Weighted Voting Games

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Computational Social Choice

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Agenda

Motivation

European Economic Community (1957)

- The "Treaty of Rome" was signed in 1957, establishing the Council of the European Economic Community
- Voting weights of the countries:

Belgium	Germany	France	Italy	Luxembourg	Netherlands
2	4	4	4	1	2

- Quota: 12 votes

Question: If you were the representative of Luxembourg, would you accept or reject this voting system?

Answer: Reject! Luxembourg does not have any influence since $10 + 1 = 11 < 12 = \mathit{quota}$.

Basic Definitions

Cooperative Games

Definition (Cooperative game)

- Cooperative game $G = (N, v)$
  - Set of players $N = \{1, \ldots, n\}$
  - Characteristic function $v : 2^N \to \mathbb{R}$
- Coalition $C \subseteq N$ is a subset of players
- Grand coalition $C = N$
- Cooperative game is **simple** if $v(C) \in \{0, 1\}$
  - C is **winning** if $v(C) = 1$, otherwise **losing**

Weighted Voting Game

Definition (Weighted voting game)

- Weighted voting game $G = [N; w; q]$
  - Set of players $N = \{1, \dots, n\}$
  - Player weights $w = (w_1, \dots, w_n) \in \mathbb{R}^n$
  - Quota q
  - Characteristic function $v : 2^N \to \{0, 1\}$:

$$v(C) = \begin{cases} 1 & \text{if } \sum_{i \in C} w_i \geq q \\ 0 & \text{otherwise.} \end{cases}$$

- Weighted voting games are *simple* cooperative games.

Solution Concepts

Solution Concepts

Players work together in a coalition C and obtain payoff p.

Question: What is a **fair** distribution of the payoff among players?

Solution Concepts:

- Define a set $\mathcal{F}$ of *allocation vectors*.
- An allocation is *fair* if it is in $\mathcal{F}$.

Examples:

- Core - based on the stability of a coalition
- Shapley value - based on the necessity of a player in a coalition

Imputations

Definition (Imputation)

- Given cooperative game $G = (N, v)$
- Imputation is a tuple $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ satisfying
  - Efficiency: $\sum_{i=1}^{n} x_i = v(N)$
  - Individual rationality: $x_i \geq v(\{i\})$ for all $i \in N$

Example

- $N = \{1, 2, 3\}$
- $v(\{1\}) = 1, \;\; v(\{2\}) = 2, \;\; v(\{3\}) = 3, \;\; v\{N\} = 10$

Are the following tuples imputations?

- $z_1 = (3, 3, 3)$ ✗
- $z_2 = (5, 3, 2)$ ✗
- $z_3 = (2, 3, 5)$ ✓

Core

Definition (Core)

The core $\mathcal{C}(G)$ is the set of imputations

$$\mathcal{C}(G) = \{\mathbf{x} \mid \forall S \subseteq N : x(S) \geq v(S)\},$$

where $x(S) = \sum_{i \in S} x_i$.

Note:

- Core contains imputations that all coalitions would agree to.
- There might be multiple imputations in the core.
- Core might be empty.

Core - Example

Recall (Core)

$$\mathcal{C}(G) = \{\mathbf{x} \mid \forall S \subseteq N : x(S) \geq v(S)\}$$

Example

Given a cooperative game $G = (N, v)$ with $N = \{1, 2, 3\}$ and v given as:

Coalition S	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	{1,2,3}
v(S)	1	2	3	4	5	6	10

Are the following imputations in the core of G?

- $z_1 = (1, 2, 7)$ ✗
- $z_2 = (2, 3, 5)$ ✓

Core - Unreasonable Imputation

Example

- Given a cooperative game $G = (N, v)$ with
  - $N = \{1, 2\}$
  - $v(\{1\}) = v(\{2\}) = 2$ and $v(N) = v(\{1, 2\}) = 10$
- Consider the imputation $\mathbf{x} = (2, 8)$.
- $\mathbf{x} = (2, 8)$ is in the core.
- If you were player 1, would you accept this imputation?

Problem: The Core might contain unreasonable imputations.

Shapley Value - Marginal Contribution

Definition (Marginal Contribution)

The **marginal contribution** $\delta_i(C)$ is the value that player i would add to the coalition $C \subseteq N \setminus \{i\}$ by joining it, i.e.

$$\delta_i(C) = v\big(C \cup \{i\}\big) - v(C).$$

Dummy player: Never contributes anything to any coalition.

Example

Luxembourg was a dummy player in our introduction example.

Shapley Value

Definition (Shapley value)

- Given a cooperative game $G = (N, v)$
- Π is the set of all permutations of players in N
- π_i is the coalition of players preceding player i in the permutation $\pi \in \Pi$
- **Shapley value** of game G is the imputation $\varphi(G) = (\varphi_1(G), \ldots, \varphi_n(G))$ with

$$\varphi_i(G) = \frac{1}{n!} \sum_{\pi \in \Pi} \delta_i(\pi_i).$$

Shapley Vlaue - Example

Recall (Shapley value)

$$\varphi_i(G) = \frac{1}{n!} \sum_{\pi \in \Pi} \delta_i(\pi_i)$$

Example

- Weighted voting game: $N = \{1, 2, 3\}$, $w = (4, 2, 3)$, $q = 4$
- Compute $\varphi_1(G)$

Π	$(1,2,3)$	$(1,3,2)$	$(2,1,3)$	$(2,3,1)$	$(3,1,2)$	$(3,2,1)$
π_1	$\emptyset$	$\emptyset$	$\{2\}$	$\{2,3\}$	$\{3\}$	$\{3,2\}$
$\delta_1(\pi_1)$	1	1	1	0	1	0

- $\varphi_1(G) = \frac{1}{3!} \cdot (1 + 1 + 1 + 0 + 1 + 0) = \frac{4}{6}$

Shapley Value

Recall (Shapley value)

$$\varphi_i(G) = \frac{1}{n!} \sum_{\pi \in \Pi} \delta_i(\pi_i)$$

- The higher the Shapley value, the higher the *power* of a player.
- Dummy players have a Shapley value of 0.

Alternative:

Definition (Shapley value)

$$\varphi_i(G) = \frac{1}{n!} \sum_{S \subsetneq N} |S|! \cdot (|N| - |S| - 1)! \cdot \delta_i(S)$$

Shapley Value - Alternative Expression

Recall (Shapley value)

$$\varphi_i(G) = \frac{1}{n!} \sum_{\pi \in \Pi} \delta_i(\pi_i)$$

- Let $N = \{1, 2, \ldots, 6\}$, compute $\varphi_1(G)$, consider the following permutations:

$$\pi^1(5, 3, 1, 6, 4, 2)$$
$$\pi^2(5, 3, 1, 6, 2, 4)$$
$$\pi^3(5, 3, 1, 4, 6, 2)$$
$$\ldots$$
$$\pi^k(3, 5, 1, 2, 4, 6)$$

Shapley Value - Alternative Expression

- In general: consider $\varphi_i(G)$ and let $\pi^* \in \Pi$ with

$$\pi^* = (\underbrace{\bullet, \bullet, \bullet,}_{S} \underset{\uparrow}{i}, \underbrace{\bullet, \bullet, \bullet}_{N \setminus \{S \cup i\}})$$

- For each coalition S, we have to multiply $\delta_i(S)$ by all possible permutations of S and $N \setminus \{S \cup i\}$.
- This leads to the alternative formulation of the Shapley value:

Definition (Shapley value)

$$\varphi_i(G) = \frac{1}{n!} \sum_{S \subsetneq N} |S|! \cdot (|N| - |S| - 1)! \cdot \delta_i(S)$$

Shapley Value - Properties

Theorem (Shapley, 1953)

Shapley values are the *only* payoff division scheme simultaneously satisfying the following properties:

- **Efficiency:** $\sum_{i=1}^{n} \varphi_i(G) = v(N)$.
- **Dummy Player:** $\varphi_i(G) = 0$ if i is a dummy player.
- **Symmetry:** $\varphi_i(G) = \varphi_j(G)$ if $\delta_i(C) = \delta_j(C)$ for all $C \subseteq N \setminus \{i, j\}$.
- **Additivity:** For $G = (N, v)$ and $G' = (N, v')$, one defines $G + G' = (N, v + v')$, where $(v + v')(S) = v(S) + v'(S)$. Then, $\varphi_i(G + G') = \varphi_i(G) + \varphi_i(G')$.

Computational Complexity

Dummy Player

Given a weighted voting game G. Deciding whether player i is a dummy player in G is particularly hard.

Theorem

DUMMY PLAYER is coNP-complete.

Note:

- $\varphi_i(G) = 0 \iff$ Player i is dummy in G
- $\Rightarrow$ Computing Shapley value must also be hard.

Shapley Value

Definition

A problem is in #P if

- $\exists$ non-deterministic polynomial time Turing machine T
- Number of accepting computations of T on a given input x gives the answer to the problem.

Theorem (Deng and Papadimitriou, 1994)

The problem of computing the Shapley value of a player i in a given weighted voting game G is $\#P$-complete.

Shapley Value - Proof

Theorem (Deng and Papadimitriou, 1994)

The problem of computing the Shapley value of a player i in a given weighted voting game G is $\#P$-complete.

Proof.

Membership:

- Consider a non-deterministic Turing, that
  - has one computation path for each permutation π
  - accepts if $\delta_i(\pi_i) = 1$
- There are $\sum_{\pi \in \Pi} \delta_i(\pi_i) = n! \, \varphi_i(G)$ accepting computation paths.

Shapley Value - Proof

Hardness:

- Reduce an instance of a (version) of the KNAPSACK problem:
    - Given positive integers $a_1, \ldots, a_m$ and a $K \in \mathbb{N}$
    - Find the number of subsets $S \subseteq \{1, \ldots, m\}$ such that $\sum_{i \in S} a_i = K$.
    - Assume $K = A/2$ where $A = \sum_{i=1}^{m} a_i$.
    - Assume that all solution subsets S have equal cardinality.
    - This version of KNAPSACK is known to be in #P.

Shapley Value - Proof

Transformation to weighted voting game $G = [N; w; q]$ with:

- $N = \{1, \ldots, m, m+1\}$
- $w = \{a_1, \ldots, a_m, 1\}$
- $q = \frac{1}{2} \sum_{i \in N} w_i = \frac{A+1}{2}$

Consider the computation of φ_{m+1}, note:

- For all $S \subseteq N$, $\delta_{m+1}(S) = 1$ iff
  - $m + 1 \notin S$
  - S is loosing
  - $S \cup \{m+1\}$ is winning
- $w_n = 1$, thus $\sum_{i \in S} w_i = A/2 = K$ and S is solution to the KNAPSACK instance!

Shapley Value - Proof

- If we were able to compute $\varphi_i(G)$ easily, then we could also solve the KNAPSACK problem.
- Remember:

$$\varphi_{m+1}(G) = \frac{1}{(m+1)!} \sum_{S \subsetneq N} |S|! \cdot (|N| - |S| - 1)! \cdot \delta_{m+1}(S)$$

- Let k denote the cardinality of each solution subset S.
- Let z denote the number of solution subsets S for the KNAPSACK problem.
- Then, $z = \varphi_{m+1}(G) \cdot (m+1)! \cdot \frac{1}{k! \cdot (m-k)!}$.

□

Paradoxes of Power

Paradoxes of Power

Voting procedures in political institutions may change.

Weight distribution Council of the European Union since 01.07.2013:

Country	Weight
Germany, France, United Kingdom, Italy	29
Poland, Spain	27
Romania	14
Netherlands	13
Belgium, Greece, Portugal, Czech Republic, Hungary	12
Bulgaria, Austria, Sweden	10
Denmark, Finland, Croatia, Ireland, Lithuania, Slovakia	7
Estonia, Latvia, Luxembourg, Slovenia, Cyprus	4
Malta	3

Question: How does the Shapley value change under game modifications?

Paradoxes of Power

How does the Shapley value of a player i change if

- quota is lifted?
- a new player is added to the game?
- player i is split into two identities and the sum of the new identities' Shapley value is considered?

Paradoxes of Power - Quota Modification

Question: Will lifting the quota increase/decrease/not change the Shapley value of the players?

Example

Consider a weighted voting game $G = [N, w, q]$ with

- $N = \{1, 2, 3, 4\}$
- $w = (3, 3, 1, 1)$
- $q = 3$

Answer: Shapley value may increase or decrease or stay the same depending on the value of the quota.

Conclusion

Conclusion

What we learned today:

- Definition of a cooperative game and a weighted voting game
- Solution concepts:
  - Core
  - Shapley value
- Computational complexity of computing Shapley values
- to be careful when changing parameters in weighted voting games

... and last but not least:

- to be smarter than Luxembourg's politicians!

Thank you for your attention!