Solutions to Weighted Tournaments

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Recommendation system

- $n$ data points in dimension $m$ (ex. gr. films)
- Another data point $x$
- **Goal:** Find points that are similar to $x$
- Each dimension is a voter and ranks data in increasing distance from $x$.
- This problem is close to a voting profile.
## Recommendation systems

<table>
<thead>
<tr>
<th>Film</th>
<th>Date</th>
<th>Action</th>
<th>Suspense</th>
<th>Humour</th>
<th>Social</th>
</tr>
</thead>
<tbody>
<tr>
<td>The Matrix</td>
<td>1999</td>
<td>8</td>
<td>5</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>A Clockwork Orange</td>
<td>1971</td>
<td>4</td>
<td>6</td>
<td>3</td>
<td>9</td>
</tr>
<tr>
<td>The Godfather</td>
<td>1972</td>
<td>6</td>
<td>7</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>Vertigo</td>
<td>1958</td>
<td>5</td>
<td>9</td>
<td>3.5</td>
<td>2</td>
</tr>
<tr>
<td>Star Wars</td>
<td>1977</td>
<td>7</td>
<td>3</td>
<td>6</td>
<td>0</td>
</tr>
<tr>
<td>Dr. Strangelove</td>
<td>1964</td>
<td>3</td>
<td>8</td>
<td>9</td>
<td>8</td>
</tr>
</tbody>
</table>

- **What films are closest to Dr No?** (Date: 1962, Action: 8, Suspense: 7, Humour: 7, Social: 0)
  - Date: Dr. Strangelove $\succ_{Date}$ Vertigo $\succ_{Date}$ A Clockwork Orange $\succ_{Date}$ The Godfather $\succ_{Date}$ Star Wars $\succ_{Date}$ The Matrix
  - Suspense: The Godfather $\succ_{Suspense}$ Dr. Strangelove $\succ_{Suspense}$ A Clockwork Orange $\succ_{Suspense}$ Vertigo $\succ_{Suspense}$ The Matrix $\succ_{Suspense}$ Star Wars
  - ...

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**Solutions to Weighted Tournaments**

Quentin Bammey
Recommendation systems

We should represent this as a tournament.

However...
Classical tournaments cannot efficiently represent this. Why?

We would be blind to the strength of arrows, which is important here.

Fortunately...
We can use weighted tournaments.
Weighted Tournament Solutions

Parameterized algorithms
Approximations
NP-Completeness
Kemeny’s rule
Maximin rule

Generalization of $C_1$ functions
(Other $C_2$ functions)
Extended Condorcet principle

If for every alternative \( a \) in a subset \( S \), for every alternative \( b \) in a subset \( T \), \( m_R(a, b) > 0 \),

\( S \) should be preferred to \( T \)
Kemeny’s rule

- Attempts to minimize disagreement between ranks
- Kemeny score of a linear order: number of disagreements between the order and profile preferences
- **Kemeny Rank Aggregation**: Given a weighted tournament $G$, find a linear order with minimal score.

Kemeny optimal aggregation is the only neutral, consistent function satisfying the extended Condorcet principle.

- **Kemeny Score**: Given a non-negative integer $k$ and a weighted tournament $G$, is there a linear order with score at most $k$?

**Kemeny Score** is NP-complete when the number of voters is even and superior to 4 or unbounded.
Approximations

- Give satisfactory results in polynomial time.
- Borda-count: 5-approximation for Kemeny Rank Aggregation
- Efficient $\frac{4}{3}$-approximation known
- EPTAS but no FPTAS:
- Hope for better efficient approximations.
Parameterized Algorithms

• Keep the algorithm polynomial in some parameters.
• Useful if the other parameters can be kept at low values.
• **Kemeny Rank Aggregation:** Limit number of alternatives?
• Naive algorithm runs in
  
  \[ O\left(\frac{m!}{nm \log m}\right) \]

  ○ Number of orders to consider
  ○ Time to compute the Kemeny Score of an order
• Can be reduced to \( O(2^m m^2 n) \).
Generalizing $C^1$ functions

- $C^1$ functions are obviously $C^2$ functions
- Many of them can be generalized so that they take majority margin into account.
Bipartisan set - Classical tournaments

• Payoff function:

\[ p : A \times A \mapsto \{-1, 0, 1\}, (x, y) \mapsto \begin{cases} 
1 & \text{if } m(x, y) > 0 \\
-1 & \text{if } m(x, y) < 0 \\
0 & \text{otherwise}
\end{cases} \]

• Unique equilibrium of the game
• Bipartisan set is the support of this equilibrium.
Bipartisan set - Weighted tournaments

- Payoff function:

\[ p : A \times A \mapsto \{-1, 0, 1\}, (x, y) \mapsto m(x, y) \]

However...

Equilibria need no longer be unique!

- Essential set: union of the equilibrium supports
Maximin Rule

- Selects alternatives \( x \) verifying

\[
\min_{z \in A} m(x, z) = \max_{y \in A} \min_{z \in A} m(y, z)
\]

\[
\begin{array}{ccccccc}
1 & 1 & 3 & 1 & 1 & 2 \\
A & A & B & C & D & D \\
B & B & C & D & A & C \\
C & D & A & A & B & A \\
D & C & D & B & C & B \\
\end{array}
\]

Does not verify the Condorcet loser criterion!
Schulze’s method

- Majority margin of a path \((z_1, \ldots, z_k)\): minimum majority margin of any of its arc \(\min_{1 \leq j < k} m(z_j, z_{j+1})\)
- Schulze function \(S(x, y)\): the maximum majority margin of any path from \(x\) to \(y\)
- Schulze’s method: selects \(\{x \in A \mid \forall y \in A \setminus \{x\}, S(x, y) \geq S(y, x)\}\)
- Can be computed in \(O(m^3)\)
- Used for internal elections by the Wikimedia Foundation, as well as several Pirate Parties
Borda’s Rule and variations

• Borda score of \( x \):

\[
\sum_{i \in N} | \{ y \in A : x \succ_i y \} |
\]

Borda’s rule is a \( C\!2 \) function. Why?

Borda’s rule fails the Condorcet criterion. Can it be fixed?

• Black: Select Condorcet winner if it exists, otherwise use Borda’s rule.

• Nanson: Successively excludes alternatives whose score is below average.
## Conclusion

<table>
<thead>
<tr>
<th>Rule</th>
<th>Complexity class</th>
<th>Verified Criteria</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kemeny’s rule</td>
<td>NP-complete</td>
<td>extended Condorcet criterion</td>
</tr>
<tr>
<td>Bipartisan set</td>
<td>P</td>
<td>monotonic, stable, consistent</td>
</tr>
<tr>
<td>Maximin rule</td>
<td>P</td>
<td>Condorcet and Majority, but not Condorcet loser</td>
</tr>
<tr>
<td>Schulze’s method</td>
<td>P</td>
<td>Same as Maximin, plus Condorcet loser</td>
</tr>
<tr>
<td>Borda’s rule</td>
<td>P</td>
<td>monotonic, consistent, Condorcet loser, (Condorcet)</td>
</tr>
</tbody>
</table>