Measuring Diversity of Preferences & Eliciting a Suitable Voting Rule

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1. Introduction
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3. Experimental analysis
Providing the scientific background of both articles

**Artificial Intelligence**
- development of computers able to engage in human-like thought-processes
- extension of human intelligence through the use of computers

**Social Choice Theory**
- studies mechanisms for collective decision making
- preference aggregation, voting, fair division

**Voting Theory**
- analysis of rules for elections

“How should we aggregate the preferences of the members of a group to obtain a “social preference”?" (Endriss, 2014)
Overview of basic notations & terminologies

<table>
<thead>
<tr>
<th>Variables in Article</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>is the finite set of $</td>
</tr>
<tr>
<td>N</td>
<td>finite set of $</td>
</tr>
<tr>
<td>$\mathcal{L}(A)$</td>
<td>set of all (strict) linear orders = preferences over the set of alternatives</td>
</tr>
<tr>
<td>$\mathcal{L}(A)^n$</td>
<td>set of all possible profiles</td>
</tr>
<tr>
<td>$[1, m]^N$</td>
<td>set of all possible rank-vectors</td>
</tr>
<tr>
<td>$X$</td>
<td>voting instance: set of all available rank-vectors in a rank-profile $x_R$</td>
</tr>
<tr>
<td>$\mathcal{X}$</td>
<td>set of all admissible voting instances</td>
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Part 1

Measuring Diversity of Preferences in a Group by Vahid Hashemi and Ulle Endriss
Measuring Diversity of Preferences in a Group

• Preferences are of particular interest in multiagent systems
• possible preferences empirically measured do not play any role in the real-world, preference profiles exhibit a certain amount of structure in practice
• domain restrictions are too narrow for an accurate description
• Basic Intuition: The less diverse the preferences in a profile are, the easier it should be to come to a mutually acceptable decision

Marie Jean Antoine Nicolas de Caritat (1743–1794), better known as the Marquis de Condorcet: Highly influential Mathematician, Philosopher, Political Scientist, Political Activist. Observed that the majority rule may produce inconsistent outcomes ("Condorcet Paradox").

Endriss (2010)
Proposal of model for diversity measurement and the preference diversity index at its centre

A preference diversity index (PDI) is a function \( \Delta \) mapping profiles to nonnegative numbers

**support-based PDI** \( \Delta_{\text{supp}}^{l=k} \)

measures diversity as a range of distinct views hold
\[
\Delta_{\text{supp}}^{l=k}(R) = |\{T \in \mathcal{L}_k(A) | T \subseteq R_i \text{ for some } i \in N\}| - \binom{m}{k}; k \leq m
\]

**distance-based PDI** \( \Delta_{\text{dist}}^{\Sigma,K} \)

measures diversity as an aggregate distance between individual views
\[
\Delta_{\text{dist}}^{\Phi,\delta}(R) = \Phi(\delta(R_i, R_{i'})) \quad i, i' \in N \text{ with } i < i'
\]
given distance \( \delta : \mathcal{L}(A) \times \mathcal{L}(A) \rightarrow \mathbb{R} \) and aggregation operator \( \Phi : \mathbb{R}^{n \times n} \rightarrow \mathbb{R} \)
Proposal of model for diversity measurement and the preference diversity index at its centre

**compromise-based PDI** $\Delta_{\text{com}}^{\Phi,F}(R)$

measures diversity as a distance of the group’s views to a single compromise view

$$\Delta_{\text{com}}^{\Phi,F}(R) = \Phi\left(K(R_i, F(R))\middle| i \in N\right)$$

given $SWF\ F : \mathcal{L}(A)^n \to 2^{A \times A}$ and aggregation operator $\Phi : \mathbb{R}^n \times n \to \mathbb{R}$

majority graph includes all profiles with references of voters who prefer $x$ over $y$ and who when counted are more than half of all voters

social welfare function (SWF) maps profiles to binary orders
The needed Axioms for the following propositions

**Anonymity**
basic symmetry requirement w.r.t voters
A PDO $\succeq$ is anonymous if, for every permutation $\sigma : N \to N$, we have $(R_1, ..., R_n) \sim (R_{\sigma(1)}, ..., R_{\sigma(n)})$ applicable for all profiles $(R_1, ..., R_n)$

**Neutrality**
basic symmetry requirement w.r.t to alternatives
A PDO $\succeq$ is neutral if, for every permutation $\tau : \chi \to \chi$, we have $(R_1, ..., R_n) \sim (\tau(R_1), ..., \tau(R_n))$

For any permutation $\tau : \chi \to \chi$ on alternatives and any preference order $R \in \mathcal{L}(\chi)$, define $\tau(R) = \{(x, y) | \tau(x)R\tau(y)\}$
The needed Axioms for the following propositions

**Strong Discernibility**
two profiles cannot be equal unless anonymity and neutrality imply so
A PDO $\succsim$ is anonymous if, for every permutation
$\sigma : N \rightarrow N$, we have $(R_1, ..., R_n)\sim(R_{\sigma(1)}, ..., R_{\sigma(n)})$ applicable for all profiles $(R_1, ..., R_n)$

**Weak Discernibility**
A PDO $\succsim$ is weakly discernible if $R$ being unanimous and $R'$ not being unanimous together imply $R' \succ R$

**Support Invariance**
level of diversity of a profile should only depend on its support
A PDO $\succsim$ is support-invariant if $SUPP(R) = SUPP(R')$ implies $R \sim R'$
Main propositions of the axiomatic analysis

Fact 1
Every PDO induced by a PDI of the form \( \Delta_{\text{supp}}^{l=k}, \Delta_{\text{dist}}^{\Phi,\delta} \) or \( \Delta_{\text{com}}^{\Phi,F} \) with \( k \in \{1, ... m\}, \Phi \in \{\Sigma, \text{max}\}, \delta \in \{K, S, D\} \) and F being anonymous and neutral SWF is anonymous, neutral and weakly discernible.

Proposition 2 Impossibility result
For \( m > 2 \) and \( n > m! \), no PDO can be both support-invariant and strongly discernible.

Proposition 3 Characterisation result
A PDO is support-invariant and weakly discernible if and only if it is the simple support-based PDO.
Overview of applied voting rules in experiment

**Copeland rule**
- Each alternative gets +1 point for every won pairwise majority contest and −1 point for every lost pairwise majority contest. The alternative with the most points wins.

**Borda rule**
- Each voter gives as many points to x as there are other alternatives below x in i’s ranking. The Borda score of x is the sum of those points.

**Plurality rule**
- Each voter submits a ballot showing the name of one alternative. The alternatives receiving the most votes win.
The main results of the experimental analysis

**Figure 1.** Preference diversity (x-axis) against frequency (y-axis) in impartial cultures and amongst AGH students. \( n = 50, \ m = 5 \)

**Figure 2.** Diversity for \( \Delta_{\text{dist}}^{\Sigma,K} \) / IC data (x-axis). Condorcet winners/cycles; agreement between voting rules; voter satisfaction (y-axis). \( n = 50, \ m = 5 \)
Part 2

Eliciting a Suitable Voting Rule via Examples by Olivier Cailloux and Ulle Endriss
Eliciting a Suitable Voting Rule via Examples

1. Specification of a voting rule by a series of examples
2. Association of alternatives with vectors of ranks
3. Each example answers the question how should a rule rank 2 alternatives given the positions at which each voter ranks the 2 alternatives
4. approximate the target rule as well as possible by defining a robust voting rule

We need to specify axioms for this voting rule.....

How can we guarantee that our axioms are not mutually incompatible?

Solution

Source of picture: https://encrypted-tbn3.gstatic.com/images?q=tbn:ANd9GcTeerLQs8qP2znY3kDv mw_oCeqWYOtLFLF6tMlxw3aVFhg9z
Overview of specific terminologies for article B

**Terminologies**

<table>
<thead>
<tr>
<th>Variable</th>
<th>Definition</th>
</tr>
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<tbody>
<tr>
<td>profile $R$</td>
<td>$R : N \rightarrow \mathcal{L}(A)$</td>
</tr>
<tr>
<td>rank-vector $x$</td>
<td>$x : N \rightarrow [1, m]$</td>
</tr>
<tr>
<td>rank-profile $x_R$</td>
<td>$x_R : A \rightarrow [1, m]^N$</td>
</tr>
<tr>
<td>voting rule $F$</td>
<td>$F : \mathcal{L}(A)^n \rightarrow \mathcal{P}(A) \setminus {\emptyset}$</td>
</tr>
<tr>
<td>rank-voting rule $F'$</td>
<td>$F'(x_R) = F(R)$</td>
</tr>
</tbody>
</table>

**Correspondence of profile and rank-profile**

*Figure 1.* A Profile and the corresponding Rank-Profile

Endriss and Cailloux (2014)
Voting rules as operators for selecting subsets from a given set of rank-vectors — The Condorcet rule

- **Condorcet winner** is a rank-vector that would beat every other rank-vector in a given set of rank-vectors in a pairwise majority contest
- A rank-vector \( x \in X \) is a Condorcet winner if \( \left| \{ i \mid x_i < y_i \} \right| > \frac{n}{2} \) for all \( y \in X \setminus \{x\} \)

- **Condorcet consistency** means the winner of all pairwise majority contests is elected whenever there is one
- A voting rule \( F \) is **Condorcet-consistent** if \( x \) being a **Condorcet Winner** for \( X \) implies \( F(x) = \{x\} \)
Voting rules as operators for selecting subsets from a given set of rank-vectors – The PSR

A voting rule $F$ is a *positional scoring rule (PSR)* with

- function $s : [1, m] \rightarrow \mathbb{R}$ mapping ranks to scores
- For every voting instance $X \in \mathcal{X}$ we get:
  $$F(x) = \arg \max_{x \in X} \left( \sum_{i \in N} s(x_i) \right)$$
- Scoring vector $s = s_1, s_2, \ldots, s_m$ with $s_1 \geq s_2 \geq \cdots \geq s_m$
- Each alternative receives $s_i$ points for every voter putting it at the $i^{th}$ position
Voting rules as operators for selecting subsets from a given set of rank-vectors – The Bucklin rule

The **Bucklin rule** is the voting rule $F$ and considering $t$ the Bucklin threshold that selects as winners the alternatives that attain the maximum score as evaluated by $r$

- $r_{\leq k}(x) = \left| \{i \in N | x_i \leq k \} \right|$
- $t = \min \left\{ k \in \mathbb{N} | \exists x \in X : r_{\leq k}(x) > \frac{n}{2} \right\}$
- $r_{\leq t} : F(x) = \arg \max_{x \in X} (r_{\leq t}(x))$
Prove that the Bucklin rule is not a PSR

1. We define a rank k (normally Bucklin rule starts with k=1 and is then increases)

2. We select $r_{\leq k}(x)$ as the number of ranks in x that are better (lower than k) remember: $r_{\leq k}(x) = |\{i \in N | x_i \leq k\}|$

3. Threshold t: $t = \min \left\{ k \in \mathbb{N} | \exists x \in X : r_{\leq k}(x) > \frac{n}{2} \right\}$
Definition and features of a preorder-based and a weak order-based voting rule

1. A preorder $\succeq$, asymmetric part is $\succsim$ symmetric part is $\sim$
2. $\mathcal{Z}$ = set of all preorders over $[1, m]^N$
3. A weak-order $\geq$ over the set of rank-vectors $[1, m]^N$, asymmetric part $\nleq$
4. $\mathcal{W}$ is the set of weak orders defined over $[1, m]^N$

Preorder-based voting rule

Preorder $\succeq$ on $[1, m]^N$. Given $X \in \mathcal{X}$, the voting rule $F_{\succeq}$ returns as winners those rank-vectors which are maximal under $\succeq$ in $X$: $F_{\succeq}(X) = \{ x \in X | \not\exists y \in X : y \succeq x \}$
Example of a preorder and weak-order based voting rule

- Voting Rule F is preorder-based if there exists a preorder $\succeq$ in $\mathcal{Z}$: $F = F_\succeq$
- Voting Rule F is weak order-based if there exists a weak order $\geq$ in $\mathcal{W}$: $F = F_{\geq}$
- Any voting rule that is weak-order based is also preorder based (the converse is not true)

**Example 1.** Consider the voting instances $X_1$ and $X_2$ as well as the preorder $\succsim$ shown below, with $n = 2$, $m = 4$. A down-arrow represents $\sim$, the transitive closure is left implicit, arrows implied by reflexivity are omitted and isolated rank-vectors are not shown.

Endriss and Cailloux (2014)
Propositions made by voting from preorder-based rules

**Proposition 2:** Every PSR is weak order-based
- Proof: $x \geq y \iff \sum_i s(x_i) \geq \sum_i s(y_i)$, then $F = F_\geq$

**Proposition 3:** The Bucklin rule is weak order-based
- Given rank $k \in [1, m]^N$, number of voters $\alpha$ define $X_{k=\alpha} \subseteq [1, m]^N: \{x \in [1, m]^N | r_{\leq k-1}(x) \leq \frac{n}{2} \text{ and } r_{\leq k}(x) = \alpha\}$
- $X_{k=\alpha}$ defines equivalence of classes of $\geq$
- $X_{k=\alpha} \not\supseteq X_{l=\beta}$, only if $k < l$ or both $k = l$ and $\alpha > \beta$
- Bucklin threshold t: $u = \max_x x \in X r_{\leq t}(x)$
- x is Bucklin winner if $x \in X_{t=u}$, which is the case iff. $x \in F_\geq(X)$
Propositions made by voting from preorder-based rules

**Proposition 4:** For $n = 3$ and $m = 4$, no Condorcet-consistent voting is preorder-based

- Voting rule $F$ is Condorcet-consistent, assumption that there exists a preorder $\succeq$ in $\mathcal{X}$: $F = F_{\succeq}$
- To have $F_{\succeq}(X_1) = \{123\}$ we must have $123 \succeq 231$ considering all voting instances we obtain $231 \succeq 321$ and $321 \succeq 123$
- Result: a cycle and no preorder

Endriss and Cailloux (2014)
Examples seen as constraints to find a voting rule

- Each example imposes a constraint on the voting rule, by fixing the relative ordering of 2 rank-vectors
- 2 binary relations: $>^C$ and $\sim^C$ on the set $[1, m]^N$ of rank-vectors
- Given constraints $C = (>^C, \sim^C)$, a preorder $\succeq \in \mathcal{Z}$ satisfies $C$ if $>^C \subseteq \succeq$ and $\sim^C \subseteq \succeq$
- $\mathcal{Z}_C$ is the set of preorders satisfying $C$, $C$ is consistent if $\mathcal{Z}_C \neq \emptyset$, similarly $\mathcal{W}_C$ satisfying $C$
Definition of robust voting rule as approximation of voting rule the committee searches for

- For any nonempty set of preorders $S \subseteq \mathcal{Z}$, the robust voting rule $F_S$ returns as winners all those rank-vectors that win under some rules associated with a preorder in $S$:

  $$S: F_S(X) = \bigcup_{\succcurlyeq \in S} F_{\succcurlyeq}(X)$$

2 ways of defining a robust voting rule, given constraints $C$

- Voting rule $F_{\mathcal{Z}_c}$: considering all preorders satisfying $C$
- Voting rule $F_{\mathcal{W}_c}$: considering only the compatible weak orders
Implementation of a voting rule for the committee

1. Committee has a weak order $\geq$ over set of rank-vectors $[1, m]^N$ and preferred voting rule $F_{\geq}$
   Definition of $F$ as resolute as possible, such that $F_{\geq} \subseteq F$

2. Committee respects Pareto-principle
   Pareto dominance over rank-vectors
   $x \succ y$ iff. $[\forall i \in N : x_i \leq y_i] \land [x \neq y] (\succ \in \geq)$

3. A weak order $\geq$ is indifferent to a permutation of the ranks in a rank-vector:
   $\forall x, y \in [1, m]^N, \forall \text{permutations } p, q :$
   $x \geq y \rightarrow p(x) \geq p(y)$

We conducted all answers of questions as constraints & formulated a robust voting rule – How can we now implement a suitable voting rule?
Implementation of a voting rule for the committee

- A question is an unordered pair of rank vectors \((x, y)\).
- Questions are answered according to their weak order: \(x \not\geq y, y \not\geq x\) or \((x \geq y) \land (y \geq x)\).
- Starting from constraints \(C_k = (\succ C_k, \sim C_k)\), after \(k\) answers
  - We build \(C_{k+1}\):
    \[ x \not\geq y: C_{k+1} = (\succ C_k \cup \{(x, y)\}, \sim C_k) \]
    \[ x \sim y: C_{k+1} = (\succ C_k, \sim C_k \cup \{(x, y)\}) \]
- To finally come to an approximated target rule
  - \(F \geq \subseteq F_{\mathcal{W}} C_{k+1} \subseteq F_{\mathcal{W}} C_k \subseteq \cdots \subseteq F_{\mathcal{W}} C_0\)

Wait! Do we actually now how to raise the right questions?

Seems like we found an excellent model on how to find a suitable voting rule …

Good point.. Let's start the final round!
Asking the right questions to elicit a voting rule

- Given set of constraints $c$ and preorder $\succeq^C$ induced by $C$: Fitness measure $fit(x, y, C) \in \mathbb{R}^+$ is defined for all pairs of rank vectors $x, y$ that are incomparable in $\succeq^C$, already known pairs are assigned to zero.
- Fitness measure is a heuristic that indicates how good we expect a question to $(x, y)$ to be.
- Elicitation strategy picks one of the maximally fit pairs.
- Optimistic: $fit_o(x, y, C) = |\succeq^C (x) \setminus \succeq^C (y)| + |\succeq^C (y) \setminus \succeq^C (x)|$
- Pessimistic: $fit_p(x, y, C) = \min\{|\succeq^C (x) \setminus \succeq^C (y)| + |\succeq^C (y) \setminus \succeq^C (x)|\}$
- Likelihood: $p$ is a probability distribution over $\mathcal{X}$: $fit_l(x, y, C) = \sum\{X \in \mathcal{X} \mid x, y \in \mathcal{W} \setminus c(x)\} p(X)$
- Random: selects randomly a pair $(x, y)$ of incomparable pairs in $\succeq^C$. 
What are the experimental results?

- $F_\geq \subseteq F_{\mathcal{W}C}$: voting rule contains target and supplementary winners

- Badness ration of number of winners: $\frac{1}{|X|} \sum_{X \in \mathcal{X}} \left| F_{\mathcal{W}C}(X) \right| / \left| F_\geq(X) \right|$

- Badness of average weak order error on a supplementary winner

Table 1. Results of the experiment

<table>
<thead>
<tr>
<th></th>
<th>n</th>
<th>m</th>
<th>q</th>
<th>fit</th>
<th>Borda nb w.</th>
<th>Borda wo su.</th>
<th>Random nb w.</th>
<th>Random wo su.</th>
<th>Ratio of number of winners &amp; average weak order error on supplementary winner</th>
</tr>
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<tbody>
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2 first columns = problem size

Fitness strategies

Number of questions

Likelihood with a sample size of 10000

Results are averaged over 10 runs
References