Two Perspectives on Manipulability

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The Olympic Badminton Scandal

In the course of the badminton tournament in the Olympics 2012, 4 badminton teams were disqualified for deliberately losing a match.
Manipulation in Sports Competitions
Strat.-Proofness and Reluctance to make Large Lies

Motivation
Tournaments
Fairness Properties
The Impossibility Result

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Tournaments

- Finite set $X$ of competitors
- $T_X \subseteq \mathcal{P}(X \times X)$ set of all total and asymmetric binary relations over $X$.
- Tournament: elements of $T_X$
Tournaments: Example

- $X = \{a, b, c\}$
- $T = \{(a, b), (a, c), (c, b)\} \in T_X$
A competition format is a function

\[ F: X^n \times T_X^m \rightarrow X \]

that takes as input
- a fixed amount of players \( x_1 \ldots x_n \)
- \( m \) tournaments \( T_1 \ldots T_m \) over \( X \), corresponding to \( m \) rounds of the competition

and returns the single winner \( x_i \), \( 1 \leq i \leq n \) of the competition.
For every $x \in X$, let $N(x) := |\{y \in X|(x, y) \in T\}|$ be the number of winning matches of $x$ in $T$.

**Definition**

$$R_1(x_1, \ldots, x_n, T) = x_i$$

iff $i$ is the smallest index satisfying $N(x) \leq N(x_i)$ for all $x \in \{x_1, \ldots, x_n\}$. 
Definition

For $n > 1$ and $x_j = R_1(x_1, \ldots, x_n, T)$, the competition format $R_2$ can be defined:

$$R_2(x_1, \ldots, x_n, T) = x_i \text{ iff } i \text{ is the smallest index satisfying }$$

$$i \neq j \text{ and } N(x_k) \leq N(x_i) \text{ for all } k \neq j.$$
Fairness Properties

- Anonymity
- Independence of Irrelevant Alternatives
- Symmetry
- Non-Imposition
- Monotonicity
Anonymity

\[ \pi: \begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix} \]
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Anonymity

\[ F(abc, T) = a \implies F(bca, T') = b \]
Anonymity

Definition

A competition format $F$ satisfies **Anonymity** iff whenever $T$ and $T'$ are isomorphic under permutation $\pi$, we have

$$\pi \left( F(x_1, \ldots, x_n, T) \right) = F(\pi(x_1), \ldots, \pi(x_n), T').$$
Independence of Irrelevant Alternatives

\[ T = \{a, b, c\} \quad T' \]
Independence of Irrelevant Alternatives

\[ F(abc, T) = F(abc, T') \]
Independence of Irrelevant Alternatives

**Definition**

A competition format $F$ satisfies IIA iff

$$\forall T, T', x_1 \ldots x_n:\ T = \{x_1, \ldots, x_n\} \implies T' \implies F(x_1 \ldots x_n, T) = F(x_1 \ldots x_n, T').$$
A competition format $F$ satisfies **Symmetry** iff

$$F(x_1, \ldots, x_k, x_{k+1}, \ldots, x_{2k}, T_1, \ldots, T_m) = F(x_{k+1}, \ldots, x_{2k}, x_1, \ldots, x_k, T_1, \ldots, T_m).$$
Non-Imposition

Definition

A competition format $F$ satisfies **Non-Imposition** iff for all $1 \leq i \leq n$ it holds that

$$\forall x_i \exists T_1, \ldots, T_m : F(x_1, \ldots, x_i, \ldots, x_n, T_1, \ldots, T_m) = x_i.$$
Monotonicity

**Definition**

$T'$ monotonically improves $T$ for $a$:

$T' \geq_a T \iff$

$(x, y) \in T \iff (x, y) \in T'$ for all $x, y \neq a$ and

$(a, x) \in T$ implies $(a, x) \in T'$.
Monotonicity

\[ T \geq_b T \]
Monotonicity

\[ T' \geq_b T \]
Monotonicity

\[ T'' \geq_b T \]
Monotonicity

Definition

A competition format satisfies **Monotonicity** iff for all

- \( a \in X \)
- \( x_1, \ldots, x_n \in X \)
- \( T_1, \ldots, T_m, T'_1, \ldots, T'_m \in T_X \),

Whenever \( F(x_1, \ldots, x_n, T_1, \ldots, T_m) = a \) and \( T'_i \geq_a T_i \forall i \), then \( F(x_1, \ldots, x_n, T'_1, \ldots, T'_m) = a \).
Impossibility Result

Theorem

If $|X| \geq 6$, there is no function $G: X^4 \times T_X \to X$ which satisfies

- Symmetry
- Non-Imposition
- Anonymity
- IIR

and for which the function

$$H^i_j \left( x_1, \ldots, x_i, x_{i+1}, \ldots, x_{i+j}, T_A, T_B, T \right) =$$

$$G\left( R^1( x_1, \ldots, x_i, T_A ), R^2( x_1, \ldots, x_i, T_A ), R^1( x_{i+1}, \ldots, x_{i+j}, T_B ), R^2( x_{i+1}, \ldots, x_{i+j}, T_B ), T \right)$$

is monotonic for all $i, j \geq 2$. 

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Outline of the proof

Symmetry, Non-Imposition, Anonymity IIA entails (I-V) for $G$.

Transform (I-V) to equivalent criteria (I’-V’) for $g: \{0, 1\}^6 \rightarrow \{1, 2, 3, 4\}$

Use a computer program to find all suitable $g$.

Show that all such $g$ would entail $G$ violates Non-Imposition.
Example condition

(I): \[ \forall T' = \{x_1, y_1, y_2\} \ T : \]

if \( G(x_1, x_2, y_1, y_2, T) = x_1 \) then \( G(x_1, x_2, y_1, y_2, T') = x_1 \).
A more succinct representation of $G$

Since $G$ sat. IIR + Non-Imposition, there is a function $g : \{0, 1\}^6 \rightarrow \{x_1, x_2, y_1, y_2\}$ such that for all $T$ we have

$$g(b_1, b_2, b_3, b_4, b_5, b_6) = G(x_1, x_2, y_1, y_2, T),$$

where

$$
\begin{align*}
    b_1 &= 1 \text{ iff } x_1 \top x_2 \\
    b_2 &= 1 \text{ iff } y_1 \top y_2 \\
    b_3 &= 1 \text{ iff } x_1 \top y_1 \\
    b_4 &= 1 \text{ iff } x_1 \top y_2 \\
    b_5 &= 1 \text{ iff } x_2 \top y_1 \\
    b_6 &= 1 \text{ iff } x_2 \top y_2.
\end{align*}
$$
Too many functions

We can transform \((\mathbf{I-V})\) to \((\mathbf{I}'-\mathbf{V}')\) s.t. 
\(g\) satisfies \((\mathbf{I}'-\mathbf{V}')\) iff \(G\) sat. \((\mathbf{I-V})\).

However, there are too many candidates for \(g\): 
\(4^{(2^6)} > 10^{37}\)
functions in the domain \(\{0, 1\}^6 \rightarrow \{x_1, x_2, y_1, y_2\}\).
Too many functions

**Solution:**

- Consider only sub-space of original function space. In this case all $g_1, g_2$ of the form

  \[
  g_1(b_3, b_4, b_5, b_6) = g(1, 1, b_3, b_4, b_5) \\
  g_2(b_3, b_4, b_5, b_6) = g(0, 0, b_3, b_4, b_5)
  \]

  and modify conditions $(I'-V')$ appropriately.

- Also: make use of the fact that $g$ is symmetrical. However, both contradict the Non-Imposition property of $G$. This completes the proof.

- This reduces the search space to the size of $4^8$ functions to consider.
The result

- The computer program yields only two possible candidate functions satisfying all conditions.
- However, both contradict the Non-Imposition property of \( G \). This completes the proof.
Strategy-Proofness and Reluctance to make Large Lies

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According to the Gibbard-Satterthwaite-Theorem, manipulability, is inherent to non-dictatorial choice rules. But what if each voter is only willing to make "small" lies? Does this rule out manipulability? We consider this for the case of weak linear orders.
As opposed to the linear-order case, indifferences are allowed:

\[ R = \begin{pmatrix} b \\ a \\ c, e \\ d \end{pmatrix} \]

\[ R = \{(b, a), (b, c), (b, e)(b, d), (a, c), (a, e), (a, d), (c, e), (e, c), (c, d), (e, d)\} \]
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Otherwise, usual notation:

- $A$ : finite set of alternatives.
- $m := |A|$ is the number of alternatives
- $R_N := (R_1, \ldots, R_n)$ denotes the preference profile.
Restricted Domains

In some cases, it is instructive to restrict the domain of possible preference orderings:

- Let \( \mathcal{W} \) be the set of all weak orders on \( A \). We call a subset \( \mathcal{D} \subseteq \mathcal{W} \) a **domain**.

- In the special case where \( \mathcal{D} = \mathcal{W} \), we speak of the **universal domain**.

- Other example domains:
  - The set of strictly linear orders.
  - Set of all linear orders where some alternative \( a \) is always on top.
How to measure Largeness of a Lie?

Definition

Let $R, R'$ two preference orderings. The Kemeny distance $d(R, R')$ is defined through the cardinality of the symmetric difference between $R$ and $R'$:

$$d(R, R') := |R \Delta R'| = |R - R' \cup R' - R|.$$
Kemeny Distance

\[ R = (a, b, c, d) \]

\[ R' = \begin{pmatrix} a \\ b, c, d \end{pmatrix} \]

\[ d(R, R') = ? \]
Kemeny Distance

\[ R = (a, b, c, d) \]
\[ R' = \begin{pmatrix} a \\ b, c, d \end{pmatrix} \]
\[ d(R, R') = |\{(b, a), (c, a), (d, a)\}| = 3 \]
Kemeny Distance

Assuming $|A| = m$, what is the maximum value of $d(R, R')$?

$$d(R, R') \leq ?$$
Assuming $|A| = m$, what is the **maximum value** of $d(R, R')$?

\[ d(R, R') \leq m \times (m - 1) \]

(in the case where $R, R'$ are both linear and $R$ is the reverse of $R'$.)
Weak orders of distance at most $k$

**Definition**

For each weak order $R \in \mathcal{W}$ and each $k$, let $D(R, k)$ denote $\{ R' \in \mathcal{W} | 1 \leq d(R, R') \leq k \}$; the set of preferences within $k$ distance from $R$. 
A choice rule $f$ on domain $D$ satisfies \textit{D(k)-proofness} if

- for each $R_N \in D^n$ and
- for each $i, \ 1 \leq i \leq n$

it holds that

$$f(R_N) R_i f(R_N[R_i := R'_i]) \ \forall R'_i \in D(R_i, k) \cap D$$

where $R_N[R_i := R'_i]$ denotes the preference profile that results from replacing $R_i$ by $R'_i$ in $R_N$. 

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\textbf{Two Perspectives on Manipulability}
Equivalence of strategy-proofness and $D(k)$-proofness

We say that $D(k)$-proofness on $\mathcal{D}$ is equivalent to strategy-proofness on $\mathcal{D}$ if each rule satisfies $D(k)$ proofness on $\mathcal{D}$, if and only if it satisfies strategy-proofness on $\mathcal{D}$. 
Theorem

On $\mathcal{W}$, $D(k)$-proofness is equivalent to strategy-proofness if and only if $k \geq m - 1$. 
Theorem

On $\mathcal{W}$, $D(k)$-proofness is equivalent to strategy-proofness if and only if $k \geq m - 1$. 
Criterion for Equivalence of D(1)-Proofness and Strategy-Proofness

**Definition**

A weak order is *almost linear* if one indifference class consists of two alternatives, and all the other consist of one.

**Definition**

A path between $R, R'$ is a sequence

$$R \xrightarrow{D(R,R_1)=1} R_1 \xrightarrow{D(R_1,R_2)=1} R_2 \ldots R_i \xrightarrow{D(R_i,R')=1} R'.$$
Criterion for Equivalence of D(1)-Proofness and Strategy-Proofness

Definition

A path of weak orders is **with restoration** if there exists \( x, y \) whose evolution along the sequence can be represented as

\[
\begin{align*}
\| x & \rightarrow xy \rightarrow \| y & \rightarrow xy \\
\| y & \rightarrow xy \rightarrow \| x & \rightarrow xy
\end{align*}
\]
Criterion for Equivalence of D(1)-Proofness and Strategy-Proofness

**Theorem**

Let $\mathcal{AL}$ be the set of linear and almost-linear orders and let $\mathcal{D} \subseteq \mathcal{AL}$. Assume that, for each pair of preference relations in $\mathcal{D}$, there exists a path in $\mathcal{D}$ **without restoration** between them. Then, each rule on $\mathcal{D}$ satisfying $D(1)$ proofness satisfies strategy-proofness.
Summary

- On $\mathcal{W}$, $D(k)$-proofness is equivalent to strategy-proofness if and only if $k \geq m - 1$.
- On the domains of almost-linear orders without restoration $D(1)$-proofness and strategy-proofness are equivalent.
- The results are still valid if $f$ satisfies Non-Imposition and Anonymity.