Identifying k-Majority Digraphs via SAT Solving

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Key question	**How many voters** are required to **obtain a certain majority relation?**
This paper	• Powerful **SAT-based technique** to solve the question of k-majority digraphs **for arbitrary k** • **Experimental** perspective • Seems like **very few voters suffice** in most cases

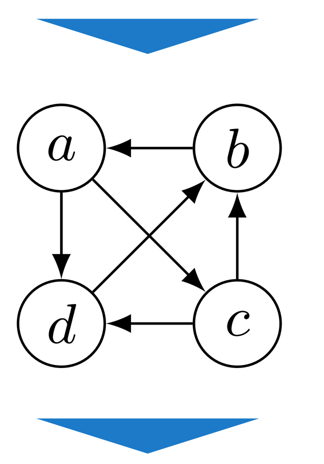
Related Work

- **Known theoretical insight: Any digraph can be realized as the majority relation of a preference profile with**
  - $O(n^2)$ voters (McGarvey, 1953)
  - $O(n/\log n)$ voters (Erdős and Moser, 1964) (non-constructive)
  - $\leq n - \log n + 1$ voters (Fiol, 1992)
- **Successful Applications of SAT in Social Choice Theory**
  - Verification of well-known impossibilities (Tang and Lin, 2009)
  - Automated theorem search for ranking sets of objects (G. and Endriss, 2011)
  - (Im)possibility theorems for strategyproof majoritarian social choice functions (B. and G., 2014)
  - Finding preference profiles of given Condorcet dimension (G., 2014)

Preliminaries

- Preference profiles $R = (R_1, R_2, \ldots, R_k)$
  - Finite set of n alternatives, k voters
  - Voters $i \in \{1, 2, \ldots, k\}$ with linear preference relations R_i over alternatives
- Majority relation $>_R$
  - $a >_R b$ iff $|\{i : a\ R_i\ b\}| > |\{i : b\ R_i\ a\}|$
  - Can be represented by a digraph G (we then say: R induces G)
- **Problem:** Given a digraph G and a positive integer k, is there a preference profile with k voters that induces G? (We then say: G is a k-majority digraph)
- **Voter complexity of G:** minimal k such that G is a k-majority digraph

2	1	1	1
b	c	a	d
a	d	d	c
c	b	c	b
d	a	b	a



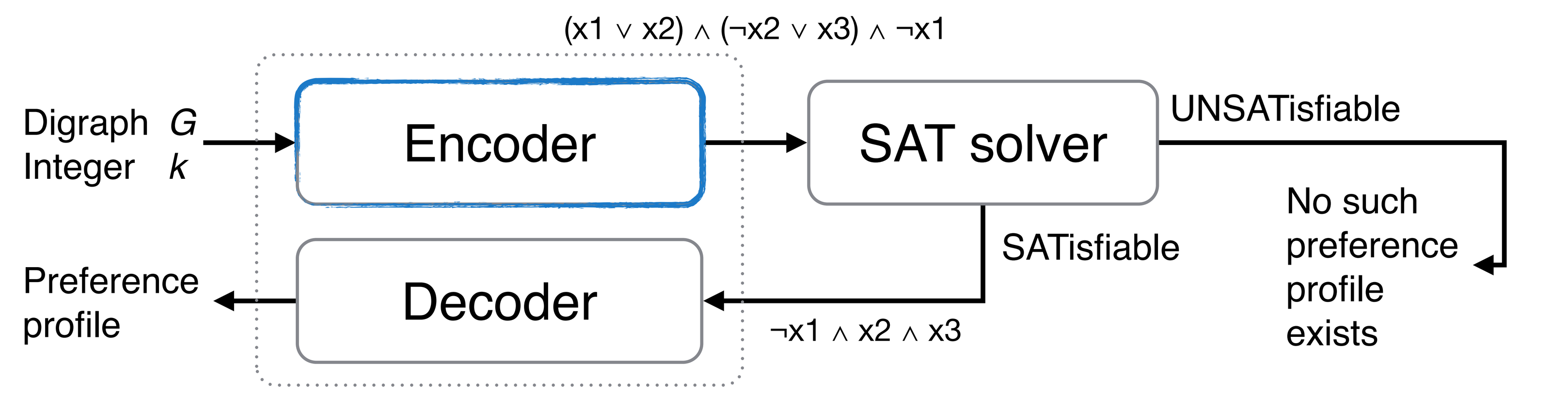
$k = 3$
?

Classical Approach: "Characterize and Conquer"

- **Lemma.** (B. et al., 2013)
  A digraph $(A, >)$ is a 3-majority digraph if and only if $>$ is complete and can be partitioned into $>_1 \cup >_2 = >$ such that
  - $(A, >_1)$ is a 2-majority digraph and
  - $>_2$ is acyclic
- Whether $(A, >_1)$ is a 2-majority digraph can be checked efficiently (Yannakakis, 1982; Dushnik and Miller, 1941)

SAT-based Approach

- Encode any given problem instance into SAT (propositional logic)



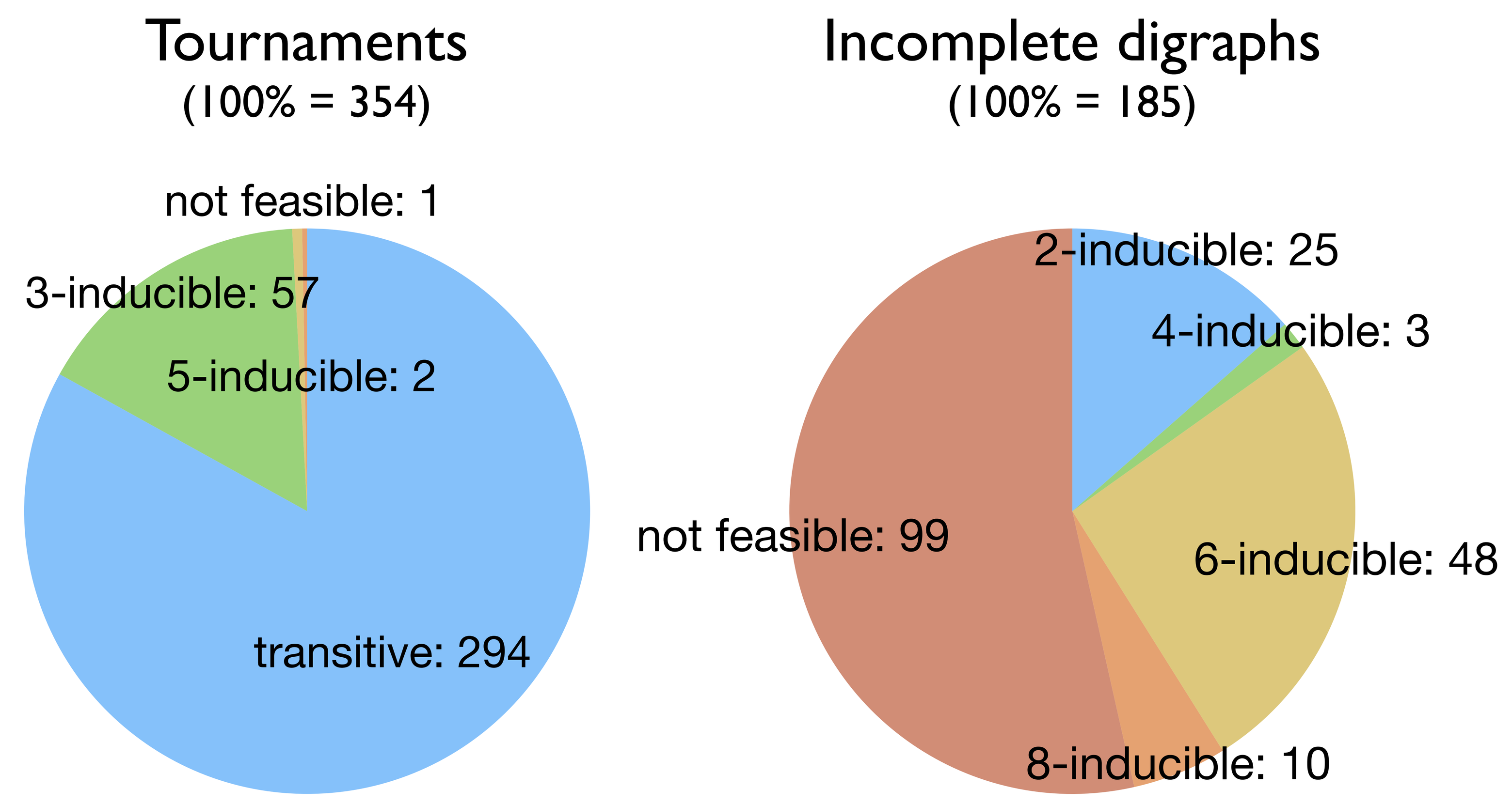
- SAT-based implementation significantly outperforms classical approaches, e.g., running times for k=3 depending on n:

algorithm	5	6	7	8	9	10	20	50	100
SAT	< 0.1s	< 0.1s	< 0.1s	< 0.1s	< 0.1s	< 0.1s	0.1s	1.5s	12.5s
2-Partition	< 0.1s	0.1s	2s	1200s	—	—	—	—	—

Exhaustive Analysis

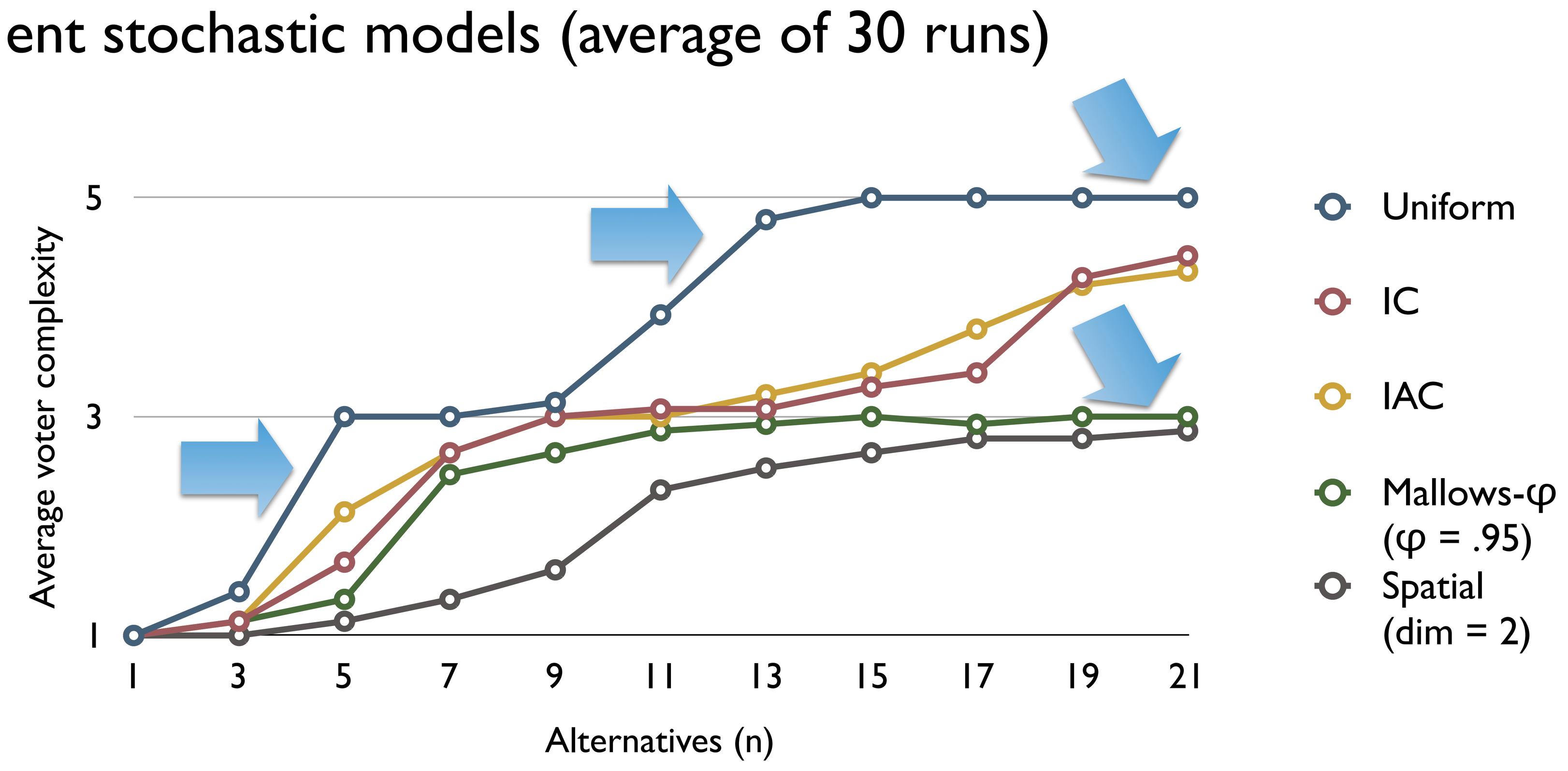
- **Tournaments that are 3-inducible**
  - All tournaments with $n \leq 7$
    (confirming a conjecture by Shepardson and Tovey, 2009)
- **Tournaments that are 5-inducible**
  - All tournaments with $n \leq 10$
  - All (semi-)regular tournaments with $n \leq 12$
  - Millions of instances of tournaments with sizes $10 \leq n \leq 100$
- **Could not find a tournament that is not 5-inducible**
  - Only aware of one concrete tournament with ~600 million nodes
  - Existence of a 42-node tournament from pigeonhole principle

Empirical Analysis (PrefLib)



Stochastic Analysis

- Sampled majority digraphs (with 51 voters) according to 5 different stochastic models (average of 30 runs)



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