Funding Public Projects:
A Case for the Nash Product Rule

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We study a mechanism design problem where a community of agents wishes to fund public projects via voluntary monetary contributions by the community members. This serves as a model for public expenditure without an exogenously available budget, such as participatory budgeting or voluntary tax programs, as well as donor coordination when interpreting charities as public projects and donations as contributions. Our aim is to identify a mutually beneficial distribution of the individual contributions. In the preference aggregation problem that we study, agents report linear utility functions over projects together with the amount of their contributions, and the mechanism determines a socially optimal distribution of the money. We identify a specific mechanism—the Nash product rule—which picks the distribution that maximizes the product of the agents' utilities. This rule is Pareto efficient, and we prove that it satisfies attractive incentive properties: it spends each agent's contribution only on projects the agent finds acceptable, and agents are strongly incentivized to participate.

1. Introduction

Italian tax payers have the option, under the *cinque per mille* program, to redirect 0.5% of their personal income tax to a non-profit organization of their choice. To participate, tax payers enter an organization's tax code into their tax return, choosing from a catalog of about 600 research organizations, 10,000 sports organizations, or 47,000 voluntary organizations.[1] Participating in this program is a good choice for anyone who believes that funding for at least one of these organizations would do more good than

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[1] https://www.agenziaentrate.gov.it/portale/web/guest/elenchi-versione-precedenti

additional tax income to the Italian government. In 2017, more than 10 million tax payers participated, for a total payout of more than 300 million euros.[2] However, one might worry that the allocation of funding to the organizations is inefficient because too little information about the participants' preferences is elicited. Each person only indicates a single organization, but presumably they would be happy to support any of several organizations. If we knew this approval information, we would likely be able to find an allocation that everyone prefers, in the sense that the money directed to approved organizations would be larger for each tax payer.

Suppose we convinced the Italian government to allow tax returns to indicate a list of organizations rather than just one. Given this information, how should we decide on the allocation of funds? A simple way to ensure a Pareto efficient outcome would be to maximize utilitarian welfare: one could define an individual's welfare as the amount of money disbursed to approved organizations and then maximize the sum of the welfare of each participating tax payer. The result would be that all the available funds would be disbursed to the (usually unique) organization that received the most "votes". While this is efficient, it fails to provide the participation incentives of the current system: one additional vote is unlikely to change which organization is most popular, and those who do not think that this organization is worth funding will choose to not participate.

It is in fact quite difficult to find an allocation mechanism that retains the strong participation incentives of the naive system (where each agent chooses just one organization) and also selects an efficient outcome. Mechanisms that spend each voter's contribution only on approved organizations tend to fail efficiency. To narrow down the search, we can observe that any mechanism that incentivizes participation must also satisfy some natural group fairness axioms: for example, if a group of voters all approve the same list of organizations, then the mechanism must spend the accumulated tax contribution of the group on organizations on this list. A result by Bogomolnaia et al. (2002) about group fairness implies that among separable social welfare functions, there is only a single candidate that might work: maximizing the *Nash product*, which selects the allocation of funds that maximizes the product (rather than the sum) of utilities. In this paper, we prove that the Nash product rule indeed incentivizes participation at least as much as the naive rule. In particular, if a tax payer chooses to participate by submitting a list of approved organizations, we can guarantee that the money allocated to those organizations grows by at least her individual tax contribution. In fact, it can grow by more than that, because the Nash product rule may choose to redirect others' contributions to these organizations.

This result makes the Nash product rule an attractive choice in many other contexts, where we wish to incentivize voluntary monetary contributions to a common pool which is to be spent on public projects in an efficient manner. Examples of communities that face this problem might include residents of an apartment complex (who want to coordinate spending on gardening in a courtyard, or on cleaning services), homeowners on a city street (to coordinate tree care, snow removal, or security patrols), or student

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[2] http://www.vita.it/it/article/2019/03/27/5-per-mille-2017-ecco-gli-elenchi-a-un-passo-dai-500-mln/151058/

clubs in a university (to coordinate funding for events and meet-ups).

Charitable donations provide another important application. Typically, these are undertaken independently without coordination among donors. As a consequence, mutual interest in the same charities goes unnoticed, even though the utility of all donors could be increased through coordination. For any given community of donors (such as the employees of a company running an annual charity matching program),[3] introducing a voting system based on the Nash product rule could produce better outcomes. The same conclusion holds in higher-stakes applications involving major philanthropic applications. Notably, the Open Philanthropy Project, which grants more than \$100 million a year to various organizations, has called on academics to develop mechanisms to combine different staff members' views on the most effective giving opportunities, and also to help coordinate the giving of different philanthropic organizations (Muehlhauser, 2017). Interest in donor coordination mechanisms has also been expressed in the effective altruism community (Peters, 2019), many of whose members have pledged to donate 10% of their income to effective charities.

While our full model allows agents to specify fine-grained utilities, for ease of exposition, we will start by assuming that each agent only submits a list of approved projects and that the agent is indifferent among these. We also ask each agent to commit a monetary amount to contribute to the funding system. The approvals are interpreted as dichotomous utility functions, so that $u_i(x) \in \{0,1\}$ is the utility per unit of money that agent i assigns to project x. For a distribution δ of the overall collected contributions to projects, agent i's utility of distribution δ is defined as $u_i(\delta) := \sum_x u_i(x)\delta(x)$, where $\delta(x)$ is the amount spent on project x. With dichotomous utilities, $u_i(\delta)$ is just the total amount of money that δ spends on projects approved by i.

Our main result concerns contribution incentives. We wish to assure agents that it is beneficial for them to contribute their money to the mechanism. In other words, if i contributes an additional amount $\Delta > 0$ of money, then the total amount spent on projects approved by i needs to increase by at least Δ. Formally, if the mechanism selects distribution δ when i contributes C_i, and selects distribution δ' when i contributes $C_i + \Delta$ with $\Delta > 0$, then we must guarantee that $u_i(\delta') \geq u_i(\delta) + \Delta$. We call this property *contribution incentive-compatibility*.

While this property may seem mild on first sight, it is difficult to satisfy together with efficiency. A naive procedure where each agent's contribution is split uniformly between her approved projects, violates efficiency. Maximizing utilitarian welfare among all distributions is efficient, but severely violates contribution incentive-compatibility, because the mechanism may spend agent i's contribution on projects that are not acceptable to i. If we constrain the welfare maximization to distributions where each agent's contribution

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[3]For example, Microsoft and Apple run such programs. In 2020, Microsoft employees donated over \$110 million to charities, which the company doubled to over \$220 million. (`https://www.microsoft.com/en-us/corporate-responsibility/philanthropies/employee-engagement`). Since its inception in 2011, Apple's charity matching program raised nearly \$600 million in total donations for more than 34,000 organizations (`https://www.apple.com/newsroom/2020/12/a-landmark-year-of-giving-from-apple/`).

is only spent on acceptable projects, contribution incentive-compatibility is satisfied but we lose efficiency. Replacing the utilitarian objective with a Rawlsian leximin objective does not work either, and mechanisms based on serial dictatorships fail, too.

Remarkably, the Nash product rule described above combines efficiency and contribution incentives. The Nash product rule selects the distribution δ that maximizes $\prod_{i \in N} u_i(\delta)^{C_i}$, where C_i is the size of i's contribution and N is the set of agents. Since this rule maximizes a monotonic function of agents' utilities, its outcome is guaranteed to be efficient. While it is easy to see that the Nash product rule satisfies $u_i(\delta') \geq u_i(\delta)$ when i contributes an additional amount Δ, it is more difficult to establish that we have $u_i(\delta') \geq u_i(\delta) + \Delta$ as required by contribution incentive-compatibility. Our main result, Theorem 2, shows that this property is satisfied by the Nash product rule. The proof is rather involved, because we need to reason about the trajectory of the maximizer of the Nash product as agent i's contribution continuously increases.

The Nash product rule is the only mechanism known to us that is both efficient and contribution incentive-compatible. It is plausible that the mechanism is characterized by these properties, but we could not establish this. However, as discussed in Section 2, two characterizations by Bogomolnaia et al. (2002) and Guerdjikova and Nehring (2014) imply that, at least when imposing further strong assumptions, the Nash product rule is characterized by contribution incentive-compatibility. The Nash product rule satisfies an additional property that is important in our model: it spends the contribution of user i only on projects that are approved by i. Formally, we say that a distribution δ is *decomposable* if we can decompose it as $\delta = \delta_1 + \cdots + \delta_n$ such that δ_i spends exactly C_i, and spends it only on projects acceptable to i. In Theorem 1, we prove that the distribution selected by the Nash product is always decomposable. This provides an intuitive justification of the chosen distribution to agents. Decomposability can also be interpreted as an incentive property: In some applications, the money is not distributed to projects by a central clearinghouse, but the mechanism's output is just used as a recommendation to agents where to direct their contribution. In this case, decomposability becomes essential, since a recommendation to send money to an unacceptable project is likely to be ignored. There do exist artificial mechanisms other than the Nash product rule that are efficient and decomposable,[4] but the Nash product rule is the only natural such mechanism that we know.

Our results generalize beyond the case of dichotomous utilities. In our formal treatment, we allow agents to indicate arbitrary utility values $u_i(x) \geq 0$ for the projects, and extend these to distributions as linear utilities as before, so $u_i(\delta) = \sum_x u_i(x)\delta(x)$. The Nash product rule works for this more general class of utilities, and in particular it retains efficiency. It also continues to satisfy contribution incentive-compatibility and decomposability, but these two properties hold in a weak sense that only distinguishes *acceptable* projects with strictly positive utility $u_i(x) > 0$ from *unacceptable*

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[4]To construct such mechanisms, note that we can modify an efficient distribution and retain efficiency as long as we do not change or only shrink its support (see, e.g., Aziz et al., 2015). So we can, for example, take the support of the Nash distribution and let every agent assign her entire contribution to one of her most preferred projects within the support. The resulting distribution is efficient and decomposable.

ones with $u_i(x) = 0$. Contribution incentive-compatibility guarantees that the amount spent on acceptable project grows by Δ when an extra amount of Δ is contributed, and decomposability guarantees that an agent's contribution is only spent on acceptable projects. In Section 5, we discuss strengthened versions of decomposability and contribution incentive-compatibility that use fine-grained utilities and provide guarantees based on an agent's most-preferred project (for example, strong decomposability requires that an agent's contribution is only spent on most-preferred projects). These two stronger properties may be desirable, but we prove impossibility theorems that show that each of the two strengthened axioms is incompatible with efficiency.

While the Nash product rule is incentive-compatible in the sense that it is decomposable and incentivizes contribution, it still allows for other strategic behavior. In particular, agents may have an incentive to misrepresent their utility functions. Because the Nash product penalizes distributions in which some agents obtain very low utility, it can be beneficial for agents to pretend to like popular projects less, or even to mark them as unacceptable. This will make the Nash product rule worry that those agents will be underserved, and thus increase the funding of other projects acceptable to them. Unfortunately, by a result due to Hylland (1980, Thm. 2), *every* efficient mechanism will be vulnerable to misrepresentation of preferences, except for dictatorships. This impossibility is robust, and analogues hold even for dichotomous utilities (Bogomolnaia et al., 2005; Duddy, 2015). Since efficiency is our main objective, we ignore possible misrepresentation of preferences in our discussion.

Overall, our discussion suggests that the Nash product rule is a prime candidate for funding public projects through voluntary individual contributions. It combines efficiency with strong incentive properties, and as detailed in Section 2, it also satisfies important fairness and proportionality properties. Finally, the rule is simple to define, can be computed to arbitrary precision using convex programming, and because it is decomposable, its distribution decisions can be easily understood by users. We are excited for the possibility of implementing a system based on the Nash product rule in the real world.

2. Related Work

The classic literature on *private provision of public goods* (e.g., Samuelson, 1954; Bergstrom et al., 1986) studies Nash equilibria in the non-cooperative setting where each agent decides how much to contribute to funding a public good. The main conclusion is that public goods will be underprovided in equilibrium, leading to inefficiency. In our model, we study cases where underprovision is less of a problem, for example because a company's matching program makes contributing a dominant strategy, or because the outside option is unattractive, such as paying taxes. Similarly, in the context of donor coordination, agents may have set aside a part of their income as a budget for charitable activities. The inefficiency that we are worried about is an inefficient allocation among different public goods.

In contrast to the above literature, we study a setting where there is an explicit coor-

dinating infrastructure or mechanism that aggregates preferences. Our model can thus be said to fall within the area of collective decision making where the set of alternatives is some subset of the Euclidian space, modeling divisible public goods or lotteries over indivisible public goods (see, e.g., Le Breton and Weymark, 2011). Two concrete applications in this context that have recently gained a lot of attention are those of *participatory budgeting* (e.g., Aziz and Shah, 2020) and *probabilistic social choice* (e.g., Brandt, 2017).

Participatory budgeting is a paradigm that allows citizens to collectively decide how a portion of a public budget ought to be spent (Cabannes, 2004). It has mostly been studied under the assumption that the budget is provided by an outside source (such as the city government). In the most common model, projects come with a fixed cost, and they can either be fully funded or not at all. Probabilistic social choice studies the aggregation of individual preferences into a lottery over alternatives. Both settings are interrelated because a division of a fixed budget among projects is equivalent to a probability distribution over alternatives. The social choice literature typically focusses on ordinal preferences. Bogomolnaia et al. (2005) have initiated the study of probabilistic social choice for dichotomous utility functions, where cardinal and ordinal preferences coincide.

The idea of maximizing the product of agents' utilities originates in the *Nash bargaining solution* and the corresponding mechanism is therefore often referred to as the Nash product rule (Nash, 1950). The Nash product rule has recently become popular in various fields, including the allocation of indivisible private items (Caragiannis et al., 2019), committee elections (Lackner and Skowron, 2018), and participatory budgeting (Fain et al., 2016, 2018). For all these settings, the Nash product rule satisfies strong fairness and proportionality properties.

In the context of dichotomous preferences, Aziz et al. (2019) showed that the Nash product rule guarantees *average fair share*: for any group of $\alpha\%$ of the agents which is *cohesive* (there is a project that they all approve), for an average group member, the Nash product spends at least $\alpha\%$ of the budget on approved projects. They also proved that the Nash product rule satisfies *strict participation*. This property, which was introduced by Brandl et al. (2015), makes sense for the fixed-budget setting, but it is relatively weak in our variable-budget setting.[5] Our main result showing that the Nash product rule is contribution incentive-compatible implies Aziz et al.'s (2019) result.

Two axiomatic characterizations of the Nash product rule are of particular interest in our context. First, Bogomolnaia et al. (2002, Prop. 6) have shown that the Nash product rule is the only rule that satisfies *unanimous fair share* (a condition weaker than decomposability and significantly weaker than contribution incentive-compatibility) among

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[5]For dichotomous preferences, strict participation implies that if before contributing, $\beta\%$ of others' money was spent on i's approved projects, then strictly more than $\beta\%$ of i's additional contribution will be spent on i's approved projects, while others' money is not spent in a worse way for i. On the other hand, contribution incentive-compatibility ensures that if agent i contributes money, *all* of it will be spent on i's approved projects, while again others' money is not spent in a worse way for i. Aziz et al., p. 768 mention that a large class of additive welfarist rules satisfy strict participation. Out of these only the Nash product rule satisfies contribution incentive-compatibility.

rules that maximize a quantity of the form $\sum_{i \in N} C_i f(u_i(\delta))$ for some function f (see also Aziz et al., 2019, p. 768). Their result was shown for the domain of dichotomous preferences, but easily extends to our more general domain due to the restricted form of mechanisms considered. Secondly, Guerdjikova and Nehring (2014) have characterized a solution concept called the *diversity value* for weighting different information sources based on their reliability. Their result can be translated into a characterization of the Nash product rule for dichotomous preferences using conditions such as convexity, continuity, reinforcement, and a core condition that is again weaker than decomposability and significantly weaker than contribution incentive-compatibility.

Fain et al. (2016) have initiated the study of a participatory budgeting setting where projects can receive an arbitrary amount of funding (like in our paper) but the budget is still exogenous and of fixed size. Fain et al. argued that allocations in *Lindahl equilibrium* (Foley, 1970) are particularly desirable. The Lindahl equilibrium is a market equilibrium in an artificial market for public goods. In these markets, each agent faces personalized prices (usually interpreted as taxes) for the public goods, and in equilibrium each agent demands the same bundle of public goods. Under standard assumptions, Foley (1970) showed that a Lindahl equilibrium exists (by reducing to the Arrow–Debreu private goods case), and is efficient. He also showed that equilibrium allocations are in the *core*: no coalition of agents can afford (using only a fraction of the budget proportional to their size) an allocation that each coalition member prefers to the equilibrium. For the case of additive linear utilities, Fain et al. (2016) proved that the Nash product rule yields an allocation in Lindahl equilibrium, and hence is in the core.[6] The core can be interpreted as guaranteeing agents proportional representation: if a fraction of $\alpha\%$ of agents assign positive utility only to some set A' of projects, then the Nash product rule will spend at least $\alpha\%$ of the budget on projects in A'.

Gul and Pesendorfer (2020) study Lindahl equilibrium as a collective choice rule. They characterize the set of all Lindahl equilibrium utility profiles as the outcomes of a bargaining solution they call the equitable solution. Every outcome of the equitable solution can be justified by being the Nash bargaining outcome of a simple related bargaining problem.

The key difference between all of the above literature and our model is that in our model, the individual contributions to the endowment are *owned* by the agents. This suggests the definitions of the axioms of decomposability and contribution incentive-compatibility, which—to the best of our knowledge—have not been considered in previous work.

Even though our model is concerned with the provision of public goods, it is unsuitable for mechanisms like the Vickrey-Clarke-Groves (VCG) mechanism. The VCG mechanism implements the utilitarian rule by imposing payments on participants under the assumption that utility functions are quasilinear. By contrast, in our setting, the monetary contribution of each agent is fixed and independent of the agent's preferences. The central issue is one of fair distribution rather than one of strategic implementation.

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[6]This mirrors the canonical result that the Nash product yields an equilibrium in Fisher markets for private goods under additive valuations (Eisenberg and Gale, 1959).

3. Model and Axioms

Let A be a finite set of m *projects* (e.g., charities or joint activities) and N a finite set of n *agents*. For each $i \in N$, agent i's *contribution* is $C_i \in \mathbb{R}_{>0}$, giving rise to a total *endowment* of $C = \sum_{i=1}^{n} C_i$.

A *distribution* δ is a function that describes how some amount V (e.g., an individual contribution or the entire endowment) is distributed among the projects, so $\delta : A \to \mathbb{R}_{\geq 0}$ such that $\sum_{x \in A} \delta(x) = V$. For convenience, we write distributions as linear combinations of projects, so that $a + 2b$ denotes for distribution δ with $\delta(a) = 1$ and $\delta(b) = 2$. The set of all distributions of value V is denoted by $\Delta(V)$. Each distribution of the endowment $\delta \in \Delta(C)$ can be divided into n individual distributions $\delta_i \in \Delta(C_i)$ such that $\delta = \sum_{i \in N} \delta_i$. Clearly, the division of a distribution into individual distributions is not unique.

Apart from her contribution, each agent possesses a utility function $u_i : A \to \mathbb{R}$, where $u_i(a)$ describes how much utility an agent derives if one unit of money goes to project a. So agent i's utility for a distribution $\delta \in \Delta(C)$ is

$$u_i(\delta) = \sum_{x \in A} \delta(x) \cdot u_i(x).$$

A project is said to be *acceptable* by an agent if it gives her positive utility, and *unacceptable* it gives her utility 0. We normalize utility functions such that the utility assigned to least-preferred acceptable projects is 1.[7] (In the special case that an agent is completely indifferent, we label all projects as acceptable and so set all utilities to 1). A utility function u_i is *dichotomous* if we have $u_i(x) \in \{0, 1\}$ for all $x \in A$, so that agent i only distinguishes between acceptable and unacceptable projects without further discriminating between the acceptable ones. The *type* of an agent i is $\theta_i = (u_i, C_i)$ and the set of all types Θ. A *profile* $\theta = (\theta_i)_{i \in N} \in \Theta^N$ assigns a type to every agent. We denote by θ_{-i} the profile including all agents but i. A *mechanism* f is a function mapping a profile θ to a distribution $\delta \in \Delta(C)$.

We now discuss the main properties of distribution mechanisms that we are interested in: efficiency, decomposability, and contribution incentive-compatibility.

A mechanism that yields high-quality distributions should, at minimum, satisfy Pareto efficiency (or Pareto optimality). Indeed, if a mechanism produces a distribution so that we could reshuffle money between projects and make every agent better off, it has not made full use of the potential for mutual gains.

Definition 1 (Efficiency). Let $\theta \in \Theta^n$ be a type profile. A distribution $\delta' \in \Delta(C)$ *Pareto dominates* another distribution $\delta \in \Delta(C)$ if $u_i(\delta') \geq u_i(\delta)$ for all $i \in N$ and $u_i(\delta') > u_i(\delta)$ for some $i \in N$. A distribution $\delta \in \Delta(C)$ is *efficient* if no distribution dominates it.

We say that a mechanism f is *efficient* if $f(\theta)$ is efficient for all $\theta \in \Theta^n$.

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[7]Since we focus on the Nash product rule (which is invariant to rescaling utility functions), this is without loss of generality. In other contexts, we can avoid the normalization by adjusting Definition 3.

Depending on the concrete application, the mechanism might operate in a decentralized setting, and may not be able to directly control the use of the agents' contributions (for example, when a donor coordination service does not actually collect money from its participants). In such cases, the mechanism's output δ is better understood as a recommendation to the agents about how they should use their resources. We would then need to decompose δ into personalized recommendations $\delta_i \in \Delta(C_i)$, so that if every agent spends her money according to δ_i, we recover δ. In order for the distribution to be decomposable, it should ask every agent to spend her contributions exclusively on acceptable projects.

Definition 2 (Decomposability). Let $\theta \in \Theta^n$ be a type profile. A distribution $\delta \in \Delta(C)$ is *decomposable* if it can be divided into individual distributions $(\delta_i)_{i \in N}$ with $\delta_i \in \Delta(C_i)$ for all $i \in N$ and $\delta = \sum_{i \in N} \delta_i$ such that for all $i \in N$, we have $\delta_i(x) > 0$ only if $u_i(x) > 0$.

We say that a mechanism f is *decomposable* if $f(\theta)$ is decomposable for all $\theta \in \Theta^n$.

In Section 5, we discuss a strengthening of decomposability which requires that $\delta_i(x) > 0$ only if i has assigned maximum utility to x, i.e., only if $u_i(x) \geq u_i(y)$ for all $y \in A$. However, this requirement turns out to be too strong; it clashes with efficiency. Alternative characterizations of decomposability and strong decomposability are given in Appendix B.

We cannot force agents to participate in the mechanism. Even when not participating, agents still benefit from the projects funded by the participating agents; this fact reduces contribution incentives (a kind of free-riding). However, there is a downside to not participating: the mechanism only considers the utility functions of those who contribute to the mechanism. Thus, agents who do not participate forego the opportunity to influence the spending of other agents.

To motivate the definition of contribution incentive-compatibility, focus on agent i, and recall that we have normalized i's reported utilities so that the utility of each acceptable project is at least 1. Agent i considers two possible actions:

(i) contribute 0 to the mechanism and instead spend the amount C_i on some outside option that gives i a utility of 1 per unit of money, for a total utility of $u_i(f(\theta_{-i})) + C_i$;

(ii) contribute C_i to the mechanism, for a total utility of $u_i(f(\theta))$.

Contribution incentive-compatibility requires that i obtains at least as much utility when choosing action *(ii)* than when choosing *(i)*.

Definition 3 (Contribution incentive-compatibility). A mechanism f is *contribution incentive-compatible* if for each $\theta \in \Theta^n$ and $i \in N$, $u_i(f(\theta)) \geq u_i(f(\theta_{-i})) + C_i$.

While contribution incentive-compatibility is phrased as a choice between contributing 0 or $C_i > 0$, it will also imply a similar property saying that it is beneficial for agents to increase their contribution from C_i to $C_i + \Delta$; this implication holds for all reasonable rules that treat agents with the same utility function as a single agent (like the Nash product rule does).

In Section 5, we again discuss a strengthening of contribution incentive-compatibility, where the quality of the outside option is higher, so that i obtains utility $u_i(f(\theta_{-i})) + C_i \cdot \max_{y \in A} u_i(y)$ when choosing action *(i)*. Again, this stronger version is incompatible with efficiency.

Decomposability and contribution incentive-compatibility are logically independent properties, even when utilities are dichotomous. It is not hard see that decomposability does not imply contribution incentive-compatibility. For example, the rule that maximizes the sum of utilities subject to decomposability and efficiency is decomposable but violates contribution incentivize-compatibility. Our example that the converse implication also fails is more cumbersome. We state it in Appendix C.

4. The Nash Product Rule

The *Nash product*, which refers to the product of agent utilities, is often seen as a compromise between utilitarian and egalitarian welfare (Moulin, 1988). Maximizing the Nash product has been found to yield fair and proportional outcomes in many preference aggregation settings, and it also turns out to be attractive in our context. Formally,

$$NASH(\theta) = \arg\max_{\delta \in \Delta(C)} \prod_{i \in N} (u_i(\delta))^{C_i} = \arg\max_{\delta \in \Delta(C)} \sum_{i \in N} C_i \log (u_i(\delta)).$$

The distribution $NASH(\theta)$ maximizes a sum of functions, namely $C_i \log(\cdot)$, that are strictly increasing in the agents' utilities. Thus, it is efficient (see, e.g., Moulin, 1988).

	$u_i(a)$	$u_i(b)$	C_i
Agent 1	1	0	1
Agent 2	1	3	1

Table 1: Type profile $\theta = (\theta_1, \theta_2)$ with contributions $C_1 = C_2 = 1$ and $NASH(\theta) = 1.5\ a + 0.5\ b$.

A simple example of $NASH$ for a type profile θ with two agents and two projects is shown in Table 1. We have

$$\delta = NASH(\theta) = \arg\max_{\delta \in \Delta(2)} \delta(a) \cdot (\delta(a) + 3\ \delta(b)) = 1.5\ a + 0.5\ b.$$

The collective distribution δ can be decomposed into individual distributions

$$\delta_1 = a \quad \text{and} \quad \delta_2 = 0.5\ (a + b).$$

Contribution incentive-compatibility is satisfied in this example because

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\begin{aligned}
u_1(\delta) &= 1.5 > 1 = u_1(b) + 1 = u_1(NASH(\theta_{-1})) + C_1 \text{ and} \\
u_2(\delta) &= 3 > 2 = u_2(a) + 1 = u_2(NASH(\theta_{-2})) + C_2.
\end{aligned}
$$

On the other hand, simply maximizing the sum of individual utilities in this example would result in $\delta' = 2\,b$, which is not decomposable, as project b is unacceptable for agent 1, and violates contribution incentive-compatibility because agent 1 would prefer an outside option to participating in the mechanism.

In general, *NASH* can be efficiently approximated using convex programming (see, e.g., Bogomolnaia et al., 2005). However, as *NASH* can return distributions with irrational values, its exact computation is not possible in general.

It is not obvious whether the distribution returned by *NASH* is always decomposable or if *NASH* incentivizes contribution. We will deal with these questions in turn and find that the answer in both cases is yes. The proof for contribution incentive-compatibility is rather involved.

The Nash distribution is the solution to an optimization problem, and thus satisfies the first-order conditions of optimality. By manipulating these conditions, we can show that the *NASH* distribution is always decomposable.[8]

Theorem 1. *NASH is decomposable.*

Proof. We have to show that there is a decomposition of $NASH(\theta)$ into $\delta_i \in \Delta(C_i)$, $i \in N$, such that $\sum_{i \in N} \delta_i(x) = \delta(x)$ for all x.

We consider the Karush–Kuhn–Tucker conditions and write the Lagrangian as

$$\mathcal{L}(\delta, \lambda, \mu_1, \ldots, \mu_m) = \sum_{i \in N} C_i \log\left(u_i(\delta)\right) + \lambda \left(C - \sum_{x \in A} \delta(x) \right) + \sum_{x \in A} \mu_x \delta(x),$$

where $\lambda \in \mathbb{R}$ is the Lagrange multiplier for the constraint $\sum_{x \in A} \delta(x) = C$ and $\mu_x \geq 0$ is the multiplier for the constraint $\delta(x) \geq 0$.

Suppose δ is an optimal solution. By complementary slackness, we must have $\mu_x = 0$ whenever $\delta(x) > 0$. Also, we must have $\partial \mathcal{L}/\partial \delta(x) = 0$, that is, $\sum_{i \in N} C_i u_i(x)/u_i(\delta) - \lambda + \mu_x = 0$. By case distinction based on whether $\delta(x) > 0$, it follows that $\lambda \delta(x) = \sum_{i \in N} C_i \delta(x) u_i(x)/u_i(\delta)$ for all $x \in A$. Hence,

$$\lambda \cdot C = \sum_{x \in A} \lambda \delta(x) = \sum_{x \in A} \sum_{i \in N} C_i \delta(x) u_i(x)/u_i(\delta) = \sum_{i \in N} C_i u_i(\delta)/u_i(\delta) = \sum_{i \in N} C_i = C.$$

So $\lambda = 1$, and hence $\sum_{i \in N} C_i u_i(x)/u_i(\delta) = 1$ for all $x \in A$ such that $\delta(x) > 0$.

Now, for each $i \in N$, define an individual distribution $\delta_i \in \Delta(C_i)$ with $\delta_i(x) = C_i \delta(x) u_i(x)/u_i(\delta)$ for all $x \in A$. Clearly, $\mathrm{supp}(\delta_i) \subseteq \{a \in A\colon u_i(a) > 0\}$ and $\delta_i \in \Delta(C_i)$, since $\sum_{x \in A} \delta(x) u_i(x) = u_i(\delta)$. To see that $\delta = \sum_{i \in N} \delta_i$, note that for $x \in A$ with $\delta(x) = 0$ we have $\delta_i(x) = 0$ for all $i \in N$, and for $x \in A$ with $\delta(x) > 0$, we have

$$\sum_{i \in N} \delta_i(x) = \sum_{i \in N} C_i \delta(x) u_i(x)/u_i(\delta) = \delta(x) \sum_{i \in N} C_i u_i(x)/u_i(\delta) = \delta(x). \qquad \square$$

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[8]This proof is similar to a result by Guerdjikova and Nehring (2014) who consider *NASH* with dichotomous preferences, and establish an equivalent property in this restricted setting.

By inspecting the proof, we see that the distribution δ_i of agent i satisfies a stronger notion of decomposability: the fraction of her contribution that she gives to project x is proportional to the utility $\delta(x)u_i(x)$ she derives from x in the Nash distribution δ (see also Guerdjikova and Nehring, 2014). For example, if half of agent i's utility $u_i(\delta)$ is due to the amount $\delta(x)$ spent on x, then she transfers half of her contribution to x. Thus, it suffices that the central clearinghouse announces the overall distribution δ. Agents can then easily compute their individual distributions δ_i without needing to know the other agents' utility functions or contributions.

The proof that $NASH$ is contribution incentive-compatible is technically involved and requires a number of lemmas, which are stated and proved in the appendix. At a high level, we estimate the rate of change of an agent's utility as her contribution increases, and integrate this quantity as she goes from not participating to participating in the mechanism to obtain the desired result.

Theorem 2. *$NASH$ is contribution incentive-compatible.*

Proof. Recall that we normalized utilities so that the utility assigned to least-preferred acceptable projects is 1 and so that the utility assigned to unacceptable projects is 0. Formally, in order to prove Theorem 2, we must show that for all $\theta \in \Theta^n$ and $i \in N$, $u_i(NASH(\theta)) \geq u_i(NASH(\theta_{-i})) + C_i$. Consider the function $g\colon \Theta^n \to \Delta(1)$ with $g(\theta) = NASH(\theta)/C_\theta$ for all $\theta \in \Theta^n$, where C_θ denotes the sum of the contributions in θ. We will show that

$$u_i(g(\theta)) \geq \frac{1}{C_\theta}((C_\theta - C_i)u_i(g(\theta_{-i})) + C_i), \tag{1}$$

which is equivalent to the inequality above for $NASH$.

Denote by $\mathcal{P}_\theta \subseteq \mathbb{R}^n$ the polytope of feasible utility profiles scaled by $1/C_\theta$, i.e., $\mathcal{P}_\theta = \{u(\delta)\colon \delta \in \Delta(1)\}$. Note that $\mathcal{P}_\theta$ is convex. For $U \in \mathcal{P}_\theta$, let $F_\theta(U) = \sum_{i \in N} C_i \log U_i$. Since by Lemma 1, F_θ has a unique maximizer for all $\theta \in \Theta^n$, we can define the function $\mathcal{U}\colon \Theta^n \to \mathbb{R}^n_{\geq 0}$ that returns this unique maximizer. Observe that $\mathcal{U}(\theta) \in \mathcal{P}_\theta$ for all $\theta \in \Theta^n$.

We prove (1) with $i = 1$ as the focal agent. The first step, and also the bulk of the proof, is to derive a lower bound on the derivative of $\mathcal{U}_1(\theta) = u_1(g(\theta))$ as a function of C_1 whenever $\mathcal{U}_1(\theta) < 1$. Then, integrating this derivative and using monotonicity of $\mathcal{U}_1$ (see Lemma 2) gives (1). (If $\mathcal{U}_1(\theta) = 1$, agent 1 already receives the highest utility possible given the normalization of g. By Lemma 2, the derivative of $\mathcal{U}_1(\theta)$ is 0. We treat this case in Step 2.)

Step 1. Let $\theta \in \Theta^n$ such that $\mathcal{U}_1(\theta) < 1$ and $U = \mathcal{U}(\theta)$. Moreover, let $\mu \in (0, 2)$ be arbitrary and let ε^* be such that the conclusion of Lemma 7 holds; let $\varepsilon \in (0, \varepsilon^*)$. Considering the Taylor expansion of the logarithm, there exists $\varepsilon' > 0$ such that for all $i \in N$ and $|r| < \varepsilon'$,

$$\left| \log(U_i + r) - \log U_i - \frac{r}{U_i} + \frac{1}{2}\left(\frac{r}{U_i}\right)^2 \right| \leq \frac{\varepsilon}{4}\left(\frac{r}{U_i}\right)^2. \tag{2}$$

Now let $C'_N \in \mathbb{R}^n_{\geq 0}$ such that $C'_1 = C_1 + dC_1$ with $0 < dC_1 < \min\{\varepsilon', \frac{\varepsilon}{(2+\varepsilon)} C_1\}$ and $C'_i = C_i$ for all $i \in N \setminus \{1\}$, and let $\theta' = (u, C')$. Consider the function $\phi \colon \mathbb{R}^n \to \mathbb{R}$ defined on dU with $|dU| < \varepsilon^*$, such that

$$\phi(dU) := F_{\theta'}(U + dU) - F_\theta(U) - dC_1 \log U_1 = \sum_{i \in N} C_i \frac{dU_i}{U_i} + dC_1 \frac{dU_1}{U_1} - \psi(dU),$$

for some $\psi \colon \mathbb{R}^n \to \mathbb{R}$ with

$$(1 - \varepsilon)\frac{1}{2} \sum_{i \in N} C_i \left(\frac{dU_i}{U_i} \right)^2 \leq \psi(dU) \leq (1 + \varepsilon)\frac{1}{2} \sum_{i \in N} C_i \left(\frac{dU_i}{U_i} \right)^2.$$

The existence of ψ is guaranteed by (2) and the bound on dC_1.

Now let $U' = \mathcal{U}(\theta')$ and $dU' = U' - U$. Note that, since the only term in $\phi(dU)$ that depends on dU is $F_{\theta'}(U + dU)$, dU' maximizes ϕ among all $dU \in \mathbb{R}^n$ with $U + dU \in \mathcal{P}_\theta$. By Lemma 3, there is $\varepsilon'' > 0$ such that, for all $dU \in \mathbb{R}^n$ with $|dU| \leq \varepsilon''$ and $U + dU \in \mathcal{P}_\theta$, we have $U + rdU \in \mathcal{P}_\theta$ for all $r \in [0, 2]$. Since $\mathcal{U}$ is continuous in C_N by Lemma 2, $|dU'|$ will be small if dC_1 is small and we can choose dC_1 to be even smaller if necessary so that $2|dU'| \leq \min(\varepsilon', \varepsilon'')$. Then, the function $\Phi \colon [0, 2] \to \mathbb{R}$ with $\Phi(r) = \phi(rdU')$ is well-defined and satisfies the prerequisites of Lemma 7 with

$$\alpha = \sum_{i \in N} C_i \frac{dU'_i}{U_i} + dC_1 \frac{dU'_1}{U_1} \quad \text{and} \quad \beta = \frac{1}{2} \sum_{i \in N} C_i \left(\frac{dU'_i}{U_i} \right)^2.$$

Hence, it follows from Lemma 7 that

$$\sum_{i \in N} C_i \frac{dU'_i}{U_i} + dC_1 \frac{dU'_1}{U_1} \geq \mu\Phi(1).$$

Since U maximizes F_θ, by Lemma 4, $\sum_{i \in N} C_i \frac{dU'_i}{U_i} \leq 0$. It follows that

$$dC_1 \frac{dU'_1}{U_1} \geq \mu\Phi(1). \tag{3}$$

Next, let $\delta = g(\theta)$. Let $H_1 = \sum_{a \in A: u_1(a) > 0} \delta(a)$ be the fraction spent on agent 1's acceptable projects, i.e., those that agent 1 assigns positive utility. Recall that $U_1 < 1$, and so $H_1 < 1$. Since $NASH$ satisfies the positive share property, i.e., it gives each agent positive utility, we have $H_1 > 0$. From $0 < H_1 < 1$, we get that $\delta(a) < 1$ for all $a \in A$. Thus, for $|t| > 0$ small enough, take the distribution δ^t with $\delta^t(a) = (1 + t)\delta(a)$ for all $a \in A$ with $u_1(x) > 0$ and $(1 - \frac{H_1}{1 - H_1} t)\delta(a)$ for all $a \in A$ with $u_1(a) = 0$. One can check that $\delta^t \in \Delta(1)$:

$$\sum_{a \in A} \delta^t(a) = \sum_{\substack{a \in A \\ u_1(a) > 0}} (1 + t)\delta(a) + \sum_{\substack{a \in A \\ u_1(a) = 0}} (1 - \frac{H_1}{1 - H_1} t)\delta(a)$$

$$= (1 + t) \sum_{\substack{a \in A \\ u_1(a) > 0}} \delta(a) + (1 - \frac{H_1}{1 - H_1} t) \sum_{\substack{a \in A \\ u_1(a) = 0}} \delta(a)$$

$$= (1 + t)H_1 + (1 - \frac{H_1}{1 - H_1} t)(1 - H_1) = 1.$$

Let $dU^t = u(\delta^t) - U$. For $|t|$ small enough, we have that $U + dU^t \in \mathcal{P}_\theta$ and $U - dU^t_\theta \in \mathcal{P}_\theta$. Indeed, $U + dU^t = u(\delta^t)$, and for the second statement we can perturb δ infinitesimally in the opposite direction. This is a valid perturbation because $\delta(a) < 1$ for all $a \in A$, and for $a \in A$ such that $\delta(a) = 0$ we have $\delta^t(a) = \delta(a)$. Thus, by Lemma 4, we have

$$\sum_{i \in N} C_i \frac{dU^t_i}{U_i} = 0.$$

So for sufficiently small $|t|$, we have

$$\phi(dU^t) = dC_1 \frac{dU^t_1}{U_1} - \psi(dU^t) \geq dC_1 \frac{dU^t_1}{U_1} - (1+\varepsilon)\frac{1}{2}\sum_{i \in N} C_i \left(\frac{dU^t_i}{U_i}\right)^2.$$

Since $dU^t_1 = u_1(\delta^t) - U_1 = (1+t)U_1 - U_1$, we have that $\frac{dU^t_1}{U_1} = t$. Similarly, it follows that $-\frac{H_1}{1-H_1}t \leq \frac{dU^t_i}{U_i} \leq t$ for all $i \in N$.

Now, by definition of H_1, we have $U_1 \geq H_1$. Thus $1 - U_1 \leq 1 - H_1$. Hence $-\frac{U_1}{1-U_1} \leq -\frac{H_1}{1-H_1}$. Thus, applying Lemma 5 with $\alpha = \frac{U_1}{(1-U_1)}t$, $\beta = t$, and $x_i = \frac{dU^t_i}{U_i}$, it follows that

$$\phi(dU^t) \geq dC_1 t - (1+\varepsilon)\frac{1}{2}\frac{U_1 C_\theta}{1-U_1}t^2.$$

Now let $t := \frac{1-U_1}{U_1 C_\theta}dC_1$. If dC_1 is small enough, then t is also small enough and, recalling that dU' maximizes ϕ among all $dU \in \mathbb{R}^n$ with $U + dU \in \mathcal{P}_\theta$, we get

$$\Phi(1) = \phi(dU') \geq \phi(dU^t) \geq \frac{1}{2}(1-\varepsilon)\frac{1-U_1}{U_1 C_\theta}(dC_1)^2.$$

Thus, by (3), we get

$$dC_1 \frac{dU'_1}{U_1} \geq \frac{\mu}{2}(1-\varepsilon)\frac{1-U_1}{U_1 C_\theta}(dC_1)^2,$$

from which it follows from $dC_1 > 0$ that

$$dU'_1 \geq \frac{\mu}{2}(1-\varepsilon)\frac{1-U_1}{C_\theta}dC_1.$$

Since $\mu \in (0, 2)$ was arbitrary and $\varepsilon > 0$ can be chosen arbitrarily small, it follows that

$$dU'_1 \geq \frac{1-U_1}{C_\theta}dC_1.$$

Step 2. Now let $\theta \in \Theta^N$ be arbitrary. We show (1). Let $C^s_N \in \mathbb{R}^N$ such that $C^s_1 = s$ and $C^s_i = C_i$ for all $i \in N \setminus \{1\}$, $\theta^s = (u, C^s_N)$, and $\tilde{\mathcal{U}}(s) = \mathcal{U}(\theta^s)$. Let $C'_1 = \sup\{s \in [0, C_1] : \tilde{\mathcal{U}}_1(s) < 1\}$.

Case 1. Consider the case $C'_1 = 0$. By Lemma 2, $\tilde{\mathcal{U}}_1$ is monotonic in s, and so

$$\tilde{\mathcal{U}}_1(C_1) \geq \tilde{\mathcal{U}}_1(0) = \frac{1}{C_\theta}\left((C_\theta - C_1)\tilde{\mathcal{U}}_1(0) + C_1 \tilde{\mathcal{U}}_1(0)\right) \geq \frac{1}{C_\theta}\left((C_\theta - C_1)\tilde{\mathcal{U}}_1(0) + C_1\right)$$

which proves (1).

Case 2. If $C_1' > 0$, let $\epsilon \in (0, C_1')$ be arbitrary. By Step 1, the lower right derivative of $\tilde{\mathcal{U}}_1$ at $s \in (\epsilon, C_1')$ is at least $\frac{1 - \tilde{\mathcal{U}}_1(s)}{C_\theta - C_1 + s}$. Integrating this estimate from ϵ to C_1' yields

$$- \int_\epsilon^{C_1'} \frac{\frac{\partial \tilde{\mathcal{U}}_1(s)}{\partial s}}{1 - \tilde{\mathcal{U}}_1(s)} ds \le - \int_\epsilon^{C_1'} \frac{1}{C_\theta - C_1 + s} ds$$

from which we get

$$\log(1 - \tilde{\mathcal{U}}_1(C_1')) - \log(1 - \tilde{\mathcal{U}}_1(\epsilon)) \le -(\log(C_\theta - C_1 + C_1') - \log(C_\theta - C_1 + \epsilon)).$$

Exponentiation yields $\frac{1 - \tilde{\mathcal{U}}_1(C_1')}{1 - \tilde{\mathcal{U}}_1(\epsilon)} \le \frac{C_\theta - C_1 + \epsilon}{C_\theta - C_1 + C_1'}$. Since ϵ was arbitrary and $\tilde{\mathcal{U}}_1$ is monotonic, we get $\frac{1 - \tilde{\mathcal{U}}_1(C_1')}{1 - \tilde{\mathcal{U}}_1(0)} \le \frac{C_\theta - C_1}{C_\theta - C_1 + C_1'}$. Rewriting this equation gives us

$$\tilde{\mathcal{U}}_1(C_1') \ge \frac{1}{C_\theta - C_1 + C_1'} \left((C_\theta - C_1) \tilde{\mathcal{U}}_1(0) + C_1' \right)$$

Finally, we note that if $C_1' < C_1$, then $\tilde{\mathcal{U}}_1(C_1') = 1$. Together with monotonicity, this gives us

$$\tilde{\mathcal{U}}_1(C_1) = \frac{1}{C_\theta} \left((C_\theta - C_1 + C_1') \tilde{\mathcal{U}}_1(C_1') + (C_1 - C_1') \tilde{\mathcal{U}}_1(C_1') \right)$$

$$\ge \frac{1}{C_\theta} \left((C_\theta - C_1) \tilde{\mathcal{U}}_1(0) + C_1' + (C_1 - C_1') \tilde{\mathcal{U}}_1(C_1') \right) = \frac{1}{C_\theta} \left((C_\theta - C_1) \tilde{\mathcal{U}}_1(0) + C_1 \right)$$

which is (1). $\qquad \square$

5. Limits of Efficient Mechanisms

In this section, we discuss the limits that we run into if we try to strengthen our notions of decomposability and contribution incentive-compatibility as described in Section 3. Specifically, we show that these strengthenings are incompatible with efficiency.

First, we consider *strong decomposability*, which requires that δ can be divided into individual distributions $(\delta_i)_{i \in N}$ where for each $i \in N$, we have $\delta_i(x) > 0$ only if i has assigned *maximum* utility to x, i.e., only if $u_i(x) \ge u_i(y)$ for all $y \in A$. In other words, each agent is only asked to spend her contributions on her favorite projects.

Proposition 1. *No efficient mechanism satisfies strong decomposability when $m \ge 3$ and $n \ge 2$.*

Proof. To see that strong decomposability is in conflict with efficiency, consider the example in Table 2 below. Here, both agents 1 and 2 have a pet project a and b, respectively, which the other agent dislikes; there is also a compromise project x, which is close to optimal for both. It is best for an agent to spend her entire contribution on her pet project independently of what the other agent is doing. So the only allocation we can implement in the above sense is $a + b$, which gives utility $1 + \varepsilon$ for both. But this fails to make use of the mutual interest in x: if they spent the total endowment of 2 on x, they could achieve utility 2 each. $\qquad \square$

	$u_i(a)$	$u_i(b)$	$u_i(x)$	C_i
Agent 1	$1 + \varepsilon$	0	1	1
Agent 2	0	$1 + \varepsilon$	1	1

Table 2: Type profile with $0 < \varepsilon < 1$ showing the incompatibility of strong decomposability and efficiency.

One can interpret the situation in Table 2 as a prisoner's dilemma in which agents cooperate by spending on x or defect (free-ride) by spending on a and b.

The strengthening of contribution incentive-compatibility we discussed in Section 3 requires that $u_i(f(\theta)) \geq u_i(f(\theta_{-i})) + C_i u_i^{\max}$, where $u_i^{\max} = \max_{y \in A} u_i(y)$. This *strong contribution incentive-compatibility* cannot be satisfied in conjunction with efficiency. The strong version makes sense if agents can use their money to fund public projects without going through the aggregation mechanism. This is typically the case for charities, but may be less applicable for some of the other scenarios discussed in Section 1, such as residents of an apartment complex.

Proposition 2. *No efficient mechanism is strongly contribution incentive-compatible when $m \geq 4$ and $n \geq 3$.*

	$u_i(a)$	$u_i(b)$	$u_i(c)$	$u_i(x)$	C_i
Agent 1	$2 - \varepsilon$	0	0	1	1
Agent 2	0	$2 - \varepsilon$	0	1	1
Agent 3	0	0	$2 - \varepsilon$	1	1

Table 3: Type profile with $0 < \varepsilon < 0.5$ used in the proof of Proposition 2.

Proof. Assume for contradiction that there exists a mechanism f that is strongly contribution incentive-compatible and efficient.

For $\theta = (\theta_1, \theta_2, \theta_3)$ as in Table 3, the distribution $\delta = f(\theta)$ should only allocate resources to at most one of a, b, and c. Otherwise, if there is any subset $\{y, z\} \subset \{a, b, c\}$, $y \neq z$ with $\delta(y) > 0$ and $\delta(z) > 0$, the distribution

$$(\delta(y) - \kappa) \; y + (\delta(z) - \kappa) \; z + (\delta(x) + 2\kappa) \; x$$

with $\kappa = \min(\delta(y), \delta(z))$ is strictly preferred by all three agents. Thus, without loss of generality, we can assume that $\delta(c) = 0$.

Starting with agent 1, we let the other agents join one after another and, using strong contribution incentive-compatibility, derive lower bounds on the resources allocated to

project x. It will turn out that after agent 3 has joined, the mechanism would have to allocate more than the total endowment of 3 to x in order to accommodate strong contribution incentive-compatibility, which is a contradiction.

Let $\theta' = (\theta_1, \theta_2)$ and $\delta' = f(\theta')$. As above, efficiency implies that either $\delta'(a) = 0$ or $\delta'(b) = 0$. Otherwise, if $\delta'(a) > 0$ and $\delta'(b) > 0$, the distribution

$$(\delta'(a) - \kappa')\, a + (\delta'(b) - \kappa')\, b + (\delta'(x) + 2\kappa')\, x$$

with $\kappa' = \min(\delta'(a), \delta'(b))$ is strictly preferred by both agents with a utility improvement of $2\kappa'\varepsilon > 0$. We assume that $\delta'(b) = 0$. Treating the case $\delta'(a) = 0$ requires no more than switching the order of agents 1 and 2.

By strong contribution incentive-compatibility, agent 2 must get at least the same utility as if both agents acted in an uncoordinated manner: $u_2(\delta') \geq u_2(a) + C_2 u_2^{\max} = u_2(a) + (2 - \varepsilon) = 2 - \varepsilon$ and with $\delta'(b) = 0$, we have $\delta'(x) \geq \frac{2-\varepsilon}{u_2(x)} = 2 - \varepsilon$.

Thus the utility of agent 3 from δ' can be bounded from below by $u_3(\delta') \geq \delta'(x)\, u_3(x) \geq (2 - \varepsilon)$.

Applying strong contribution incentive-compatibility for agent 3 yields $u_3(\delta) \geq u_3(\delta') + C_3 u_3^{\max} \geq (2 - \varepsilon) + 1 \cdot (2 - \varepsilon) = 4 - 2\varepsilon$. As $\delta(c) = 0$, agent 3 can only get positive utility from project x, and thus $\delta(x) = \frac{u_3(\delta)}{u_3(x)} \geq 4 - 2\varepsilon > 3$ for $0 < \varepsilon < 0.5$, which exceeds the total endowment of 3. □

The reason for this incompatibility is structurally similar to that for decomposability: efficiency requires spending resources on the compromise project x, but strong contribution incentive-compatibility can only be satisfied if the pet projects a, b, and c are funded.

When only allowing dichotomous utility functions, both decomposability and contribution incentive-compatibility coincide with their strong counterparts. Hence, Propositions 1 and 2 do not apply. In this restricted setting, which has been well-studied by Bogomolnaia et al. (2005), Duddy (2015), and Aziz et al. (2019), *NASH* becomes an even stronger candidate mechanism, though Duddy's (2015) *conditional utilitarian rule*, which returns the decomposable distribution with the highest utilitarian welfare, constitutes an attractive alternative.

Acknowledgements

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A. Omitted proofs

Lemma 1. *For all $\theta \in \Theta^n$, F_θ has a unique maximizer $U \in \mathcal{P}_\theta$. Moreover, $C_i \leq U_i \leq C_\theta u_i^{\max}$ for all $i \in N$.*

Proof. Assume for contradiction that there are two distinct $U', U'' \in \mathcal{P}_\theta$ which maximize F_θ. As a positive linear combination of strictly concave functions, F_θ is a strictly concave function. Hence, for $U = \frac{1}{2}(U' + U'') \in \mathcal{P}_\theta$, by strict concavity of F_θ, we have

$$F_\theta(U) > \frac{1}{2}\left(F_\theta(U') + F_\theta(U'')\right) = F_\theta(U'),$$

which contradicts the assumption that U' maximizes F_θ over $\mathcal{P}_\theta$.

Now let $i \in N$. Clearly, U_i is upper bounded by $C_\theta u_i^{\max}$. For the lower bound on U_i, recall from Theorem 1 that $NASH$ is decomposable. Thus the distribution $NASH(\theta)$ is the sum of distributions δ_j, $j \in N$, such that $\delta_j \in \Delta(C_j)$ and $\delta_j(x) > 0$ only if $u_j(x) > 0$. In particular, $u_i(\delta_i) \geq C_i$, since we have normalized utility functions so that the lowest positive utility of each agent is 1. $\qquad\square$

Recall that $\mathcal{U} \colon \Theta^n \to \mathbb{R}_{\geq 0}^n$ returns the unique maximizer of F_θ. By Lemma 1, $\mathcal{U}$ is well-defined. We show that $\mathcal{U}(\theta)$ depends continuously on the vector of contributions $C_N \in \mathbb{R}_{\geq 0}^n$, where $\theta = (u, C_N)$. Moreover, the utility of every agent is weakly increasing in her contribution.

Lemma 2. *$\mathcal{U}$ is continuous in C_N on $\mathbb{R}_{>0}^n$ and $\mathcal{U}_i$ is weakly increasing in C_i for all $i \in N$.*

Proof. First we show that $\mathcal{U}$ is continuous in C_N on $\mathbb{R}_{>0}^n$. Let $\theta \in \Theta^n$, $\theta = (u, C_N)$ with $C_N \in \mathbb{R}_{>0}^n$, $(C_N^k)_{k \in \mathbb{N}} \subseteq \mathbb{R}_{>0}^n$ converging to C_N, and $\theta^k = (u, C_N^k)$. Further, let $U^k = \mathcal{U}(\theta^k)$ and $U = \mathcal{U}(\theta)$. Observe that since C_N^k converges to C_N, by Lemma 1, $0 < \lambda \leq U_i^k \leq \Lambda$ for all i and some $\lambda, \Lambda > 0$ and large enough k. Hence, by passing to a subsequence if necessary, we may assume that U^k converges to U^* for some $U^* \in \mathcal{P}_\theta$. Since the family of functions F_θ, F_{θ^k}, $k \in \mathbb{N}$, is uniformly equicontinuous on $[\lambda, \Lambda]^n$, it follows that $F_{\theta^k}(U^k)$ converges to $F_\theta(U^*)$. Moreover, as U^k maximizes F_{θ^k}, we have $F_{\theta^k}(U^k) \geq F_{\theta^k}(U)$, which converges to $F_\theta(U)$. Hence, U^* maximizes F_θ, which, since F_θ has a unique maximizer by Lemma 1, implies that $U^* = U$. Hence, U^k converges to U.

To prove that $\mathcal{U}_i$ is monotonically increasing in C_i, let $\theta, \theta' \in \Theta^n$ and $t > 0$ such that $\theta' = (u, C')$ with $C'_1 = C_1 + t$ and $C_i = C'_i$ for all $i \in N \setminus \{1\}$. We show that $\mathcal{U}_1(\theta') \geq \mathcal{U}_1(\theta)$. Let $U = \mathcal{U}_1(\theta)$ and $U' = \mathcal{U}_1(\theta')$ and assume for contradiction that $U'_1 < U_1$. Then,

$$F_{\theta'}(U') = \sum_{i \in N} C_i \log U'_i + t \log U'_1 < \sum_{i \in N} C_i \log U_i + t \log U_1 = F_{\theta'}(U),$$

where the inequality follows from the assumption that $U'_1 < U_1$ and the fact that U is a maximizer of F_θ. This contradicts the assumption that U' maximizes $F_{\theta'}$. $\qquad\square$

Lemma 3. *For every $\theta \in \Theta^n$ and $U \in \mathcal{P}_\theta$, there is $\varepsilon > 0$ such that for all $dU \in \mathbb{R}^n$ with $|dU| \leq \varepsilon$ and $U + dU \in \mathcal{P}_\theta$, we have $U + tdU \in \mathcal{P}_\theta$ for all $t \in [0, 2]$.*

Proof. Since $\mathcal{P}_\theta$ is a polytope, it is an intersection of a finite number of closed half-spaces H_i. Observe that the desired property holds for each H_i. Indeed, if the point U is in the interior of H_i, we can take ε to be half of the distance from U to the boundary of H_i, while if U is on the boundary of H_i, the entire ray $\{U + tdU \mid t \geq 0\}$ is contained in H_i and we can take ε to be any positive real number. It follows that the desired property also holds for the intersection of the half-spaces H_i, which is $\mathcal{P}_\theta$. $\qquad\square$

The next three lemmas will be useful for analyzing error terms obtained in the main analysis.

Lemma 4. *Let $\theta \in \Theta^n$, $U = \mathcal{U}(\theta)$, and $dU \in \mathbb{R}^n$ such that $U + dU \in \mathcal{P}_\theta$. Then,*

$$\sum_{i \in N} C_i \frac{dU_i}{U_i} \leq 0.$$

If also $U - dU \in \mathcal{P}_\theta$, then equality holds.

Proof. Consider the function $\tau \colon [0, 1] \to \mathbb{R}$ with $\tau(t) = F_\theta(U + tdU)$ and observe that τ attains its maximum at 0. Since $U_i > 0$ for all $i \in N$ by Lemma 1, τ is differentiable at 0. Hence, the right derivative of τ at 0 is non-positive, i.e.,

$$\frac{\partial \tau}{\partial t}\Big|_{t=0} = \frac{\partial}{\partial t}\left(\sum_{i \in N} C_i \log(U_i + tdU_i)\right)\Big|_{t=0} = \sum_{i \in N} C_i \frac{dU_i}{U_i} \leq 0.$$

If additionally $U - dU \in \mathcal{P}_\theta$, the first part implies $-\sum_{i \in N} C_i \frac{dU_i}{U_i} \leq 0$, from which equality follows. $\qquad\square$

Lemma 5. *Let $\theta \in \Theta^n$, $x \in \mathbb{R}^n$, and $\alpha, \beta > 0$ such that $\sum_{i \in N} C_i x_i = 0$ and $-\alpha \leq x_i \leq \beta$ for all $i \in N$. Then,*

$$\sum_{i \in N} C_i x_i^2 \leq \alpha\beta \sum_{i \in N} C_i.$$

Proof. Since $-\alpha \leq x_i \leq \beta$, we have $\left|x_i - \frac{\beta-\alpha}{2}\right| \leq \frac{\beta+\alpha}{2}$. It follows that

$$
\begin{aligned}
\sum_{i \in N} C_i x_i^2 &= \sum_{i \in N} C_i \left(x_i - \frac{\beta-\alpha}{2}\right)^2 - \left(\frac{\beta-\alpha}{2}\right)^2 \sum_{i \in N} C_i \\
&\leq \left(\frac{\beta+\alpha}{2}\right)^2 \sum_{i \in N} C_i - \left(\frac{\beta-\alpha}{2}\right)^2 \sum_{i \in N} C_i \\
&= \alpha\beta \sum_{i \in N} C_i,
\end{aligned}
$$

as claimed. $\qquad\square$

For the proof of Lemma 7, we need the following auxiliary lemma.

Lemma 6. *Let $\lambda^* \in (0, \frac{1}{2})$. Then, there are $\varepsilon^* \in (0,1)$ and $t \in [1,2]$ such that*

$$t - \lambda \frac{1+\varepsilon}{1-\varepsilon} t^2 > 1 - \lambda \quad \text{for all } \lambda \in [0, \lambda^*] \text{ and } \varepsilon \in (0, \varepsilon^*).$$

Proof. The inequality in the statement can be rewritten as $\lambda < \frac{t-1}{\frac{1+\varepsilon}{1-\varepsilon} t^2 - 1}$. Choose an arbitrary $t \in (1, \frac{1}{\lambda^*} - 1)$. We have $t \in [1,2]$ and $\lambda^* < \frac{1}{1+t}$. Since $\lim_{\varepsilon \to 0} \frac{t-1}{\frac{1+\varepsilon}{1-\varepsilon} t^2 - 1} = \frac{1}{1+t}$, we can choose $\varepsilon^* \in (0,1)$ such that $\lambda^* < \frac{t-1}{\frac{1+\varepsilon}{1-\varepsilon} t^2 - 1}$ for all $\varepsilon \in (0, \varepsilon^*)$. It follows that $\lambda < \frac{t-1}{\frac{1+\varepsilon}{1-\varepsilon} t^2 - 1}$ for all $\lambda \in [0, \lambda^*]$ and $\varepsilon \in (0, \varepsilon^*)$, as desired. □

Lemma 7. *For all $\mu \in (0,2)$ there is $\varepsilon^* \in (0,1)$ with the following property: For any $\Phi \colon [0,2] \to \mathbb{R}$ such that $\Phi(1) = \max_{t \in [0,2]} \Phi(t)$ and*

$$\alpha t - (1 + \varepsilon)\beta t^2 \le \Phi(t) \le \alpha t - (1 - \varepsilon)\beta t^2,$$

for some $\alpha, \beta \ge 0$ and $\varepsilon \in (0, \varepsilon^*)$ and all $t \in [0,2]$, it holds that $\alpha \ge \mu\Phi(1)$.

Proof. If $\mu \le 1$, then by choosing any $\varepsilon^* \in (0,1)$, we have $\mu\Phi(1) \le \Phi(1) \le \alpha$ by assumption. Assume henceforth that $\mu > 1$. For any $\varepsilon^* \in (0,1)$ that we choose later, note that if $\alpha = 0$, then taking the given $\varepsilon \in (0, \varepsilon^*)$ yields $\Phi(1) \le 0$, so $\alpha \ge \mu\Phi(1)$ always holds. Hence it suffices to consider $\alpha > 0$. Let $\lambda^* := 1 - \frac{1}{\mu} > 0$ and choose $\varepsilon^* > 0$ and $t^* \in [1,2]$ such that

$$t^* - \lambda \frac{1+\varepsilon}{1-\varepsilon} (t^*)^2 > 1 - \lambda$$

for all $\lambda \in [0, \lambda^*]$ and $\varepsilon \in (0, \varepsilon^*)$, which is possible by Lemma 6.

Let $\lambda := \frac{\alpha - \Phi(1)}{\alpha} \ge 0$. Assume for the sake of contradiction that the desired conclusion is not true, i.e., $\alpha < \mu\Phi(1)$. This is equivalent to $\lambda < \lambda^*$. Since the function $\Psi(t) := \alpha t - \Phi(t)$ satisfies $\beta(1 - \varepsilon)t^2 \le \Psi(t) \le \beta(1 + \varepsilon)t^2$, by substituting $t = t^*$ and $t = 1$, we have $\Psi(t^*) \le \Psi(1)\frac{1+\varepsilon}{1-\varepsilon}(t^*)^2$. It follows that

$$\Phi(t^*) = \alpha t^* - \Psi(t^*) \ge \alpha \left(t^* - \frac{\Psi(1)}{\alpha} \frac{1+\varepsilon}{1-\varepsilon}(t^*)^2 \right) = \alpha \left(t^* - \lambda \frac{1+\varepsilon}{1-\varepsilon}(t^*)^2 \right) > \alpha(1-\lambda) = \Phi(1).$$

This contradicts the assumption that $\Phi(1) = \max_{t \in [0,2]} \Phi(t)$. □

B. Characterization of Decomposability

Recall that for $i \in N$, $A_i = \{a \in A \colon u_i(a) > 0\}$ denotes the support of u_i. Moreover, let $\bar{A}_i = \arg\max\{u_i(a) \colon a \in A\}$.

Proposition 3. *Let $\delta \in \Delta(C)$ be a distribution. Then, δ is*

(i) *decomposable if and only if for every $N' \subseteq N$, $\sum_{x \in \bigcup_{i \in N'} A_i} \delta(x) \ge \sum_{i \in N'} C_i$;*

(ii) *strongly decomposable if and only if for every $N' \subseteq N$, $\sum_{x \in \bigcup_{i \in N'} \bar{A}_i} \delta(x) \geq \sum_{i \in N'} C_i$.*

Proof. We prove (i). It is easy to see that the inequalities hold if δ is decomposable. We prove the converse direction by an application of the max-flow min-cut theorem. We start by setting up a flow network, that is, a weighted digraph (V, E, c), where V is a set of vertices, E is a set of (directed) edges, and c is a function that assigns to each edge a capacity. Set $V = N \cup A \cup \{s, t\}$, where s and t are the source and the sink, respectively. For each $i \in N$, add an edge with capacity C_i from s to i and, for each $x \in A_i$, an edge with capacity C_i from i to x. For each $x \in A$, add an edge with capacity $\delta(x)$ from x to t.

A flow in this network with value $C = \sum_{i \in N} C_i$ provides a decomposition of $\delta = \sum_{i \in N} \delta_i$, where $\delta_i(x)$ is the flow along the edge from i to x. We prove that there exists such a flow by showing that every cut has value at least C. The proof is illustrated in Figure 1.

A cut is a partition (V_1, V_2) of V such that $s \in V_1$ and $t \in V_2$. The value $v(V_1, V_2)$ of a cut is the sum of the capacities of edges from V_1 to V_2, that is, $v(V_1, V_2) = \sum_{e \in (V_1 \times V_2) \cap E} c(e)$. Given a cut (V_1, V_2), let $N_j = N \cap V_j$ and $B_j = A \cap V_j$ for $j = 1, 2$. We decompose N_1 further into $N_1' = \{i \in N_1 : A_i \cap B_2 = \emptyset\}$ and $N_1'' = N_1 \setminus N_1'$. Notice that, by assumption, $\sum_{x \in B_1} \delta(x) \geq \sum_{i \in N_1'} C_i$ since no agent in N_1' approves a project in B_2. We can see that

$$v(V_1, V_2) = \sum_{x \in B_1} \delta(x) + \sum_{i \in N_1''} \sum_{x \in A_i \cap B_2} C_i + \sum_{i \in N_2} C_i \geq \sum_{i \in N_1'} C_i + \sum_{i \in N_1''} C_i + \sum_{i \in N_2} C_i = C$$

The first sum counts edges from B_1 to t, the double sum counts edges from N_1 to B_2 (all of which start in N_1''), and the third sum counts edges from s to N_2. For the inequality we use that $\sum_{x \in B_1} \delta(x) \geq \sum_{i \in N_1'} C_i$ as noted above and the fact that $|A_i \cap B_2| \geq 1$ for $i \in N_1''$. Thus, every cut has value at least C, which implies that there is a flow of value C.

The proof for (ii) proceeds along the same lines. The only difference is that now there is an edge from agent i to project x if $x \in \bar{A}_i$. $\qquad \square$

Proposition 3 implies that, for the special case of dichotomous preferences, decomposability is equivalent to the *fair group share* axiom introduced by Bogomolnaia et al. (2002) and latter dubbed *proportional sharing* by Duddy (2015).

C. Contribution Incentive-Compatibility Does Not Imply Decomposability

We construct a mechanism that is contribution incentive-compatible but violates decomposability for $N = \{1, 2, 3\}$ and $A = \{a, b, c, d\}$. This mechanism can be straightforwardly extended to more agents and projects. First, consider the uncoordinated
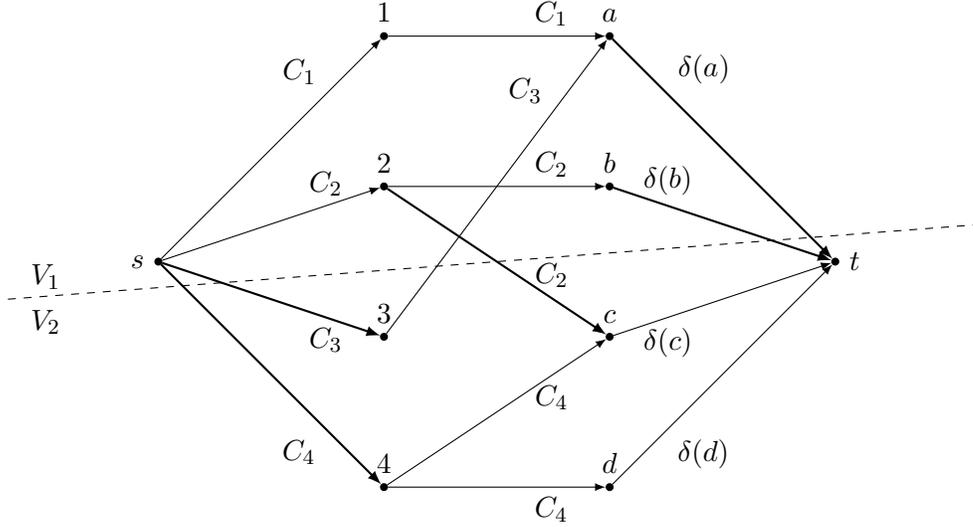
Figure 1: Illustration of the proof of Proposition 3. In this example, $A_1 = \{a\}$, $A_2 = \{b,c\}$, $A_3 = \{a\}$, and $A_4 = \{c,d\}$. The dashed line indicates a cut (V_1, V_2). Hence, $N_1' = \{1\}$, $N_1'' = \{2\}$, $N_2 = \{3,4\}$, $B_1 = \{a,b\}$, and $B_2 = \{c,d\}$. Edges that start in V_1 and end in V_2 count toward the value of the cut and are drawn thicker.

rule UNC, where every agent distributes her contribution uniformly over the projects to which she assigns the highest utility. That is, for a profile $\theta \in \Theta^N$, $UNC(\theta) = \sum_{i \in N} \frac{C_i}{|\arg\max u_i|} \sum_{x \in \arg\max u_i} x$. Notice that UNC (trivially) is decomposable and contribution incentive-compatible.

Now we define $f(\theta)$ as follows. For $B \subset A$, denote by 1_B the dichotomous utility function with $1_B(a) = 1$ for $a \in B$ and $1_B(a) = 0$ for $a \notin B$. If $u_1 = 1_{\{a,b\}}$, $u_2 = 1_{\{a,c\}}$, and $u_3 = 1_{\{d\}}$, let $f(\theta) = UNC(\theta) + \frac{1}{2}\min\{C_1, C_2\} \cdot (a + d - b - c)$ (we refer to this as the "exceptional case") and $f(\theta) = UNC(\theta)$ otherwise.

The outcome of f is not decomposable since in the exceptional case, if, for example, $C_i = 1$ for all $i \in N$, then $f(\theta) = 1.5\,a + 1.5\,d$. On the other hand, f is contribution incentive-compatible. For $i \in \{1,2\}$, this follows from the fact that for any profile $\theta \in \Theta^N$, $u_i(f(\theta)) = u_i(UNC(\theta))$. To see this, observe that in the exceptional case,

$$u_1(f(\theta)) = u_1(UNC(\theta)) + \frac{1}{2}\min\{C_1, C_2\} \underbrace{(u_1(a) - u_1(b))}_{=0} = u_1(UNC(\theta))$$

and similarly for agent 2. In the non-exceptional case, $f(\theta) = UNC(\theta)$ and so $u_i(f(\theta)) = u_i(UNC(\theta))$ trivially holds. Now consider agent 3. In the exceptional case, we have $u_3(f(\theta)) = C_3 + \frac{1}{2}\min\{C_1, C_2\}$ and $u_3(f(\theta_{-3})) = 0$ and so f is contribution incentive-compatible in that case. In the non-exceptional case, we have $f(\theta) = UNC(\theta)$ and $f(\theta_{-3}) = UNC(\theta_{-3})$ and so f is contribution incentive-compatible since UNC is.