Majority Graphs of Assignment Problems and Properties of Popular Random Assignments

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• 2 is more popular than 1
• An assignment is popular if no more popular assignment exists (Gärdenfors, 1975)
How to find popular assignments: the majority graph

- Directed (weighted) graph $G$
  - one vertex per assignment
  - majority margin as edge weights

- Popular assignments are **weak Condorcet winners** in the majority graph
  - do not have to exist...
  - ...but randomized versions exist
Popular random assignments

- **Random assignment**: probability distribution over assignments
- A random assignment is **popular** if no other random assignment is preferred by an expected majority of agents
  - existence guaranteed by Minimax Theorem (Kavitha et al., 2011)

- Computation only requires majority graph
  - Which assignment problems induce identical majority graphs?

- Set of popular random assignments is convex
  - When is there a unique popular random assignment?

- Popularity is incompatible with strong envy-freeness and strong strategyproofness (Aziz et al., 2013)
  - What about weak envy-freeness and weak strategyproofness?
Identical majority graphs: decomposition

- **Decomposition**: Partition houses $H$ into maximal number of subsets $H_1, \ldots, H_k$ s.t.
  \[ H_i > H_j \iff i < j \]

- Two decompositions are **rotation equivalent** if one can be obtained from the other by rotation of $H_1, \ldots, H_k$

- **Theorem**: Two assignment problems induce identical majority graphs iff their decompositions are rotation equivalent
**Identical majority graphs: decomposition**

- **Theorem**: Two assignment problems induce identical majority graphs iff their decompositions are rotation equivalent.

- Check in poly. time if two assignment problems induce identical majority graphs.

- Check in poly. time if given majority graph is induced by some assignment problem.

- Rotation equivalent decompositions imply identical popular random assignments.

- Vast majority of profiles induce unique majority graph.
Uniqueness of popular random assignments

• Assumption: agents share identical preferences

• **Theorem**: A random assignment $p$ is popular
  iff $p_{ij} = p_{i,j+2}$ for all $i, j$

  - Odd $n$: unique popular random assignment: $p_{ij} = \frac{1}{n}$
  - Even $n$: infinitely many popular random assignments, for $n = 4$ e.g. $q$ and $q'$

\[
q = \begin{pmatrix}
\frac{1}{2} & 0 & \frac{1}{2} & 0 \\
\frac{1}{2} & 0 & \frac{1}{2} & 0 \\
0 & \frac{1}{2} & 0 & \frac{1}{2} \\
0 & \frac{1}{2} & 0 & \frac{1}{2}
\end{pmatrix}
\]

\[
q' = \begin{pmatrix}
\frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\
\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\
\frac{5}{12} & \frac{5}{12} & \frac{5}{12} & \frac{1}{12} \\
0 & \frac{1}{2} & 0 & \frac{1}{2}
\end{pmatrix}
\]
Uniqueness of popular random assignments

- No obvious criterion for uniqueness

- Explicit computation infeasible for larger $n$
  - $\sim 10^{17}$ profiles for $n = 6$

- Computer experiments to gain an insight
  - preferences sampled by Impartial Culture or Spatial (2-dim euclidean)
  - 10 000 samples for each $n$

- Fraction of assignment problems admitting unique popular random assignment decreases exponentially in $n$
Popularity vs. envy-freeness

• **Strongly envy-free**: own assignment preferred to all other for all vNM fct.’s
  ‣ violated if someone’s assignment preferred according to some vNM fct.

• **Weakly envy-free**: no one’s assignment preferred to own for all vNM fct.’s
  ‣ violated if someone’s assignment preferred according to all vNM fct.’s

• Theorem (Aziz et al., 2013): Popularity and strong envy-freeness are incompatible for some assignment problem \((n \geq 3)\)

• **Theorem**: Popularity and weak envy-freeness are incompatible for some assignment problem \((n \geq 5)\)
Popularity vs. strategyproofness

- **Strongly strategyproof**: Truth-telling is preferred to lying for all vNM fct.’s
  - violated if manipulation possible for some vNM fct.

- **Weakly strategyproof**: Lying is never preferred to truth-telling (for all vNM fct.’s)
  - violated if manipulation possible for all vNM fct.’s

- Theorem (Aziz et al., 2013):
  Popularity and strong strategyproofness are incompatible ($n \geq 3$)

- Theorem: Popularity and weak strategyproofness are incompatible ($n \geq 7$)
Conclusion

• **Main contributions**
  ‣ equivalence theorem linking assignment problems and majority graphs
  ‣ analysis of uniqueness of popular random assignments
  ‣ solution to two open problems by Aziz et al. (2013) regarding weak envy-freeness and weak strategyproofness

• **Open problem**
  ‣ Does the impossibility for weak strategyproofness also hold for efficiency w.r.t. *pairwise comparison* instead of popularity?