Optimal Strategies for a Generalized "Scissors, Paper, and Stone" Game

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Example: Scissors, Paper, Stone

Visualization

Scissors

Paper

Stone

Explanation of the Scissors, Paper, Stone game

- Scissors cuts paper
- Paper smothers stone
- Stone smashes scissors
Game with five nodes

Structure

Explanation

- Both of the players each chose a node
- If both chosen nodes are equal: game is tied
- Else: node at the end of the edge wins
**Tournament**

**Modelling as directed graph**
- Nodes equal options
- Directed edges equal a 'is beaten by'-relation
- Graph is complete

**Definition: Tournament**
A directed graph $G(V, E)$ with exactly one directed edge $e_{ij} = (v_i, v_j) \in E$ between each pair of nodes $v_i, v_j \in V$ is called a tournament (-game).
Main idea

Idea
- Specifying strategy as vector \( p = (p_1, \cdots, p_n) \)
- The node is always randomly chosen
- Each node has its own probability

Definition of strategy
- Vector \( p = (p_1, p_2, \cdots, p_n) \)
- \( \sum_{i=1}^{n} p_i = 1 \)
- \( 1 \geq p_i \geq 0 \ \forall i \)
Example

Graphs with 6, 7 and 8 Nodes

Strategy vectors

- $p = (0, \frac{1}{3}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{3})$
- $p = \left(\frac{9}{35}, \frac{1}{7}, \frac{1}{5}, \frac{1}{35}, \frac{1}{35}, \frac{1}{5}, \frac{1}{7}\right)$
- $p = (0, 0, 0, \frac{1}{3}, 0, \frac{1}{3}, 0, \frac{1}{3})$
Example

Graph with 8 Nodes

Strategy vector

- Blue nodes build a 3-cycle (see red edges)
- The green nodes are beaten by the cycle 3 out of 3 times
- The orange nodes are beaten by the cycle 2 out of 3 times

\[
p = \left(0, 0, 0, \frac{1}{3}, 0, \frac{1}{3}, 0, \frac{1}{3}\right)
\]
Payoff Matrix

Definition of the Payoff Matrix

\[ k_{ij} = \begin{cases} 
0 & \text{if } i = j \\
1 & \text{if } (i, j) \text{ is an arc} \\
-1 & \text{if } (j, i) \text{ is an arc} 
\end{cases} \]

\[ K(T) = \begin{pmatrix} 
0 & 1 & -1 & -1 \\
-1 & 0 & 1 & 1 \\
1 & -1 & 0 & -1 \\
1 & -1 & 1 & 0 
\end{pmatrix} \]
### Definition of the Payoff Matrix

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0 & \text{if } i = j \\
1 & \text{if } (i, j) \text{ is an arc} \\
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\end{cases}$$

$$K(T) = \begin{pmatrix}
0 & 1 & -1 & -1 \\
-1 & 0 & 1 & 1 \\
1 & -1 & 0 & -1 \\
1 & -1 & 1 & 0
\end{pmatrix}$$

### Annotation

$K(T)$ is a skew-symmetric matrix.

### Derivation

- Player $B$ plays node $i$ and player $A$ uses strategy $p$
  \Rightarrow average win: $(K(T)p)_i$

- Player $B$ wants to minimize the win of player $A$
  \Rightarrow average win: $\min_i (K(T)p)_i$
Maximin problem

**Derivation**

- Player $B$ plays node $i$ and player $A$ uses strategy $p$
  \[ \Rightarrow \text{average win: } (K(T)p)_i \]
- Player $B$ wants to minimize the win of player $A$
  \[ \Rightarrow \text{average win: } \min_i (K(T)p)_i \]

**Maximin problem**

\[ v \equiv \max_{p \geq 0; 1^T p = 1} \left( \min_i (K(T)p)_i \right) \]

**Annotation**

Player $B$ may use the same strategy as player $A$

\[ \Rightarrow v = 0 \]
Maximin problem

\[ v \equiv \max_{p \geq 0; 1^T p = 1} \left( \min_i (K(T)p)_i \right) \]

Optimal Strategy

The Maximin problem can be simplified: \( p \) is an optimal Strategy, if following conditions are fulfilled:

\[
\begin{align*}
K(T)p & \geq 0 \\
p & \geq 0 \\
1^T p & = 1
\end{align*}
\]
Optimal Strategy - Example

Example with 4 nodes

\[ K(T) = \begin{pmatrix}
0 & 1 & -1 & -1 \\
-1 & 0 & 1 & 1 \\
1 & -1 & 0 & -1 \\
1 & -1 & 1 & 0
\end{pmatrix} \quad p = \begin{pmatrix}
\frac{1}{3} & \text{paper} \\
\frac{1}{3} & \text{scissors} \\
\frac{1}{3} & \text{well} \\
0 & \text{stone}
\end{pmatrix} \]

Annotation

The well dominates the stone \( \Rightarrow \) The 3-cycle paper/scissors/well beats all other nodes at least 2 out of 3 times.
Lemma 1

Lemma (1)

Let $p$ and $q$ be optimal strategies for a tournament game on a tournament, $T$. Then $q_i > 0$ implies $(K(T)p)_i = 0$.

Conclusion

- Player A plays optimal strategy $\Rightarrow$ Winnings $\geq 0$ (B minimizes the winnings)
- Select nodes which make winnings zero $\Rightarrow$ Subtournaments
Positive Tournaments

Definition

A tournament, $T$, is **positive** if there is a positive vector, $p$ with $(K(T)p) = 0$.

Example

$$K(T) = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}, \quad p = \left( \frac{1}{3} \frac{1}{3} \frac{1}{3} \right)$$
Theorem 1

Let $n$ be an odd number. Then the number of $n$-node positive tournaments is

$$\sum \frac{2^{f(d_1, d_3, \cdots, d_n)}}{d_1! \cdot 1^{d_1} \cdot d_3! \cdot 2^{d_3} \cdots d_n! \cdot n^{d_n}}$$

where

$$f(d_1, d_3, \cdots, d_n) = 1 + \frac{1}{2} \sum_{k=1,3,\cdots,n} d_k \left( -3 + d_1 \gcd(k, 1) + d_3 \gcd(k, 3) + \cdots + d_n \gcd(k, n) \right)$$

and the outer summation is over all nonnegative $d_1, d_3, \cdots, d_n$ with

$$d_1 + 3d_3 + 5d_5 + \cdots + nd_n = n$$

Conclusion

Calculation of the number of positive Tournaments in a game with an odd number of nodes is now possible.
Theorem 1 - Example

Example for 5 nodes

<table>
<thead>
<tr>
<th>$d_1, d_3, d_5$</th>
<th>$f(d_1, d_3, d_5)$</th>
<th>Summand</th>
</tr>
</thead>
<tbody>
<tr>
<td>0, 0, 1</td>
<td>$1 + \frac{1}{2} [1 (-3 + 1 \cdot 5)] = 2$</td>
<td>$\frac{2^2}{1! \cdot 5^1} = \frac{4}{5}$</td>
</tr>
<tr>
<td>2, 1, 0</td>
<td>$1 + \frac{1}{2} [2 (-3 + 2 \cdot 1 + 1 \cdot 1) + 1 (-3 + 2 \cdot 1 + 1 \cdot 3)] = 2$</td>
<td>$\frac{2^2}{2! \cdot 1^2 \cdot 3^1} = \frac{2}{3}$</td>
</tr>
<tr>
<td>5, 0, 0</td>
<td>$1 + \frac{1}{2} [5 (-3 + 5 \cdot 1)] = 6$</td>
<td>$\frac{2^6}{5! \cdot 1^5} = \frac{8}{15}$</td>
</tr>
</tbody>
</table>

Number of positive Tournaments on 5 nodes: $\frac{4}{5} + \frac{2}{3} + \frac{8}{15} = 2$

The 2 tournament graphs

![Two tournament graphs for 5 nodes](image_url)
Theorem (2)

Let $T$ be a tournament on $n$ nodes. Then

$$
\text{rank}(K(T)) = \begin{cases} 
  n & \text{if } n \text{ is even} \\
  n - 1 & \text{if } n \text{ is odd}
\end{cases}
$$
Theorem 2 - Example

**Theorem (2)**

Let $T$ be a tournament on $n$ nodes. Then

$$\text{rank}(K(T)) = \begin{cases} 
  n & \text{if } n \text{ is even} \\
  n - 1 & \text{if } n \text{ is odd}
\end{cases}$$

**Example**

$$K(T) = \begin{pmatrix}
  0 & 1 & -1 & -1 \\
  -1 & 0 & 1 & 1 \\
  1 & -1 & 0 & -1 \\
  1 & -1 & 1 & 0
\end{pmatrix}$$

$$\text{det}(K(T)) = 1 \cdot \begin{pmatrix}
  -1 & 1 & 1 \\
  1 & 0 & -1 \\
  1 & 1 & 0
\end{pmatrix} - 1 \cdot \begin{pmatrix}
  -1 & 0 & 1 \\
  1 & -1 & -1 \\
  1 & -1 & 0
\end{pmatrix} - 1 \cdot \begin{pmatrix}
  -1 & 0 & 1 \\
  1 & -1 & 0 \\
  1 & -1 & 1
\end{pmatrix} \neq 0$$

$$\Rightarrow \text{rank}(K(T)) = 4$$
Theorem (2)

Let $T$ be a tournament on $n$ nodes. Then

$$\text{rank}(K(T)) = \begin{cases} n & \text{if } n \text{ is even} \\ n - 1 & \text{if } n \text{ is odd} \end{cases}$$

Proof - Idea

Case 1: $n$ is even

- For better understanding: Laplace’s formula
- Number of nonzero terms equals derangements $D_n$
- $D_n$ is odd which means $D_n \neq 0$
- $\Rightarrow \det(K(T))$ is odd
- $\text{rank}(K(T)) = n$
**Theorem 2 - Idea of the Proof**

**Theorem (2)**

Let $T$ be a tournament on $n$ nodes. Then

$$\text{rank}(K(T)) = \begin{cases} 
  n & \text{if } n \text{ is even} \\
  n - 1 & \text{if } n \text{ is odd}
\end{cases}$$

**Proof - Idea**

Case 2: $n$ is odd

- $K(T)$ is skew-symmetric: $K(T)^T = -K(T)$
- $\det(K(T)) = \det(K(T)^T) = \det(-K(T)) = (-1)^n \det(K(T))$
- $\Rightarrow \det(K(T)) = 0$
- $\text{rank}(K(T)) = n - 1$
Corollary (1)

Positive Tournaments have an odd number of nodes.

Example: 3 nodes

\[
K(T) = \begin{pmatrix}
0 & 1 & -1 \\
-1 & 0 & 1 \\
1 & -1 & 0
\end{pmatrix},
p = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)
\]
The tournament game on an \( n \) node tournament \( T \) has a unique optimal strategy, \( p \), such that \( p_i > 0 \) on a positive subtournament (which must have an odd number of nodes).
Theorem (3) - Idea of the Proof

The tournament game on an $n$ node tournament $T$ has a unique optimal strategy, $p$, such that $p_i > 0$ on a positive subtournament (which must have an odd number of nodes).

Proof: Idea

Use of lemma (1) and theorem (2) combined with the definitions of subtournaments and definition (2) of optimal strategies.
Theorem (3)

The tournament game on an n node tournament $T$ has a unique optimal strategy, $p$, such that $p_i > 0$ on a positive subtournament (which must have an odd number of nodes).

Proof: Idea

Use of lemma (1) and theorem (2) combined with the definitions of subtournaments and definition (2) of optimal strategies.

Main conclusion

There is a positive subtournament which allows us to dominate all the other nodes.
## Constraints

\[
K(T)p \geq 0 \\
p \geq 0 \\
1^T p = 1
\]
Two Different Strategies

Strategy 1

- Compute the unique optimal strategy
- Use strategy p (choosing a random node while taking the probabilities into account)
Two Different Strategies

**Strategy 1**
- Compute the unique optimal strategy $p$
- Use strategy $p$ (choosing a random node while taking the probabilities into account)

**Strategy 2**
- Given the opponents calculate $q$
- Compute $q^T K(T)$ then $p$ (each round)
- $p$ changes over time
Example of practical usage: Financial Investments

Model
- Each investment as a single node
- Set of quality criteria
- Edge as the result of criteria comparison
- \( K(T) \) can be computed

Usage
- Compute optimal strategy \( p \)
- Spend money according to \( p \)
Thank you for your Attention!
References


