Introduction to the Theory of Voting

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   Motivation

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Voting

Definition

Voting is the process of achieving a *collective decision* based on the *preferences* among all *alternatives* from the *voters*.
Definitions

- $N = \{1, 2, \ldots, n\}$ a finite set of voters.
- $A$ a finite set of $m$ alternatives, $m \geq 2$
- A ballot cast by voter $i \in N$, which is a linear ordering $\succeq_i$ of $A$ (transitive, complete, reflexive and antisymmetric) and induces a preference ranking
- A profile $P = (\succeq_1, \succeq_2, \ldots, \succeq_n)$, which specifies a ballot for each voter $i \in N$

<table>
<thead>
<tr>
<th>102</th>
<th>101</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>$b$</td>
<td>$c$</td>
</tr>
<tr>
<td>$b$</td>
<td>$c$</td>
<td>$b$</td>
</tr>
<tr>
<td>$c$</td>
<td>$a$</td>
<td>$a$</td>
</tr>
</tbody>
</table>
Definitions

- A **social choice function** (SCF) \( f : \mathcal{L}(A)^n \to \mathcal{C}(A) \) that returns the winning alternatives for each profile of strict preferences.

- A **social welfare function** (SWF) yields a weak ranking \( f : \mathcal{L}(A)^n \to \mathcal{R}(A) \) of the set alternatives (**pre-linear** ordering) for each profile of strict preferences.
From school: Elections

Definition

A fair election should be free, equal, secret, general, direct, public/transparent and effective.
# 2015 UK General Elections

<table>
<thead>
<tr>
<th>2015 election</th>
<th>Vote share (%)</th>
<th>Seats won</th>
<th>&quot;Danish&quot; system</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conservatives</td>
<td>36.8</td>
<td>331</td>
<td>240</td>
</tr>
<tr>
<td>Labour</td>
<td>30.4</td>
<td>232</td>
<td>198</td>
</tr>
<tr>
<td>Ukip</td>
<td>12.6</td>
<td>1</td>
<td>82</td>
</tr>
<tr>
<td>Liberal Democrats</td>
<td>7.9</td>
<td>8</td>
<td>51</td>
</tr>
<tr>
<td>SNP</td>
<td>4.7</td>
<td>56</td>
<td>32</td>
</tr>
<tr>
<td>Greens</td>
<td>3.8</td>
<td>1</td>
<td>25</td>
</tr>
<tr>
<td>Plaid Cymru</td>
<td>0.6</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>Northern Ireland parties</td>
<td>2.3</td>
<td>18</td>
<td>18</td>
</tr>
<tr>
<td>Other parties</td>
<td>0.9</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
2015 UK General Elections

First-Past-The-Post system (FPTP)

- 650 constituencies
- between 60,000 and 80,000 voters per constituency
- 1 seat per constituency for candidate with the most votes

Is used, because of uneven population distribution, otherwise major part of representatives would be from London and thus legislation biased towards it.
2015 UK General Elections

Issues:

- not representative
- favors major parties
- not every vote is equal

Possible changes:

- have larger constituencies and give multiple seats (better representation)
- do not use constituencies (favor of denser populated areas)
- develop some system which distributes seats over constituencies after the fact
New voting system (since 2011)

- two lists: candidates and parties
- 5 votes per list
- Kumulieren (accumulation) and Panaschieren (vote splitting)

Issue: Too complicated for many voters!
Introduction

Axioms I

Voting Rules I

Axioms II

Impossibilities

Anonymity, Neutrality and Pareto Property

Anonymity

**Definition**

An SCF \( f \) is **anonymous** if each pair of voters play interchangeable roles: \( f(P) = f(P^*) \) holds whenever \( P^* \) is obtained from \( P \) by swapping the ballots cast by two voters.

**Example (of nonanonymity)**

- voters with veto rights, for example the 5 permanent members of the UNO in the UN Security Council
- votes, which have to pass the two houses of parliament
- any organization where board members have more votes than regular members (or they count more).
Neutrality

Definition

An SCF $f$ is **neutral** if each pair of alternatives are interchangeable: when $P^\dagger$ is obtained from $P$ by swapping the positions of two alternatives in every ballot and $f(P^\dagger)$ is obtained from $f(P)$ via a similar swap.

Example (of nonneutrality)

- legislative voting systems, where the YES alternative needs a 2/3 majority.
Pareto Property

Definition

In $P$ alternative $x$ *Pareto dominates* alternative $y$ if every voter ranks $x$ over $y$; $y$ is *Pareto dominated* if such an $x$ exists. Then an SCF $f$ is **Pareto (optimal)** if $f(P)$ never contains a Pareto dominated alternative.
Introduction to the Theory of Voting
Thieme Taube

Issues

Theorem (Moulin, 1983)

Let $m \geq 2$ be the number of alternatives and $n$ be the number of voters. If $n$ is divisible by any integer $r$ with $1 < r \leq m$, then no neutral, anonymous and Pareto SCF is resolute (single valued).

Example

- obviously for two alternatives, where the number of voters is even, ties may exist

How do you break ties?
How do you break ties?

Example

- In 2013 the mayoral election in San Teodoro, Philippines, the tie was settled via coin toss.
- The French electoral code states, that ties are broken in favor of the older candidate (consequently in some elections parties favor older candidates).
Condorcet Extensions

**Definition**

A *Condorcet winner* for $P$ is an alternative $x$ that defeats every other alternative in the strict pairwise majority sense: $x >_P^\mu y$ for all $y \neq x$. Pairwise Majority Rule (PMR) declares the winning alternative to be the Condorcet winner and is undefined when a profile has no such winner.

An SCF is a **Condorcet extension** (or *Condorcet consistent*) if it selects the Condorcet winner alone, when it exists.

Problems are caused by majority cycles (e.g. $a >_P^\mu b$, $b >_P^\mu c$ and $c >_P^\mu a$). These are known as *Condorcet’s voting paradox* and relate to the intransitivity of $>_P^\mu$. 
Condorcet’s Voting Paradox

\[
\begin{array}{cccc}
102 & 101 & 100 & 1 \\
a & b & c & c \\
b & c & a & b \\
c & a & b & a \\
\end{array}
\]

Reminder:

\[\text{Net}_p(a > b) = | \{ j \in N | a \succ_j b \} | - | \{ j \in N | b \succ_j a \} | \]
Condorcet Extensions

**Example**

- The Copeland score of a Condorcet winner is $m - 1$ and uniquely highest, thus is a Condorcet extension.
- Borda can fail to elect an alternative, which is top-ranked by a majority of voters and is thus not a Condorcet extension.

**Reminder:**

- $Net_P(a > b) = |\{j \in N| a \succ_j b\}| - |\{j \in N| b \succ_j a\}|$
- $Copeland(x) = |\{y \in A|x \succ^\mu y\}| - |\{y \in A|y \succ^\mu x\}|$
  Where $x \succ^\mu y$ denotes that $Net_P(x > y) > 0$
- $Borda^\text{sym}_P(x) = \sum_{y \in A} Net_P(x > y)$
Scoring Rules

Definition

A score vector \( w = (w_1, w_2, \ldots, w_m) \) consists of real number scoring weights and is proper if \( w_1 \geq w_2 \geq \cdots \geq w_{m-1} \geq w_m \) and \( w_1 > w_m \). The weights are awarded as points to each alternative from top ranked to lowest. The winner is the alternative with the largest sum of awarded points.

A proper scoring rule is induced by a proper score vector.

Example

- Plurality with \( w = (1, 0, \ldots, 0) \)
- Borda count with \( w = (m - 1, m - 2, \ldots, 1, 0) \)
Instant Run-off Voting

Definition (Instant Run-off Voting)

At each stage the alternative with the lowest plurality score is dropped from all ballots and at the first stage at which and alternative \( x \) sits atop the majority of ballots, it is declared the winner.

Example (Irish presidential election, 1990)

<table>
<thead>
<tr>
<th>Candidate</th>
<th>Round 1</th>
<th>Round 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mary Robinson</td>
<td>612,265 (38.9%)</td>
<td>817,830 (51.6%)</td>
</tr>
<tr>
<td>Brian Lenihan</td>
<td>694,484 (43.8%)</td>
<td>731,273 (46.2%)</td>
</tr>
<tr>
<td>Austin Currie</td>
<td>267,902 (16.9%)</td>
<td>-</td>
</tr>
</tbody>
</table>
Reinforcement

**Definition**

**Reinforcement** means that the common winning alternatives (if they exist) of two disjoint sets of voters be exactly those chosen by the union: $f(s) \cap f(t) \neq \emptyset \Rightarrow f(s + t) = f(s) \cap f(t)$ for all voting situations $s$ and $t$.

**Example**

- Scoring rules are reinforcing, since if an alternative has the highest score for $s$ and $t$, it also has the highest for $s + t$ (the sum of the scores for $s$ and $t$).
- Condorcet extensions for three or more alternatives violate reinforcement.
Monotonicity

**Definition**

A resolute SCF (no ties) $f$ satisfies *(weak) monotonicity* if whenever $P$ is modified to $P'$ by having one voter $i$ switch $≿_i$ to $≿'_i$ by lifting the winning alternative $x = f(P)$ simply (increasing its relative preference without changing others), $f(P') = f(P)$.

**Example**

- Copeland and all proper Scoring Rules are monotonic.
Strategy-Proofness

Definition

A resolute SCF (no ties) \( f \) satisfies **strategy-proofness** if whenever a profile \( P \) is modified to \( P' \) by having one voter \( i \) switch \( \succ_i \) to \( \succ_i' \), \( f(P) \succ_i f(P') \).

Other monotonicity properties exist, such as *Maskin*, *Down*, *One-way*, *Half-way monotonicity* and *Participation*. For all of these including (weak) monotonicity it holds, that they are weak forms of strategy proofness.
Issues

Definition

A resolute SCF (no ties) \( f \) satisfies **Half-way monotonicity** if whenever \( P \) is modified to \( P' \) by having one voter \( i \) switch \( \succ_i \) to \( \succ_i^{\text{rev}} \) (\( z \succ w \iff w \succ^{\text{rev}} z \)), \( f(P) \succ_i f(P') \).

Theorem

*Let* \( f \) *be a resolute SCF for* \( m \geq 4 \) *alternatives and sufficiently large odd* \( n \). *If* \( f \) *is neutral and anonymous and has a Condorcet winner, then either* \( f \) *fails to be strategy proof or* \( f \) *violates half-way monotonicity.*
Definition

A resolute SCF (no ties) $f$ satisfies **Participation** (the absence of no show paradoxes) if whenever $P$ is modified to $P'$ by adding one voter $i$ with ballot $≿_i$ to the electorate, $f(P') ≿_i f(P)$.

Theorem

*Let $f$ be a resolute Condorcet extension for $m \geq 4$ alternatives. Then $f$ violates*

- participation (if $f$ is a variable-electorate SCF)
- half-way monotonicity (if $f$ is a fixed-electorate SCF for sufficiently large $n$)
Strategic manipulation

\[
\begin{array}{ccc}
2 & 3 & 2 \\
e & d & a \\
c & e & b \\
a & b & c \\
d & c & d \\
b & a & e
\end{array}
\]

Reminder:
- \(Net_P(a > b) = |\{j \in N | a \succ_j b\}| - |\{j \in N | b \succ_j a\}|\)
- \(Copeland(x) = |\{y \in A | x >^\mu y\}| - |\{y \in A | y >^\mu x\}|\)
  Where \(x >^\mu y\) denotes that \(Net_P(x > y) > 0\)
- \(Borda^\text{sym}_P(x) = \sum_{y \in A} Net_P(x > y)\)

Copeland and Borda are single voter manipulable.
Introduction to the Theory of Voting

Impossibilities

**Definition (Independence of Irrelevant Alternatives (IIA), Kenneth Arrow)**

The collective voter opinion to the relative merits of two alternatives should not be influenced by the individual opinion towards an “irrelevant” third.

**Theorem (Arrow Impossibility)**

*Every weakly-paretian SWF for \( m \geq 3 \) alternatives either violates IIA or is a dictatorship.*

*(Where weakly-paretian means, that if every voter strictly prefers \( a \) over \( b \), then the SWF ranks \( a \) strictly over \( b \).*
Impossibilities

Theorem (Gibbard-Satterhwaite (GST))

Any resolute, nonimposed (no alternative is unelectable) and strategy-proof SCF for $m \geq 3$ alternatives must be a dictatorship.
Not everything is hopeless!

In practice the voter needs

- to know the intended ballot of the other voters.
- to be sure that no other voter will similarly engage in strategic vote-switching
- the computational resources to predict whether some switch in her ballot can change the outcome into one she prefers