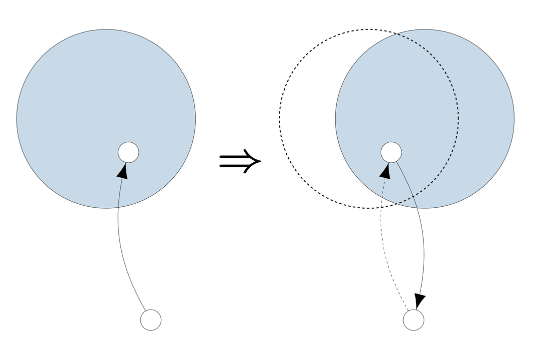
Let S be a tournament solution and let $[W]$ be the set of orientations of weak tournament W.

$$[S](W) = \bigcup_{T \in [W]} S(T)$$

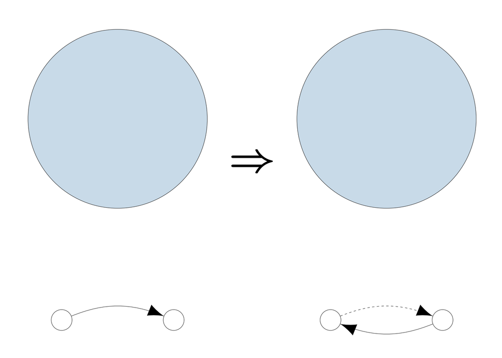


$[S](W) = \bigcup_{T\in[W]} S(T) = S(T_1) \cup S(T_2) \cup S(T_3) \cup S(T_4) = \{a, b, c\}$

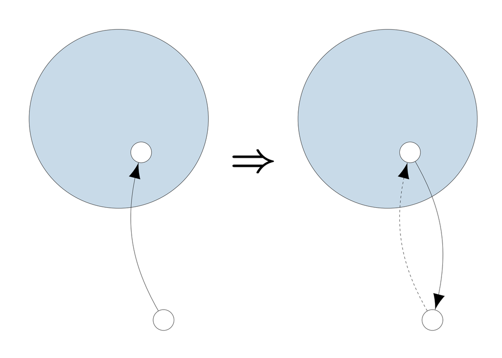
Inheritance of Properties



Monotonicity

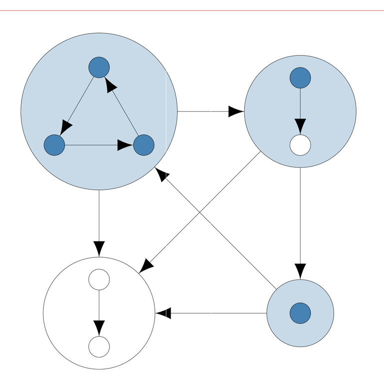


Ind. of Unchosen Alternatives

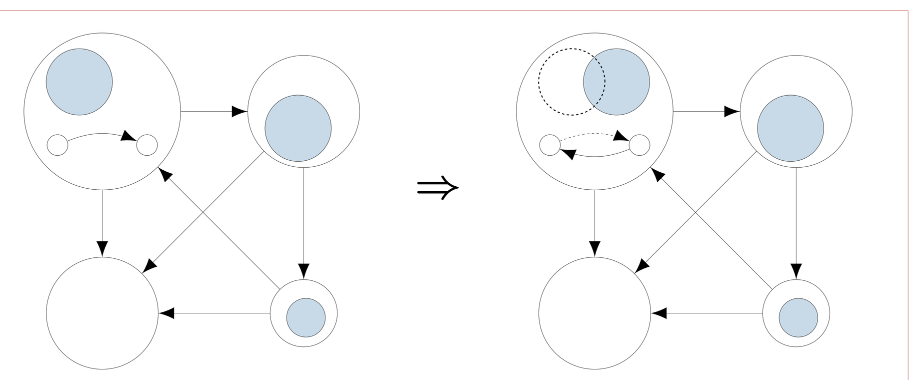


Set-Monotonicity

Proposition The following properties are inherited from S to $[S]$:
- monotonicity
- independence of unchosen alternatives
- set-monotonicity

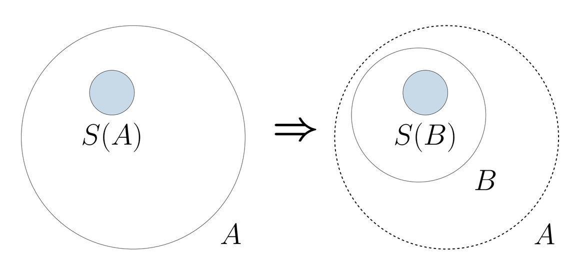


Composition-Consistency

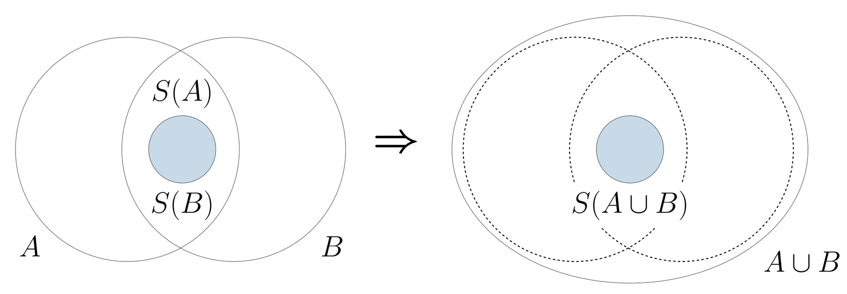


Weak Composition-Consistency

Proposition The following properties are inherited from S to $[S]$:
- composition-consistency
- weak composition-consistency



Property $\widehat{\alpha}$



Property $\widehat{\gamma}$

Proposition The following properties are inherited from S to $[S]$:
- property $\widehat{\alpha}$
- stability (property $\widehat{\alpha}$ and property $\widehat{\gamma}$ jointly)

Complexity and Possible Winners

Proposition There is a tournament solution S such that the winner determination problem is in P for S, and NP-complete for $[S]$.

Computing $[S]$ is equivalent to computing the *possible winners* in a partial tournament. Results by Cook et al. (1998), Lang et al. (2012), and Aziz et al. (2012) can thus be leveraged:

Proposition Computing $[CO]$, $[TC]$, and $[UC]$ are all in P.

Lemma If winner determination for S is NP-complete, so is winner determination for $[S]$.

As a corollary, computing $[BA]$ is NP-complete.

Open Problem Is computing $[BP]$ and $[MC]$ in P?

Comparison to other Extensions

Copeland Set (CO)

Under CO^α, each tie contributes α points ($0 \leq \alpha \leq 1$).

Proposition $CO^\alpha \subset [CO]$ if and only if $\frac{1}{2} \leq \alpha \leq 1$.

Top Cycle (TC)

$GETCHA$ selects maximal elements of the transitive closure of $\succsim$, and $GOTCHA$ selects maximal elements of the transitive closure of $\succ$ (Schwartz, 1972, 1986).

Proposition $GOCHA \subset GETCHA = [TC]$.

Bipartisan Set (BP)

The *essential set* ES collects alternatives in the support of *some* Nash equilibrium of the weak tournament game (Dutta and Laslier, 1999).

Proposition $ES \neq [BP]$.

Open problem $ES \subset [BP]$?

Uncovered Set (UC)

UC_M and UC_D are extensions of UC based on *McKelvey covering* (McKelvey, 1986) and *deep covering* (Duggan, 2013).

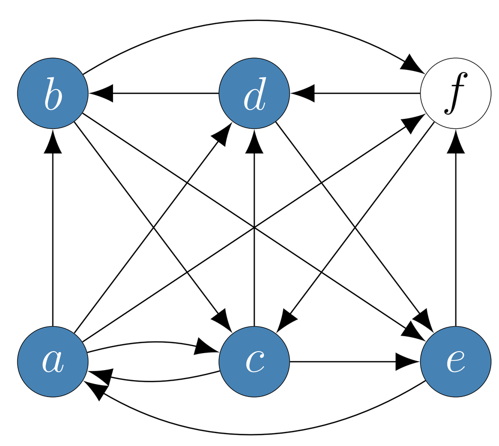
Proposition $UC_M \subset UC_D = [UC]$.

Minimal Covering Set (MC)

MC_M and MC_D are extensions of MC based on *McKelvey* and *deep* covering.

Proposition
- $[MC] \subset MC_D$
- $[MC] \not\subset MC_M$
- $MC_M \not\subset [MC]$

$[MC]$ is a new extension of MC satisfying stability!



$[MC](W) = \{a, b, c, d, e\}$

Banks Set (BA)

Four extensions of BA were suggested by Banks and Bordes (1999).

Proposition $BA_m \subset [BA]$ for all $m \in \{1, 2, 3, 4\}$.

Tournament Equilibrium Set (TEQ)

None of the six extensions of TEQ suggested by Schwartz (1990) coincides with $[TEQ]$.