Negotiating Socially Optimal Resource Allocations
Seminar Economics and Computation

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Notation

- $\mathcal{A}$: finite set of agents
- $\mathcal{R}$: finite set of (indivisible) resources
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- $\mathcal{A}$: finite set of agents
- $\mathcal{R}$: finite set of (indivisible) resources

Definition
**Allocation**: a function $A : \mathcal{A} \rightarrow 2^{\mathcal{R}}$, such that for $i \neq j : A(i) \cap A(j) = \emptyset$ and $\bigcup_{i \in \mathcal{A}} A(i) = \mathcal{R}$. 
Utility functions

**Definition**

Utility function for agent $i$: $u_i : 2^\mathcal{R} \rightarrow \mathbb{R}$

Determines the utility the agent gets from holding a certain bundle of resources.
Types of utility functions

Let $R, R_1, R_2 \subseteq \mathcal{R}$. $u_i$ is called

- **non-negative (positive)**: $u_i(R) \geq (>)0$
Types of utility functions

Let $R, R_1, R_2 \subseteq \mathcal{R}$. $u_i$ is called

- non-negative (positive): $u_i(R) \geq (>) 0$
- monotonic: $R_1 \subseteq R_2 \Rightarrow u_i(R_1) \leq u_i(R_2)$
Types of utility functions

Let $R, R_1, R_2 \subseteq \mathcal{R}$. $u_i$ is called

- **non-negative (positive)**: $u_i(R) \geq (>)0$
- **monotonic**: $R_1 \subseteq R_2 \Rightarrow u_i(R_1) \leq u_i(R_2)$
- **additive**: $u_i(R) = \sum_{r \in R} u_i(\{r\})$
Types of utility functions

Let $R, R_1, R_2 \subseteq \mathcal{R}$. $u_i$ is called

- non-negative (positive): $u_i(R) \geq (>)0$
- monotonic: $R_1 \subseteq R_2 \Rightarrow u_i(R_1) \leq u_i(R_2)$
- additive: $u_i(R) = \sum_{r \in R} u_i(\{r\})$
- 0-1 function: iff it’s additive and $u_i(\{r\}) \in \{0, 1\} \forall r \in \mathcal{R}$
Types of utility functions

Let \( R, R_1, R_2 \subseteq \mathcal{R} \). \( u_i \) is called

- **non-negative (positive)**: \( u_i(R) \geq (>)0 \)
- **monotonic**: \( R_1 \subseteq R_2 \Rightarrow u_i(R_1) \leq u_i(R_2) \)
- **additive**: \( u_i(R) = \sum_{r \in R} u_i(\{r\}) \)
- **0-1 function**: iff it’s additive and \( u_i(\{r\}) \in \{0, 1\} \forall r \in \mathcal{R} \)
- **dichotomous** iff \( u_i(R) \in \{0, 1\} \)
What is a 'Socially Optimal' outcome?

We need some kind of measure like a 'utility function for society.'
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We need some kind of measure like a 'utility function for society.'

→ *Social Welfare Function*
What is a 'Socially Optimal' outcome?

We need some kind of measure like a 'utility function for society.'

→ Social Welfare Function

**Definition**

A SWF is a function

\[ sw : \{A | A \text{ is allocation}\} \rightarrow \mathbb{R} \]

that measures the welfare of a society.
In a utilitarian setting, the welfare of society is the total utility of all its agents.

\[ sw_u(A) = \sum_{i \in A} u_i(A) \]
**Deals**

**Definition**

**Deal:** a pair $\delta = (A, A')$ whith $A \neq A'$. 
Deals

Definition

Deal: a pair $\delta = (A, A')$ with $A \neq A'$.

The set of agents involved in $\delta$: $A^\delta := \{i \in A | A(i) \neq A'(i)\}$. 
**Definition**

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The set of agents involved in \( \delta \): \( A^\delta := \{ i \in A | A(i) \neq A'(i) \} \).

For two deals \( \delta_1 = (A, A'), \delta_2 = (A, A'') \)
\( \delta_1 \circ \delta_2 := (A, A'') \) is the *composition* of \( \delta_1, \delta_2 \)
Deals

**Definition**

**Deal:** a pair \( \delta = (A, A') \) with \( A \neq A' \).

The set of *agents involved in* \( \delta \): \( A^\delta := \{ i \in A | A(i) \neq A'(i) \} \).

For two deals \( \delta_1 = (A, A'), \delta_2 = (A, A'') \), \( \delta_1 \circ \delta_2 := (A, A'') \) is the *composition* of \( \delta_1, \delta_2 \).

**Definition**

A deal is called **independently decomposable** iff it can be broken down into two deals, such that no agent is involved in both of them: \( \exists \delta_1, \delta_2 : \delta = \delta_1 \circ \delta_2 \) and \( A^{\delta_1} \cap A^{\delta_2} = \emptyset \).
In the framework a deal is called 'acceptable' if it fulfills a certain criterion.
Example criterion: *individual rational deal*: All involved agents are made strictly better off by a deal.
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Example criterion: *individual rational deal*: All involved agents are made strictly better off by a deal.

We assume that each agent will realize every acceptable deal that is offered to them!
Goal of the Paper

Find criteria for acceptability to achieve Social Optimum.
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Find criteria for acceptability to achieve Social Optimum.

Finding valid acceptable deals can be complex and is NOT part of this framework!
### Table: Overview of results

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<th>Necessity</th>
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<td>Yes</td>
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<tr>
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</table>
Outline for section 2

1. Introduction

2. Utilitarian SWF with Side Payments
   - Payments
   - Indiv. Rationality and Utilitarian SWF
   - Necessity of Deals
   - Additive Scenarios

3. Utilitarian SWF without Side Payments

4. Social Welfare Functions

5. Egalitarian SW

6. Lorenz Optimality
Payment functions and Rationality

**Definition**

**Payment function** $p : \mathcal{A} \to \mathbb{R}$, such that $\sum_{i \in \mathcal{A}} p(i) = 0$. This is essentially money.
Payment functions and Rationality

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**Payment function** $p : \mathcal{A} \to \mathbb{R}$, such that $\sum_{i \in \mathcal{A}} p(i) = 0$.

→ This is essentially money.
Individual Rationality

Definition
A deal is called **individually rational** iff
\[ \exists p : u_i(A') - u_i(A) > p(i) \quad \forall i \in A, \]
except possibly \( p(i) = 0 \) for \( i \notin A^\delta \).

This implies that each agent has unlimited money available.
<table>
<thead>
<tr>
<th>$u_1({}})$</th>
<th>$u_2({}})$</th>
<th>$u_1({r_1}})$</th>
<th>$u_2({r_1}})$</th>
<th>$u_1({r_2}})$</th>
<th>$u_2({r_2}})$</th>
<th>$u_1({r_1, r_2}})$</th>
<th>$u_2({r_1, r_2}})$</th>
</tr>
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<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>7</td>
<td>8</td>
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Lemma 1

\( \delta = (A, A') \) is individually rational iff \( sw_u(A') > sw_u(A) \)
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\( \delta = (A, A') \) is individually rational iff \( sw_u(A') > sw_u(A) \)

Proof
\( \Rightarrow \): \( u_i(A') > u_i(A) + p(i) \). Simply sum over \( A \)
\( \Leftarrow \): Construct appropriate payment function.
Theorem 1

Any sequence of individually rational deals will eventually result in an allocation with maximal utilitarian social welfare.
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Any sequence of individually rational deals will eventually result in an allocation with maximal utilitarian social welfare.

Proof sketch

- Every sequence must terminate. (Strict increase in each step, finite $A, R$)
- Assume final state $A$ isn’t optimal. Then
  $\exists A': sw_u(A') > sw_u(A)$. But then $\delta := (A, A')$ is individual rational by Lemma 1.
Necessity of Deals

Theorem 2
Let the \( \mathcal{A}, \mathcal{R} \) be fixed. Then for every deal \( \delta \) that is not independently decomposable, there exist utility functions and an initial allocation such that any sequence of individually rational deals leading to an allocation with maximal utilitarian social welfare must include \( \delta \).
This continues to be the case even when either all utility functions are required to be monotonic or all utility functions are required to be dichotomous.
Theorem 2

Let the $\mathcal{A}, \mathcal{R}$ be fixed. Then for every deal $\delta$ that is not independently decomposable, there exist utility functions and an initial allocation such that any sequence of individually rational deals leading to an allocation with maximal utilitarian social welfare must include $\delta$.

This continues to be the case even when either all utility functions are required to be monotonic or all utility functions are required to be dichotomous.

Is this good or bad? Why?
1-Deals in additive scenarios

Definition

A deal is called a 1-deal if it involves exactly one resource (and thus exactly 2 agents.)
1-Deals in additive scenarios

Definition
A deal is called a 1-deal if it involves exactly one resource (and thus exactly 2 agents.)

Theorem 3
In additive scenarios, any sequence of individually rational 1-deals will eventually result in an allocation with maximal utilitarian social welfare.
## Results

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**Table:** Overview of results
Outline for section 3

1. Introduction

2. Utilitarian SWF with Side Payments

3. Utilitarian SWF without Side Payments
   - What’s changed?
   - Cooperative Rationality and Pareto Optimality
   - 0-1 Scenarios

4. Social Welfare Functions

5. Egalitarian SW

6. Lorenz Optimality
Example: Why Money is useful

\[
\begin{align*}
    u_1(\{\} ) &= 0 \\
    u_1(\{r\} ) &= 4 \\
    u_2(\{\} ) &= 0 \\
    u_2(\{r\} ) &= 7
\end{align*}
\]
A deal is called **cooperatively rational** if no agent is made worse off, and at least one player is made strictly better off.
Cooperative Rationality and Pareto Optimality

**Definition**
A deal is called *cooperatively rational* if no agent is made worse off, and at least one player is made strictly better off.

**Theorem 4**
Any sequence of cooperatively rational deals will eventually result in a Pareto optimal allocation of resources.
Cooperative Rationality and Pareto Optimality

Definition

A deal is called **cooperatively rational** if no agent is made worse off, and at least one player is made strictly better off.

Theorem 4

Any sequence of cooperatively rational deals will eventually result in a Pareto optimal allocation of resources.

Theorem 5

Necessity holds as well.
Theorem 6

In 0-1 scenarios, any sequence of cooperatively rational 1-deals will eventually result in an allocation with maximal utilitarian social welfare.
Theorem 6

In 0-1 scenarios, any sequence of cooperatively rational 1-deals will eventually result in an allocation with maximal utilitarian social welfare.

Possible Interpretation: 1 iff agent $i$ ’needs’ a resource, 0 otherwise.
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Outline for section 4

1. Introduction
2. Utilitarian SWF with Side Payments
3. Utilitarian SWF without Side Payments
4. Social Welfare Functions
5. Egalitarian SW
6. Lorenz Optimality
7. Conclusion
What other types of SWFs exist?

Do they make any sense?
Outline for section 5

1. Introduction

2. Utilitarian SWF with Side Payments

3. Utilitarian SWF without Side Payments

4. Social Welfare Functions

5. Egalitarian SW
   - Pigou-Dalton transfers and equitable deals

6. Lorenz Optimality

7. Conclusion
Egalitarian Social Welfare

\[ sw_e(A) := \min_{i \in A} u_i(A) \]

Social welfare is determined by the utility of the 'weakest' member of society.
Pigou-Dalton transfers

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<td>A deal is called <strong>Pigou-Dalton transfer</strong> iff</td>
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<tr>
<td>(i) Only two agents are involved: $A^\delta = i, j$</td>
</tr>
<tr>
<td>(ii) $\delta$ is mean preserving: $u_i(A) + u_j(A) = u_i(A') + u_j(A')$, and</td>
</tr>
<tr>
<td>(iii) <em>delta</em> reduces inequality: $</td>
</tr>
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</table>
PD transfers are not enough!

\[
\begin{align*}
 u_1(\{\} & ) = 0 & u_2(\{\} & ) = 0 \\
 u_1(\{r_1\} & ) = 3 & u_2(\{r_1\} & ) = 5 \\
 u_1(\{r_2\} & ) = 12 & u_2(\{r_2\} & ) = 7 \\
 u_1(\{r_1, r_2\} & ) = 15 & u_2(\{r_1, r_2\} & ) = 17
\end{align*}
\]
A deal is **Equitable** iff the 'worst off' involved agent is 'better off' after the deal.

\[
\min\{u_i(A) \mid i \in A^\delta\} < \min\{u_i(A') \mid i \in A^\delta\}
\]
Equitable Deal

Theorem 7
Any sequence of equitable deals will eventually result in an allocation of resources with maximal egalitarian social welfare.

**Definition**
A deal is **Equitable** iff the 'worst off' involved agent is 'better off' after the deal.

\[
\min\{u_i(A) | i \in \mathcal{A}^\delta\} < \min\{u_i(A') | i \in \mathcal{A}^\delta\}
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Definition

A deal is **Equitable** iff the 'worst off' involved agent is 'better off' after the deal.

\[\min\{u_i(A) | i \in A^\delta\} < \min\{u_i(A') | i \in A^\delta\}\]

Theorem 7

Any sequence of equitable deals will eventually result in an allocation of resources with maximal egalitarian social welfare.

Necessity (Theorem 8) holds as well.
Outline for section 6

1. Introduction

2. Utilitarian SWF with Side Payments

3. Utilitarian SWF without Side Payments

4. Social Welfare Functions

5. Egalitarian SW

6. Lorenz Optimality

7. Conclusion
Leximin Ordering and Lorenz Domination

Definition

Allocation $A$ is Lorenz dominated by $A'$ iff

$$\sum_{i=1}^{k} \tilde{u}_i(A) \leq \sum_{i=1}^{k} \tilde{u}_i(A')$$

for all $1 \leq k \leq n$ and strict inequality for at least one $k$. 

Let's 'define' $\tilde{u}$ by example.
Leximin Ordering and Lorenz Domination

Definition

Allocation $A$ is Lorenz dominated by $A'$ iff

$$\sum_{i=1}^{k} \bar{u}_i(A) \leq \sum_{i=1}^{k} \bar{u}_i(A')$$

for all $1 \leq k \leq n$ and strict inequality for at least one $k$.

Let’s ’define’ $\bar{u}$ by example.
Lorenz Domination: An Example

\[
\begin{align*}
  u_1(\{\} &= 0 & u_2(\{\} &= 0 & u_3(\{\} &= 0 \\
  u_1(\{r_1\}) &= 6 & u_2(\{r_1\}) &= 1 & u_3(\{r_1\}) &= 1 \\
  u_1(\{r_2\}) &= 1 & u_2(\{r_2\}) &= 6 & u_3(\{r_2\}) &= 1 \\
  u_1(\{r_1, r_2\}) &= 7 & u_2(\{r_1, r_2\}) &= 7 & u_3(\{r_1, r_2\}) &= 10
\end{align*}
\]
A sufficient criterion for Lorenz optimality

**Definition**

δ is a **Simple Pareto-Pigou-Dalton deal** iff it is

(i) a 1-deal and

(ii) either cooperatively rational or a Pigou-Dalton transfer.
A sufficient criterion for Lorenz optimality

Definition

δ is a **Simple Pareto-Pigou-Dalton deal** iff it is
(i) a 1-deal and
(ii) either cooperatively rational or a Pigou-Dalton transfer.

Theorem 9

In 0-1 scenarios, any sequence of simple Pareto-Pigou-Dalton deals will eventually result in a Lorenz optimal allocation of resources.
Outline for section 7

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2. Utilitarian SWF with Side Payments
3. Utilitarian SWF without Side Payments
4. Social Welfare Functions
5. Egalitarian SW
6. Lorenz Optimality
7. Conclusion
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Table: Overview of results
Enforcing certain types of deals is enough to achieve social optimality in many situations. Actually finding feasible deals is often hard.
Conclusions

- Enforcing certain types of deals is enough to achieve social optimality in many situations.
- Actually finding feasible deals is often hard.

Thank you for your attention!