

# Arrovian Aggregation of Convex Preferences and Pairwise Utilitarianism

Florian Brandl      Felix Brandt  
Technische Universität München

{brandlf1,brandtf}@in.tum.de

We consider social welfare functions that satisfy Arrow’s classic axioms of *independence of irrelevant alternatives* and *Pareto optimality* when the outcome space is the convex hull over some finite set of alternatives. Individual and collective preferences are assumed to be continuous and convex, which guarantees the existence of maximal elements and the consistency of choice functions that return these elements, even without insisting on transitivity. We provide characterizations of both the domains of preferences and the social welfare functions that allow for Arrovian aggregation. The domains allow for arbitrary preferences over alternatives, which in turn completely determine an agent’s preferences over all outcomes. The only Arrovian social welfare functions on these domains constitute an intriguing combination of utilitarianism and pairwiseness. When also assuming anonymity, Arrow’s impossibility turns into a complete characterization of a unique social welfare function, which can be readily applied in settings that allow for lotteries or divisible goods such as time or money.

## 1. Introduction

A central concept in welfare economics are social welfare functions (SWFs) in the tradition of Arrow, i.e., functions that map a collection of individual preference relations over some set of alternatives to a social preference relation over the alternatives. Arrow’s seminal theorem states that every SWF that satisfies Pareto optimality and independence of irrelevant alternatives is dictatorial (Arrow, 1951). This sweeping impossibility significantly strengthened an earlier observation by Condorcet (1785) and sent shockwaves throughout economics as well as political philosophy and political theory (see, e.g., Maskin and Sen, 2014). A large body of subsequent work has studied whether more positive results can be

obtained by modifying implicit assumptions on the domain of admissible preferences, both individually and collectively. Two main approaches can be distinguished.

The first approach, pioneered by Sen (1969), has been to *weaken the assumption of collective transitivity* to quasi-transitivity, acyclicity, path independence or similar conditions. Although this does allow for non-dictatorial aggregation functions that meet Arrow’s criteria, these functions turned out to be highly objectionable, usually on grounds of involving weak kinds of dictatorships or violating other conditions deemed to be indispensable for reasonable preference aggregation (for an overview of the extensive literature, see Kelly, 1978; Sen, 1977, 1986; Schwartz, 1986; Campbell and Kelly, 2002). Particularly noteworthy are results about acyclic collective preference relations (e.g., Mas-Colell and Sonnenschein, 1972; Brown, 1975; Blau and Deb, 1977; Banks, 1995) because acyclicity is necessary and sufficient for the existence of maximal elements when there is a finite number of alternatives. Sen (1995) concludes that “the arbitrariness of power of which Arrow’s case of dictatorship is an extreme example, lingers in one form or another even when transitivity is dropped, so long as *some* regularity is demanded (such as the absence of cycles).”

Another stream of research has analyzed the implications of *imposing additional structure on the individual preferences*. This has resulted in a number of positive results for restricted domains, such as dichotomous or single-peaked preferences, which allow for attractive SWFs (e.g., Black, 1948; Arrow, 1951; Sen and Pattanaik, 1969; Ehlers and Storcken, 2008). Many economic domains are concerned with an infinite set of outcomes, which satisfies structural restrictions such as compactness and convexity. Preferences over these outcomes are typically assumed to satisfy some form of continuity and convexity, which roughly imply that preferences are robust with respect to minimal changes in outcomes and with respect to convex combinations of outcomes. Various results have shown that Arrow’s impossibility remains intact under these assumptions (e.g., Kalai et al., 1979; Border, 1983; Bordes and Le Breton, 1989, 1990a,b; Campbell, 1989; Redekop, 1995). Le Breton and Weymark (2011) provide an overview and conclude that “economic domain restrictions do not provide a satisfactory way of avoiding Arrovian social choice impossibilities, except when the set of alternatives is one-dimensional and preferences are single-peaked.”

The point of departure for the present approach is the observation that all impossibilities involve some form of transitivity (e.g., acyclicity), even though no such assumption is necessary to guarantee the existence of maximal elements in domains of continuous and convex preferences. Sonnenschein (1971) has shown that all continuous and convex preference relations admit a maximal element in every non-empty, compact, and convex set of outcomes. Moreover, returning maximal elements under the given conditions satisfies standard properties of choice consistency introduced by Sen (1969, 1971). Continuous and convex preference relations can thus be interpreted as rationalizing relations for the choice behavior of rational agents. Consequently, there is little justification for demanding transitivity, which has come under independent attack in normative and descriptive decision theory (see, e.g., May, 1954; Fishburn, 1970; Bar-Hillel and Margalit, 1988; Fishburn, 1991;

Anand, 1993, 2009).<sup>1</sup> For example, the *preference reversal phenomenon*, first described by Grether and Plott (1979), shows systematic experimental failures of transitivity where one lottery is preferred to another but the certainty equivalent of the latter is preferred to that of the former. As Anand (2009) writes, “once considered a cornerstone of rational choice theory, the status of transitivity has been dramatically reevaluated by economists and philosophers in recent years.”

We not only show that Arrow’s theorem ceases to hold on convex domains when dispensing with transitivity, but, moreover, Arrow’s axioms, along with some weak technical assumptions, narrow down the choice of a suitable SWF to an intriguing combination of pairwise and utilitarianism. The SWF is pairwise because it merely takes into account the agents’ pairwise comparisons between alternatives. At the same time, it has the flavor of utilitarianism because collective preferences are obtained by adding the canonical skew-symmetric bilinear (SSB) utility functions representing the voters’ ordinal preferences. SSB utility are more general than traditional linear utility functions and assign a utility value to each *pair* of alternatives.

More precisely, we consider a convex set of outcomes consisting of all probability measures on some abstract set of alternatives, which we refer to as pure outcomes. Examples of such outcome sets are allocations of divisible public goods, lotteries, time shares, monetary shares, etc. Individual and collective preference relations over these outcomes are assumed to satisfy continuity, convexity, and symmetry. We then show that there is a unique inclusion-maximal Cartesian domain of preference profiles that allows for anonymous Arrowian aggregation while satisfying minimal richness conditions. This domain allows for arbitrary preferences over pure outcomes, which in turn completely determine an agent’s preferences over all remaining outcomes. When interpreting outcomes as lotteries, this preference extension has a particularly simple and intuitive explanation: an agent prefers one lottery to another if and only if the former is more likely to return a more preferred alternative. Incidentally, this preference extension, which constitutes a central special case of SSB utility functions as introduced by Fishburn (1982), has been supported by recent experimental evidence.

We then prove that the only Arrowian SWFs are affine utilitarian with respect to the underlying SSB utility functions. As a consequence, there is a unique anonymous Arrowian SWF, which compares outcomes by the sign of the bilinear form given by the pairwise majority margins. The resulting collective preference relation over *pure* outcomes coincides with the majority relation and the corresponding choice function is therefore consistent with

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<sup>1</sup>Another frequently cited reason to justify transitivity is the *money pump*, where an agent with cyclic preferences over three outcomes is deceived into paying unlimited amounts of money in an infinite series of cyclical exchanges. As Fishburn (1991) notes, however, the money pump “applies transitive thinking [in the form of money] to an intransitive world”. Another issue with the money pump in our framework is that it cleverly avoids convexity of the feasible set by splitting it up into three subsets whose union is not convex. If the agent were confronted with a choice from the convex hull of the three original outcomes, he could simply pick his (unique) most-preferred mixed outcome and would not be tempted to exchange it when offered any other outcome in the future.

Condorcet’s principle of selecting a pure outcome that is majority-preferred to every other pure outcome whenever this is possible.<sup>2</sup> This relation is naturally extended to mixed outcomes such that, by the Minimax Theorem, every compact and convex set of outcomes admits a collectively most preferred outcome.<sup>3</sup>

Our results challenge the traditional—transitive—way of thinking about preferences, which has been largely influenced by the pervasiveness of scores and grades. A compelling opinion on transitivity, which nicely matches the narrative of this paper, is expressed in the following quote by decision theorist Peter Fishburn: “Transitivity is obviously a great practical convenience and a nice thing to have for mathematical purposes, but long ago this author ceased to understand why it should be a cornerstone of normative decision theory. [...] The presence of intransitive preferences complicates matters [...] however, it is not cause enough to reject intransitivity. An analogous rejection of non-Euclidean geometry in physics would have kept the familiar and simpler Newtonian mechanics in place, but that was not to be. Indeed, intransitivity challenges us to consider more flexible models that retain as much simplicity and elegance as circumstances allow. It challenges old ways of analyzing decisions and suggests new possibilities” (Fishburn, 1991).

## 2. Related Work

A special case of our setting, which has been particularly well studied, deals with individual preferences over lotteries that satisfy the *von Neumann-Morgenstern (vNM) axioms*, i.e., preferences that can be represented by assigning cardinal utilities to alternatives such that lotteries are compared based on the expected utility they produce. Samuelson (1967) conjectured that Arrow’s impossibility still holds under these assumptions and Kalai and Schmeidler (1977) showed that this is indeed the case when there are at least four alternatives. Hylland (1980) later pointed out that a continuity assumption made by Kalai and Schmeidler is not required. There are various versions of Arrow’s impossibility for vNM preferences, which differ in modeling assumptions and whether SWFs aggregate cardinal utilities or the preference relations represented by these utilities (see Sen, 1970; Hylland, 1980; Mongin, 1994; Dhillon and Mertens, 1997). The one closest to the framework of this paper has been shown by Le Breton (1986).<sup>4</sup>

Our results apply to Arrovian aggregation of preferences over lotteries under much loosened assumptions about preferences over lotteries. In particular, the axioms we presume entail that preferences over lotteries can be represented by *skew-symmetric bilinear (SSB)*

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<sup>2</sup>It is therefore in line with Dasgupta and Maskin (2008) who, also based on Arrow’s axioms, have forcefully argued in favor of majority rule in domains where Condorcet winners are guaranteed to exist. Our arguments extend to unrestricted preferences over pure outcomes.

<sup>3</sup>Since our domain allows for *arbitrary* preference relations over pure outcomes, it encapsulates well-known domains that allow for Arrovian aggregation in non-convex settings (such as dichotomous and single-peaked preferences). In these subdomains, the collective preference relation over pure outcomes happens to be transitive and therefore guarantees maximal pure outcomes.

<sup>4</sup>See also Le Breton and Weymark (2011, p. 214).

*utility functions*, which assign a utility value to each pair of lotteries. One lottery is preferred to another lottery if the SSB utility for this pair is positive. SSB utility theory is a generalization of linear expected utility theory due to von Neumann and Morgenstern (1947), which does not require the controversial independence axiom and transitivity (see, e.g., Fishburn, 1982, 1984b, 1988). Independence requires that a lottery  $p$  is preferred to lottery  $q$  if and only if a coin toss between  $p$  and a third lottery  $r$  is preferred to a coin toss between  $q$  and  $r$  (with the same coin used in both cases). There is experimental evidence that independence is systematically violated by human decision makers. The Allais Paradox is perhaps the most famous example (Allais, 1953). Detailed reviews of such violations, including those reported by Kahnemann and Tversky (1979), have been provided by Machina (1983, 1989) and McClennen (1988).<sup>5</sup>

Our characterization of Arrovian SWFs is related to Harsanyi’s *Social Aggregation Theorem* (Harsanyi, 1955), which shows that, for von Neumann-Morgenstern preferences over lotteries, affine utilitarianism already follows from Pareto indifference (see Fleurbaey et al., 2008, for an excellent exposition and various extensions of this theorem). Harsanyi’s theorem is a statement about *Bergson-Samuelson social welfare functions*, i.e., a single preference profile is considered in isolation. As a consequence, the weights assigned to the agents’ utility functions may depend on their preferences. This can be prevented by adding axioms that connect the collective preferences across different profiles. The SWF that derives the collective preferences by adding up utility representations normalized to the unit interval is known as *relative utilitarianism* (Dhillon, 1998; Dhillon and Mertens, 1999; Börgers and Choo, 2015, 2017). It was characterized by Dhillon and Mertens (1999) using essentially *independence of redundant alternatives* (a weakening of independence of irrelevant alternatives) and monotonicity (a weakening of a Pareto-type axiom). As shown by Fishburn and Gehrlein (1987) and further explored by Turunen-Red and Weymark (1999), aggregating SSB utility functions is fundamentally different from aggregating von Neumann-Morgenstern utility functions in that Harsanyi’s Pareto indifference axiom (and strengthenings thereof) do not imply affine utilitarianism. As we show in this paper, this can be rectified by considering Arrow’s multi-profile framework and assuming independence of irrelevant alternatives.

The probabilistic voting rule that returns the maximal elements of the unique anonymous Arrovian SWF is known as *maximal lotteries* (Kreweras, 1965; Fishburn, 1984a) and was recently axiomatized using two consistency conditions (Brandl et al., 2016b). Independently, maximal lotteries have also been studied in the context of randomized matching and assignment (see, e.g., Kavitha et al., 2011; Aziz et al., 2013).

When the set of outcomes cannot be assumed to be convex (for example, because it is finite), a common approach to address the intransitivity of collective preferences is to define alternative notions of maximality, rationalizability, or welfare, leading to concepts such as *transitive closure maximality* or the *uncovered set* (see, e.g., Laslier, 1997; Brandt and Harrenstein, 2011; Brandt et al., 2017; Nishimura, 2017). Interestingly, the support

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<sup>5</sup>Fishburn and Wakker (1995) give an interesting historical perspective on the independence axiom.

of maximal lotteries, which is known as the *bipartisan set* or the *essential set* (Laffond et al., 1993; Laslier, 2000), also appears in this literature, even though this approach is fundamentally different from the one pursued in this paper.

### 3. Preliminaries

Let  $U$  be a non-empty, finite universal set of alternatives. By  $\Delta$  we denote the set of all probability measures on  $U$ . For  $X \subseteq U$ , let  $\Delta_X$  be the set of probability measures in  $\Delta$  with support in  $X$ , i.e.,  $\Delta_X = \{p \in \Delta : p(X) = 1\}$ . We will refer to elements of  $\Delta$  as outcomes and one-point measures in  $\Delta$  as pure outcomes. Furthermore, let  $\succ$  be an asymmetric binary relation over  $\Delta$ , which is interpreted as the preference relation of an agent. Given two outcomes  $p, q \in \Delta$ , we write  $p \sim q$  when neither  $p \succ q$  nor  $q \succ p$ , and  $p \succsim q$  if  $p \succ q$  or  $p \sim q$ . For  $p \in \Delta$ , let  $U(p) = \{q \in \Delta : q \succ p\}$  and  $L(p) = \{q \in \Delta : p \succ q\}$  be the strict upper and strict lower contour set of  $p$  with respect to  $\succ$ ;  $I(p) = \{q \in \Delta : p \sim q\}$  denotes the indifference set of  $p$ . For  $X \subseteq \Delta$ ,  $\succ|_X = \{(p, q) \in \succ : p, q \in X\}$  is the preference relation  $\succ$  restricted to outcomes in  $X$ .

We will consider preference relations that are continuous, i.e., small changes in outcomes do not result in a reversal of preference. One of several possibilities to define continuity is the Archimedean axiom, which requires that, for any given outcome  $p$ , the convex hull of a more preferred outcome and a less preferred outcome also contains an equally preferred outcome. A preference relation  $\succ$  is continuous if, for all  $p, q, r \in \Delta$ ,

$$p \succ q \succ r \text{ implies } \lambda p + (1 - \lambda)r \sim q \text{ for some } \lambda \in (0, 1). \quad (\text{Continuity})$$

Another standard assumption is that preferences are convex. We will use convexity as defined by Fishburn (1982).  $\succ$  is convex if, for all  $p, q, r \in \Delta$  and  $\lambda \in (0, 1)$ ,

$$\begin{aligned} p \succ q \text{ and } p \succsim r &\text{ imply } p \succ \lambda q + (1 - \lambda)r, \\ q \succ p \text{ and } r \succsim p &\text{ imply } \lambda q + (1 - \lambda)r \succ p, \text{ and} \\ p \sim q \text{ and } p \sim r &\text{ imply } p \sim \lambda q + (1 - \lambda)r. \end{aligned} \quad (\text{Convexity})$$

Equivalently, one could require that the indifference set for an outcome  $p$  is a hyperplane through  $p$ ; the upper and lower contour sets are the corresponding half spaces. Note that convexity implies that upper contour sets, lower contour sets, and indifference sets are convex. Moreover, upper contour and lower contour sets are open and indifference sets are closed.

The existence of maximal elements is usually quoted as the main reason for insisting on transitivity of preference relations. It was shown by Sonnenschein (1971) that continuity and convexity are already sufficient for the existence of maximal elements, even when preferences are intransitive (see also Bergstrom, 1992; Llinares, 1998).<sup>6</sup> For a preference relation  $\succ$  and a subset of outcomes  $X \subseteq \Delta$ , let  $\max_{\succ}(X) = \{x \in X : x \succsim y \text{ for all } y \in X\}$ .

<sup>6</sup>Sonnenschein (1971) only required that upper contour sets are convex and that lower contour sets are open. Also his notion of continuity is weaker than ours when the set of alternatives is finite.

**Proposition 1.** (*Sonnenschein, 1971*) *If  $\succ$  is a continuous and convex preference relation, then  $\max_{\succ}(X) \neq \emptyset$  for all non-empty, compact, and convex sets  $X \subseteq \Delta$ .*

Sen (1969, 1971) has shown that two intuitive choice consistency conditions are equivalent to choosing maximal elements according to an acyclic relation. These conditions are known as *Sen's  $\alpha$*  (or *contraction*) and *Sen's  $\gamma$*  (or *expansion*). Contraction requires that if an outcome is chosen from some set, then it is also chosen from any subset that it is contained in. This condition is satisfied by  $\max_{\succ}$  without imposing any restrictions on  $\succ$ . Expansion prescribes that an outcome that is chosen from two sets  $X$  and  $Y$ , should also be chosen from their union  $X \cup Y$ . Since we are only interested in choosing from convex sets, we strengthen this condition by taking the *convex hull*  $\text{conv}(X \cup Y)$  in the consequence.  $\max_{\succ}$  satisfies this condition whenever  $\succ$  is convex. To see this, consider  $X, Y \subseteq \Delta$  and assume that  $p \in \max_{\succ} X \cap \max_{\succ} Y$ . Then,  $p \succeq q$  for all  $q \in X \cup Y$  and since  $\succ$  satisfies convexity, we have  $p \succeq q$  for all  $q \in \text{conv}(X \cup Y)$ . Thus,  $p \in \max_{\succ}(\text{conv}(X \cup Y))$ . Sen's proof can even be adapted to show that *every* choice function satisfying contraction and expansion is of the form  $\max_{\succ}$  for some  $\succ$  with convex weak upper and lower contour sets.<sup>7</sup>

Convexity of preferences implies that the corresponding indifference curves are straight lines. The symmetry axiom introduced by Fishburn (1982) prescribes that either all indifference curves are parallel or meet at one point (which may be outside of  $\Delta$ ). For all  $p, q, r \in \Delta$  and  $\lambda \in (0, 1)$ ,

$$\begin{aligned} & \text{if } p \succ q \succ r, p \succ r, \text{ and } q \sim 1/2 p + 1/2 r, \text{ then} \\ & [\lambda p + (1 - \lambda)r \sim 1/2 p + 1/2 q \text{ if and only if } \lambda r + (1 - \lambda)p \sim 1/2 r + 1/2 q]. \end{aligned} \quad (\text{Symmetry})$$

Fishburn (1984b) justifies this axiom by stating that “the degree to which  $p$  is preferred to  $q$  is equal in absolute magnitude but opposite in sign to the degree to which  $q$  is preferred  $p$ .” He continues by writing that he is “a bit uncertain as to whether this should be regarded more as a convention than a testable hypothesis – much like the asymmetry axiom [...], which can almost be thought of as a definitional characteristic of strict preference.”

By  $\mathcal{R}$  we denote the set of all continuous, convex, and symmetric preference relations. Despite the richness of  $\mathcal{R}$ , preference relations therein admit a particularly nice representation. It was shown by Fishburn (1982) that if  $\succ \in \mathcal{R}$ , then there is a skew-symmetric and bilinear (SSB) utility function  $\phi: \Delta \times \Delta \rightarrow \mathbb{R}$  such that, for all  $p, q \in \Delta$ ,<sup>8</sup>

$$p \succ q \text{ if and only if } \phi(p, q) > 0.$$

Moreover,  $\phi$  is unique up to scalar multiplication. We therefore write  $\phi \equiv \hat{\phi}$  if and only if there is some  $\alpha > 0$  such that  $\phi = \alpha \cdot \hat{\phi}$ . Let  $\Phi$  denote the set of all SSB functions on

<sup>7</sup>There are also stronger versions of expansion, which, together with contraction, are equivalent to the *weak axiom of revealed preference* or *Arrow's choice axiom* (Samuelson, 1938; Arrow, 1959). These conditions imply rationalizability via a *transitive* relation and are therefore not generally satisfied when choosing maximal elements of convex relations.

<sup>8</sup>A function  $\phi$  is skew-symmetric if  $\phi(p, q) = -\phi(q, p)$  for all  $p, q \in \Delta$ .  $\phi$  is bilinear if it is linear in both arguments.

$\Delta \times \Delta$ . For outcomes with finite support,  $\phi(p, q)$  can be written as a convex combination of the values of  $\phi$  for pure outcomes (Fishburn, 1984b). For this purpose, we identify every alternative  $a \in U$  with the pure outcome that assigns probability 1 to  $a$ . Then, for all  $p, q \in \Delta$ ,

$$\phi(p, q) = \sum_{a, b \in U} p(a)q(b)\phi(a, b).$$

We will often represent SSB functions restricted to  $\Delta_X$  for  $X \subseteq U$  as skew-symmetric matrices in  $\mathbb{R}^{X \times X}$ .

When requiring transitivity on top of continuity, convexity, and symmetry, the four axioms characterize *weighted linear (WL)* utility functions as introduced by Chew (1983).<sup>9</sup> When additionally requiring independence, then  $\phi$  is separable, i.e.,  $\phi(p, q) = u(p) - u(q)$ , where  $u$  is a linear von Neumann-Morgenstern utility function representing  $\succ$ . For independently distributed outcomes (as considered in this paper), SSB utility theory coincides with regret theory as introduced by Loomes and Sugden (1982) (see also Loomes and Sugden, 1987; Blavatsky, 2006).

Through the representation of  $\succ \in \mathcal{R}$  restricted to a finite  $X \subseteq U$  by a skew-symmetric matrix, it becomes apparent that the Minimax Theorem implies the existence of maximal elements of  $\succ$  on  $\Delta_X$ . This was noted by Fishburn (1984b, Theorem 4) and already follows from Proposition 1.

## 4. Social Welfare Functions

For the remainder of the paper we deal with the problem of aggregating the preferences of multiple agents into a collective preference relation. The set of agents is  $N = \{1, \dots, n\}$  for some  $n \geq 2$ . The preference relations of agents belong to some *domain*  $\mathcal{D} \subseteq \mathcal{R}$ . A function from the set of agents to the domain  $R \in \mathcal{D}^N$  is a preference profile. We will write preference profiles as tuples with indices in  $N$ . A *social welfare function (SWF)*  $f: \mathcal{D}^N \rightarrow \mathcal{R}$  maps a preference profile to a collective preference relation.

Arrow's impossibility theorem shows that the only SWFs that satisfy two desirable properties, Pareto optimality and independence of irrelevant alternatives, are dictatorial functions. Pareto optimality prescribes that a unanimous preference of one outcome over another in the individual preferences should be reflected in the collective preference. An SWF  $f$  is *Pareto optimal* if, for all  $p, q \in \Delta$ ,  $R \in \mathcal{D}^N$ , and  $f(R) = \succ$ ,

$$\begin{aligned} p \succ_i q \text{ for all } i \in N &\text{ implies } p \succ q, \text{ and} \\ \text{if additionally } p \succ_i q &\text{ for some } i \in N \text{ then } p \succ q. \end{aligned} \tag{Pareto optimality}$$

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<sup>9</sup>A WL function is characterized by a linear utility function and a non-vanishing weight function. The utility of an outcome is the utility derived by the linear utility function weighted according to the weight function. Thus, WL functions are more general than linear utility functions, as every linear utility function is equivalent to a WL function with constant weight function. See also Fishburn (1983).



The indifference part of Pareto optimality, which merely requires that  $p \sim_i q$  for all  $i \in N$  implies  $p \sim q$ , is usually referred to as *Pareto indifference*.

Independence of irrelevant alternatives demands that collective preferences over some feasible set of outcomes should only depend on the individual preferences over this set (and not on the preferences over outcomes outside this set). In our framework, we will assume that feasible sets are based on the availability of alternatives and are therefore of the form  $\Delta_X$  for  $X \subseteq U$ . Formally, we say that an SWF  $f$  satisfies *independence of irrelevant alternatives (IIA)* if, for all  $R, \hat{R} \in \mathcal{D}^N$  and  $X \subseteq U$ ,

$$R|_{\Delta_X} = \hat{R}|_{\Delta_X} \text{ implies } f(R)|_{\Delta_X} = f(\hat{R})|_{\Delta_X}. \quad (\text{IIA})$$

Any SWF that satisfies Pareto optimality and IIA will be called an *Arrovian SWF*. Arrow has shown that, when no structure is imposed on preference relations and feasible sets, every Arrovian SWF is dictatorial, i.e., the preference relation of one fixed agent is a sub-relation of the collective preference relation (formally, there is  $i \in N$  such that for all  $R \in \mathcal{R}^N$ ,  $\succ_i \subseteq f(R)$ ). Dictatorships are examples of SWFs that are extremely biased towards one agent. In many applications, *any* differentiation between agents is unacceptable and all agents should be treated equally. This property is known as anonymity. We denote by  $\Pi_N$  the set of all permutations on  $N$ . For  $\pi \in \Pi_N$  and a preference profile  $R \in \mathcal{D}^N$ ,  $R^\pi = R \circ \pi$  is the preference profile where agents are renamed according to  $\pi$ . Then, an SWF  $f$  is *anonymous* if for all  $R \in \mathcal{D}^N$  and  $\pi \in \Pi_N$ ,

$$f(R) = f(R^\pi). \quad (\text{Anonymity})$$

Anonymity is obviously a stronger requirement than non-dictatorship.

In order to prove our characterization, we need to assume that any domain  $\mathcal{D} \subseteq \mathcal{R}$  satisfies certain richness conditions. First, we require that the domain is neutral in the sense that it is not biased towards certain alternatives. For  $\pi \in \Pi_U$  and  $p \in \Delta$ , let  $p^\pi \in \Delta$  such that  $p^\pi(\pi(a)) = p(a)$  for all  $a \in U$ . Then, for  $\succ \in \mathcal{R}$ , we define  $\succ^\pi$  such that  $p^\pi \succ^\pi q^\pi$  if and only if  $p \succ q$  for all  $p, q \in \Delta$ . It is assumed that  $\succ \in \mathcal{D}$  if and only if  $\succ^\pi \in \mathcal{D}$  for all  $\succ \in \mathcal{D}$  and  $\pi \in \Pi_U$ . Second, it should also be possible for agents to declare completely opposed preferences. For  $\succ \in \mathcal{D}$ ,  $\succ^{-1}$  is the inverse of  $\succ$ , i.e.,  $p \succ^{-1} q$  if and only if  $q \succ p$  for all  $p, q \in \Delta$ . Then  $\succ \in \mathcal{D}$  implies  $\succ^{-1} \in \mathcal{D}$  for all  $\succ \in \mathcal{R}$ . Note that this condition is not implied by the previous neutrality condition because not only the preferences over alternatives, but also the preferences over outcomes are inverted. Finally, we demand that for every transitive and asymmetric relation on five pure outcomes,  $\mathcal{D}$  contains at least one extension of this relation to all outcomes in  $\Delta$ .<sup>10</sup>

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<sup>10</sup>To derive the conclusion of Theorem 1, a weaker condition suffices: if  $\succ \in \mathcal{D}$  with  $a \succ b \succ c$  and  $a \succ c$  for some  $a, b, c \in U$ , then there is some  $\succ' \in \mathcal{D}$  with  $a \succ' b \succ' c \succ' x$  and  $a \succ' c$  for some  $x \in U$ . This condition also covers the domain of dichotomous preferences.

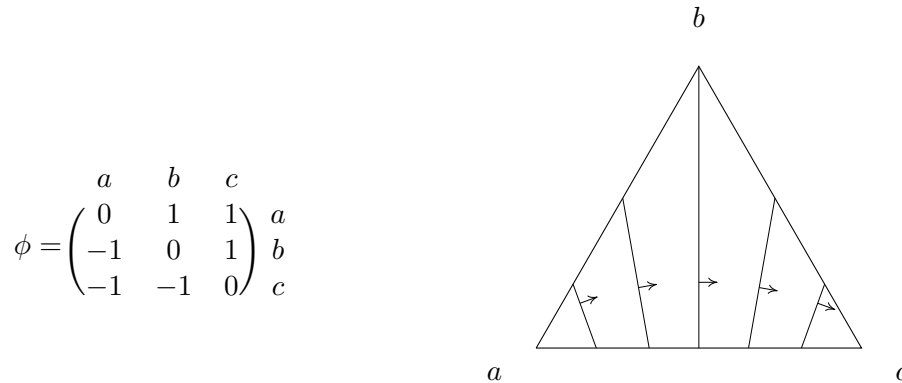


Figure 1: Illustration of preferences based on pairwise comparisons when preferences on pure outcomes are given by the transitive relation  $a \succ b \succ c$ . The left-hand side shows the corresponding SSB function and the right-hand side the Marschak-Machina probability triangle. The arrows represent the normal vectors to the indifference curves through the base of the arrow (pointing towards the lower contour set). Each indifference curve separates the corresponding upper and lower contour set.

## 5. Characterization of the Domain

Non-dictatorial Arrovian aggregation on the full domain  $\mathcal{R}$  is impossible because it is already impossible in the subdomain of vNM preferences (see Section 2). On the other hand, interesting possibilities emerge in restricted domains such as in that of dichotomous vNM preferences where each agent can only assign two different utility values. In this domain, every SWF based on affine welfare (with positive weights) satisfies IIA and Pareto optimality. The only anonymous Arrovian SWF on this domain corresponds to *approval voting* and ranks alternatives based on pairwise majority comparisons, which happen to be transitive for dichotomous preferences.

In this section, we characterize the (unique) largest domain  $\mathcal{D} \subseteq \mathcal{R}$  for which anonymous Arrovian SWFs exist.<sup>11</sup> We say that  $\phi \in \Phi$  is based on *pairwise comparisons* if  $\phi(a, b) \in \{-1, 0, 1\}$  for all  $a, b \in U$  and denote the set of SSB functions that are based on pairwise comparisons by  $\Phi^{PC} \subset \Phi$ .

**Theorem 1.** *Let  $f$  be an anonymous Arrovian SWF on some domain  $\mathcal{D}$ . Then  $\mathcal{D} \subseteq \Phi^{PC}$ .*

Preferences based on pairwise comparisons are quite natural and can be seen as the canonical SSB representation consistent with a given ordinal preference relation over alternatives. For a preference relation  $\succ$  that can be represented using an SSB function in

<sup>11</sup>The characterization also holds when slightly loosening the definition of SWFs because the proof of Theorem 1 does not require collective preferences to satisfy symmetry.

	$a$	$b$	$c$	$d$	
$\phi =$	0	1	1	1	$a$
	-1	0	1	1	$b$
	-1	-1	0	1	$c$
	-1	-1	-1	0	$d$

	$a$	$b$	$c$	$d$
$p$	0	0	1	0
$q$	$2/5$	0	0	$3/5$
$r$	0	$3/5$	0	$2/5$

Figure 2: Illustration of preferences based on pairwise comparisons when preferences on pure outcomes are given by the transitive relation  $a \succ b \succ c \succ d$ . Preferences between the three outcomes  $p$ ,  $q$ , and  $r$ , defined in the table on the right-hand side, are cyclic:  $\phi(p, q) = p^T \phi q = 3/5 - 2/5 = 1/5 > 0$ ,  $\phi(q, r) = q^T \phi r = 2/5 - (3/5)^2 = 4/25 > 0$ , and  $\phi(r, p) = r^T \phi p = 3/5 - 2/5 = 1/5 > 0$ .

$\Phi^{PC}$  and two outcomes  $p, q \in \Delta$  we thus have that

$$p \succ q \quad \text{iff} \quad p^T \phi q > 0 \quad \text{iff} \quad \sum_{a,b: a \succ b} p(a) \cdot q(b) > \sum_{a,b: a \succ b} q(a) \cdot p(b).$$

If  $p$  and  $q$  are interpreted as lotteries,  $p$  is preferred to  $q$  if and only if  $p$  is more likely to return a more preferred alternative than  $q$ . Alternatively, the terms in the inequality above can be associated with the probability of *ex ante* regret. Then,  $p$  is preferred to  $q$  if its choice results in less *ex ante* regret.

Preferences based on pairwise comparisons have previously been considered in decision theory (Blyth, 1972; Packard, 1982; Blavatskyy, 2006). Packard (1982) calls them the *rule of expected dominance* and Blavatskyy (2006) refers to them as a *preference for the most probable winner*. Aziz et al. (2015, 2016) and Brandl et al. (2016a) have studied efficiency, strategyproofness, and related properties with respect to these preferences.

Figure 1 illustrates preferences based on pairwise comparisons for three transitively ordered alternatives. Blavatskyy (2006) gives an axiomatic characterization using Fishburn’s SSB axioms and an additional axiom called *fanning-in*, which essentially prescribes that indifference curves are not parallel, but fanning in (see Figure 1). As a corollary of Theorem 1, fanning-in is implied by Fishburn’s SSB axioms and Arrow’s axioms. Blavatskyy cites extensive experimental evidence showing that indifference curves are indeed fanning in.

When there are at least four alternatives, preferences based on pairwise comparisons can be cyclic even when preferences over pure outcomes are transitive. This phenomenon, known as the *Steinhaus-Trybula paradox*, is illustrated in Figure 2 (see, e.g., Steinhaus and Trybula, 1959; Blyth, 1972; Packard, 1982; Rubinstein and Segal, 2012; Butler et al., 2016). Butler et al. (2016) have conducted an extensive experimental study of the Steinhaus-Trybula paradox and found significant evidence for preferences based on pairwise comparisons.

## 6. Characterization of the Social Welfare Function

Theorem 1 has established that Arrovian aggregation is only possible if individual preferences are based on pairwise comparisons. In the remainder of this paper, we will—with slight abuse of notation—treat Arrovian SWFs as functions from  $\mathcal{D}^N$  to  $\Phi$  with  $\mathcal{D} \subseteq \Phi^{PC}$ .

It turns out that an SWF satisfies IIA and Pareto indifference if and only if outcomes are compared by considering a linear combination of the individual SSB functions. Coefficients of the individual SSB functions may be zero or even negative. This, for example, allows for dictatorial SWFs where the collective preference is identical to the preference relation of one pre-determined agent. When assuming full Pareto optimality, the coefficients assigned to these SSB functions have to be positive, which rules out dictatorial SWFs.

**Theorem 2.** *Let  $f$  be an Arrovian SWF. Then, there are  $w_1, \dots, w_n \in \mathbb{R}_{>0}$  such that*

$$f(R) \equiv \sum_{i \in N} w_i \phi_i \text{ for all } R \in \mathcal{D}^N.$$

This can be seen as a multi-profile version of Harsanyi's Social Aggregation Theorem (see Section 2) for SSB utilities, where IIA allows us to connect coefficients across different profiles.

When furthermore assuming anonymity, the coefficients of all SSB functions have to be identical and we obtain the following complete characterization.

**Corollary 1.** *Let  $f$  be an anonymous Arrovian SWF. Then,*

$$f(R) \equiv \sum_{i \in N} \phi_i \text{ for all } R \in \mathcal{D}.$$

The unique anonymous Arrovian SWF is computationally tractable: two outcomes can be compared by straightforward matrix-vector multiplications while a maximal outcome can be found using linear programming. For illustrative purposes, consider the classic Condorcet example where there are three agents with the following transitive preferences over pure outcomes:  $a \succ_1 b \succ_1 c$ ,  $b \succ_2 c \succ_2 a$ , and  $c \succ_3 a \succ_3 b$ . These preferences are represented by  $\phi_1, \phi_2, \phi_3 \in \Phi^{PC}$  where

$$\phi_1 = \begin{pmatrix} a & b & c \\ 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix} \begin{matrix} a \\ b \\ c \end{matrix}, \quad \phi_2 = \begin{pmatrix} a & b & c \\ 0 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{matrix} a \\ b \\ c \end{matrix}, \quad \text{and} \quad \phi_3 = \begin{pmatrix} a & b & c \\ 0 & 1 & -1 \\ -1 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix} \begin{matrix} a \\ b \\ c \end{matrix}.$$

Note that the pairwise majority relation is cyclic, since there are majorities for  $a$  over  $b$ ,  $b$  over  $c$ , and  $c$  over  $a$ . The unique anonymous Arrovian SWF  $f$  aggregates preferences by

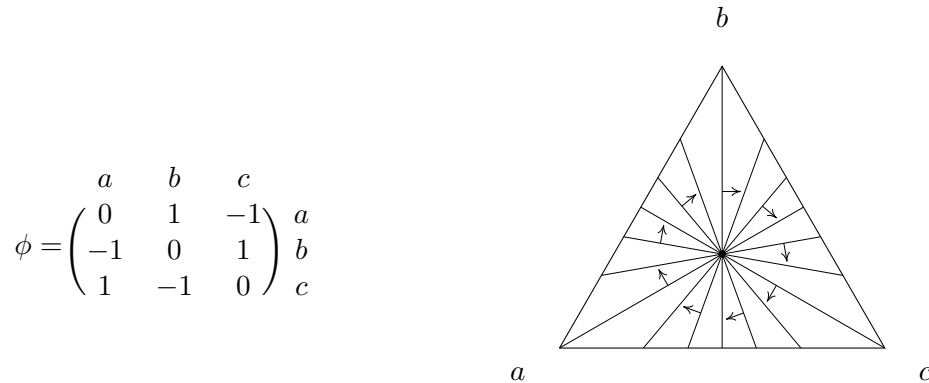


Figure 3: Illustration of collective preferences returned by the unique anonymous Arrovian SWF in the case of Condorcet’s paradox. The left-hand side shows the collective SBB function and the right-hand side the Marschak-Machina probability triangle. The arrows represent the normal vectors to the indifference curves through the base of the arrow (pointing towards the lower contour set). Each indifference curve separates the corresponding upper and lower contour set. The unique most preferred outcome is  $1/3 a + 1/3 b + 1/3 c$ .

adding the individual utility representations, i.e.,

$$f(R) \equiv \sum_{i \in N} \phi_i = \begin{pmatrix} a & b & c \\ 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{matrix} a \\ b \\ c \end{matrix} .$$

Figure 3 shows the collective preference relation represented by this matrix. The unique most preferred outcome is  $1/3 a + 1/3 b + 1/3 c$ .<sup>12</sup>

Theorem 2 can be viewed as an intermediary between Harsanyi’s Social Aggregation Theorem and Arrow’s Impossibility Theorem: it uses Arrow’s axioms to derive Harsanyi’s utilitarian consequence. Clearly, the form of utilitarianism characterized in Theorem 2 is rather restricted as, due to Theorem 1, it does not allow for intensities of individual preferences. Curiously, our result entails that Arrow’s axioms rule out intensities of individual preferences, but do allow for—and in fact require—intensities of collective preferences.

## 7. Remarks

We conclude the paper with a number of remarks.

<sup>12</sup>This outcome represents a somewhat unusual unique maximal element because it is not *strictly* preferred to any of the other outcomes. This is due to the contrived nature of the example and only happens if the support of a maximal outcome contains all alternatives.

**Remark 1.** When also requiring transitivity, our result immediately turns into an impossibility, which follows from Theorem 1 and the example given in Figure 2. This implies the impossibility of anonymous Arrovian aggregation of vNM preferences, which already follows from other results (see Section 2).

**Remark 2.** Theorem 1 does not hold without assuming anonymity. Let  $U = \{a, b, c, d, e\}$  and consider the SSB function

$$\phi|_U = \begin{pmatrix} 0 & 1 & 2 & 5 & 7 \\ -1 & 0 & 1 & 4 & 6 \\ -2 & -1 & 0 & 3 & 5 \\ -5 & -4 & -3 & 0 & 2 \\ -7 & -6 & -5 & -2 & 0 \end{pmatrix}.$$

(Observe that  $\phi$  can be represented by a linear utility function.) Let  $\succ \in \mathcal{R}$  be the preference relation represented by  $\phi$  and  $\mathcal{D}$  the domain that contains all permutations of  $\succ$  with respect to alternatives and their inverses, i.e.,  $\mathcal{D} = \{\succ^\pi, (\succ^\pi)^{-1} : \pi \in \Pi_U\}$ . By construction,  $\mathcal{D}$  satisfies the domain requirements specified in Section 4. For some small  $\epsilon > 0$  and all  $i \in N$ , let  $w_i = \epsilon^i$ . Then, the SWF  $f: \mathcal{D} \rightarrow \mathcal{R}$  that returns the preference relation represented by  $\sum_{i \in N} w_i \phi_i$  satisfies Pareto optimality and IIA but is not anonymous.

**Remark 3.** Theorem 2 does not hold if  $|U| \leq 4$ . Let  $U = \{a, b, c, d\}$ ,  $\mathcal{D} = \Phi^{PC}$ , and  $R \in \mathcal{D}^N$  such that

$$\hat{R}|_U = \left( \underbrace{\begin{pmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ -1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{pmatrix}}_{\{1,2\}}, \dots, \underbrace{\begin{pmatrix} 0 & 1 & -1 & -1 \\ -1 & 0 & -1 & -1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & -1 & 0 \end{pmatrix}}_{\{3\}}, \underbrace{\begin{pmatrix} 0 & 1 & -1 & -1 \\ -1 & 0 & -1 & -1 \\ 1 & 1 & 0 & -1 \\ 1 & 1 & 1 & 0 \end{pmatrix}}_{\{4\}}, \dots \right),$$

and every SSB function in  $\mathcal{D} \setminus \{0\}$  appears exactly once in the preferences of the agents in  $N \setminus \{1, 2, 3, 4\}$ . Then Pareto optimality has no implications for  $\hat{R}$ . Let  $f: \mathcal{D}^N \rightarrow \Phi$ ,  $f(R) = \sum_{i \in N} \phi_i$  except that

$$f(\hat{R}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Then,  $f$  satisfies Pareto optimality and IIA. The proof of Theorem 2 fails at Lemma 7.

**Remark 4.** Fishburn (1984c) shows that under additional technical assumptions about the measure space and  $\succ$ , the SSB representation holds for probability measures over arbitrary (possibly infinite) sets of alternatives. Our results extend to this framework without modifications to the proofs.

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## APPENDIX

### A. Characterization of the Domain

The first prove a useful lemma, which shows that continuous and convex preference relations are completely determined by their symmetric part up to orientation.

**Lemma 1.** *Let  $\succ, \hat{\succ}$  be continuous and convex preference relations such that  $\sim \subseteq \hat{\sim}$ . Then,  $\hat{\succ} \in \{\succ, \succ^{-1}, \emptyset\}$ .*

*Proof.* It can be shown that if a preference relation is continuous and convex, then upper contour sets and lower contour sets are open.<sup>13</sup> We will make use of this statement in the rest of the proof. We first prove an auxiliary statement: if  $\succ$  is continuous and convex and  $p \in \Delta$  such that  $I(p)$  contains a non-empty open set, then  $I(p) = \Delta$ . Assume for contradiction that  $I(p) \neq \Delta$  or, equivalently,  $U(p) \cup L(p) \neq \emptyset$  and let  $q \in I(p)$  such that a neighborhood of  $q$  is contained in  $I(p)$ . Consider the case when  $U(p) \neq \emptyset$  and let  $r \in U(p)$ . Then convexity implies that  $(1 - \lambda)q + \lambda r \in U(p)$  for all  $\lambda \in (0, 1)$ . This contradicts the fact that a neighborhood of  $q$  is contained in  $I(p)$ . The case  $L(p) \neq \emptyset$  is symmetric.

Now let  $\succ, \hat{\succ}$  be continuous and convex such that  $\sim \subseteq \hat{\sim}$ . Let  $p \in \Delta$ . By assumption, we have  $I(p) \subseteq \hat{I}(p)$ . Moreover,  $\Delta$  is the disjoint union of  $I(p), U(p), L(p)$  and  $\hat{I}(p), \hat{U}(p), \hat{L}(p)$ , respectively. This implies that  $\hat{U}(p) \cup \hat{L}(p) \subseteq U(p) \cup L(p)$ . Assume for contradiction that  $\hat{U}(p) \cap U(p) \neq \emptyset$  and  $\hat{U}(p) \cap L(p) \neq \emptyset$ . Let  $q \in \hat{U}(p) \cap U(p)$  and  $r \in \hat{U}(p) \cap L(p)$ . Continuity implies that  $\text{conv}(\{q, r\}) \cap I(p) \neq \emptyset$ . Convexity of  $\hat{\succ}$  implies that  $\text{conv}(\{q, r\}) \subseteq \hat{U}(p)$ . Hence,  $\emptyset \neq \text{conv}(\{q, r\}) \cap I(p) \subseteq \hat{U}(p)$ , which contradicts  $I(p) \subseteq \hat{I}(p)$ . Hence,  $\hat{U}(p) \subseteq U(p)$  or  $\hat{U}(p) \subseteq L(p)$ . Similarly,  $\hat{L}(p) \subseteq L(p)$  or  $\hat{L}(p) \subseteq U(p)$ .

If  $\hat{U}(p) \cup \hat{L}(p) \subseteq U(p)$  or  $\hat{U}(p) \cup \hat{L}(p) \subseteq L(p)$ , then  $\hat{I}(p)$  contains either  $L(p)$  or  $U(p)$ . If  $L(p) \subseteq \hat{I}(p)$  and  $L(p)$  is non-empty or  $U(p) \subseteq \hat{I}(p)$  and  $U(p)$  is non-empty, then  $\hat{I}(p)$  contains an open set. Hence,  $\hat{I}(p) = \Delta$ . Thus, for all  $p \in \Delta$  with  $L(p) \neq \emptyset$  and  $U(p) \neq \emptyset$ , either  $\hat{U}(p) \subseteq U(p)$  and  $\hat{L}(p) \subseteq L(p)$  or  $\hat{U}(p) \subseteq L(p)$  and  $\hat{L}(p) \subseteq U(p)$ . In the first case, we say that  $\hat{\succ}$  is oriented positively at  $p$ , in the latter case oriented negatively. If  $\hat{I}(p) = \Delta$ ,  $\hat{\succ}$  is oriented both, positively and negatively at  $p$ .

Let  $\Delta^+ = \{p \in \Delta : U(p) = \emptyset \text{ and } L(p) \neq \emptyset\}$  and  $\Delta^- = \{p \in \Delta : L(p) = \emptyset \text{ and } U(p) \neq \emptyset\}$ . Let  $p, q \in \Delta^+, r \in \text{conv}(\{p, q\})$ . For all  $s \in \Delta, p, q \in I(s) \cup U(s)$  by definition of  $\Delta^+$ . Convexity of  $\succ$  then implies that  $r \in I(s) \cup U(s)$ . Since  $L(p) \neq \emptyset, L(p) \cap (I(q) \cup L(q)) \neq \emptyset$ . For  $s \in L(p) \cap (I(q) \cup L(q))$ , convexity implies that  $s \in L(r)$ . Hence,  $\Delta^+$  is convex. Moreover, for all  $p \in \Delta^+, \Delta^+ \subseteq I(p)$ , as otherwise there is  $p \in \Delta^+$  such that  $U(p) \neq \emptyset$ . Similarly,  $\Delta^-$  is convex and, for all  $p \in \Delta^-, \Delta^- \subseteq I(p)$ . Since, for all  $p \in \Delta^+ \cup \Delta^-, I(p) \neq \Delta$ , neither of  $\Delta^+$  and  $\Delta^-$  contains a non-empty open set. Then convexity of  $\Delta^+$  and  $\Delta^-$  implies that  $\Delta^+ \cup \Delta^-$  does not contain a non-empty open set. Let  $\Delta^* = \Delta \setminus (\Delta^+ \cup \Delta^-)$ . Note that  $\Delta^*$  contains a non-empty open set.

<sup>13</sup>This statement requires the set of alternatives to be finite.

Let  $p, q \in \Delta^*$  such that  $q \in \hat{U}(p)$  and  $\hat{\succ}$  is oriented positively at  $p$ . Hence,  $\hat{U}(p) \subseteq U(p)$  and  $q \in \hat{U}(p)$  imply that  $p \in L(q)$ . Also  $q \in \hat{U}(p)$  is equivalent to  $p \in \hat{L}(q)$ . Hence,  $\hat{\succ}$  is oriented positively at  $q$ . Similarly, if  $q \in \hat{L}(p)$  or if  $p, q \in \Delta^*$  such that  $q \in \hat{U}(p) \cup L(p)$  and  $\hat{\succ}$  is oriented negatively at  $p$ , then  $\hat{\succ}$  is oriented negatively at  $q$ .

If  $\hat{\succ} = \emptyset$ , the statement of the lemma holds trivially. Assume that  $\hat{\succ} \neq \emptyset$ . Since  $\Delta^*$  contains an open set and  $\{p \in \Delta: I(p) = \emptyset\}$  does not, there is  $p \in \Delta^*$  with  $\hat{I}(p) \neq \Delta$ . From before, it follows that  $\hat{\succ}$  is oriented positively or negatively at  $p$  (but not both). Assume that  $\hat{\succ}$  is oriented positively at  $p$ . Let  $q \in \Delta^*$ . If there is  $r \in \Delta^* \setminus (\hat{I}(p) \cup \hat{I}(q))$ , then two applications of what we have proven above yield that  $\hat{\succ}$  is oriented positively at  $q$ . If  $\Delta^* \subseteq \hat{I}(p) \cup \hat{I}(q)$ , then either  $\hat{I}(p)$  or  $\hat{I}(q)$  contains an open set. Hence either  $\hat{I}(p) = \Delta$  or  $\hat{I}(q) = \Delta$ . Since  $\hat{I}(p) \neq \Delta$  by assumption, we have that  $\hat{I}(q) = \Delta$ . Then  $\hat{\succ}$  is trivially oriented positively (and negatively) at  $q$ . Together we get that  $\hat{\succ}$  is oriented positively at all  $p \in \Delta^*$ .

Let  $\succ_p$  denote the restriction of  $\succ$  to those comparisons involving  $p$ , i.e.,  $\succ_p = \succ \cap (\{p\} \times \Delta \cup \Delta \times \{p\})$ . By continuity and convexity of  $\hat{\succ}$ ,  $\hat{I}(p)$  is the hyperplane separating  $\hat{U}(p)$  and  $\hat{L}(p)$ . Similarly,  $I(p)$  is the hyperplane separating  $U(p)$  and  $L(p)$ . Since  $I(p) \subseteq \hat{I}(p)$ , it follows that  $I(p) = \hat{I}(p)$ . Since  $\hat{\succ}$  is oriented positively at  $p$ , we get that  $\hat{\succ}_p = \succ_p$ . Now let  $q \in \Delta^*$  be arbitrary. If  $\hat{I}(q) \neq \Delta$ , we have that  $\hat{I}(p) \cup \hat{I}(q) \neq \Delta$ , as this would imply that either  $\hat{I}(p)$  or  $\hat{I}(q)$  contains an open set in which case  $\hat{I}(p) = \Delta$  or  $\hat{I}(q) = \Delta$ , respectively. Hence,  $\Delta^* \setminus (\hat{I}(p) \cup \hat{I}(q)) \neq \emptyset$ . Let  $r$  be an element thereof. Since  $r \in \Delta^* \setminus \hat{I}(p) = \Delta^* \setminus I(p)$ , it follows that  $\hat{\succ}_r = \succ_r$ . Also, since  $r \in \Delta^* \setminus \hat{I}(q)$ , i.e.,  $q \in \Delta^* \setminus \hat{I}(r) = \Delta^* \setminus I(r)$ , it follows that  $\hat{\succ}_q = \succ_q$ . Now consider the case when  $\hat{I}(q) = \Delta$ . If  $I(q) = \Delta$ , then  $\hat{\succ}_q = \succ_q$  follows trivially. If  $I(q) \neq \Delta$ , it holds that  $\Delta^* \setminus I(q) \neq \emptyset$ . For  $r \in \Delta^* \setminus I(q)$ , we have  $\hat{I}(r) \neq \Delta$  which implies  $\hat{\succ}_r = \succ_r$  by the previous case. This implies  $r \in \Delta^* \setminus \hat{I}(q)$ , which contradicts  $\hat{I}(q) = \Delta$ . Hence this case cannot occur.

Lastly, consider  $q \in \Delta \setminus \Delta^*$ . If  $q \in \Delta^+$ , then  $U(q) = \emptyset$  and from before we know that  $\hat{U}(q) \cup \hat{L}(q) \subseteq L(q)$ . If  $\hat{U}(q) \neq \emptyset$ , then  $\hat{U}(q)$  is open and hence intersects with  $\Delta^*$ . For  $r \in \hat{U}(q) \cap \Delta^*$  we know that  $\hat{\succ}_r = \succ_r$ . This means that  $r \in U(q)$  and  $r \in L(q)$ , which is a contradiction. Hence  $\hat{U}(q) = \emptyset$ , which means that  $\hat{\succ}$  is oriented positively at  $q$ . From before it follows that  $\hat{\succ}_q = \succ_q$ . Similarly for  $q \in \Delta^-$ .

Together, we have that  $\hat{\succ}_q = \succ_q$  for all  $q \in \Delta$ , i.e.,  $\hat{\succ} = \succ$ . If  $\hat{\succ}$  is oriented negatively at  $p$ , we get  $\hat{\succ} = \succ^{-1}$  by an analogous argument. □

Lemma 1 is a generalization of Theorem 2 by Fishburn and Gehrlein (1987). The proof only requires continuity and convexity, but not symmetry, of  $\mathcal{R}$ .<sup>14</sup>

<sup>14</sup>Lemma 1 does not hold if convexity is weakened to the assumption that  $U(p)$ ,  $L(p)$ , and  $I(p)$  need to be convex for all  $p \in \Delta$ . To see this, consider the following preference relations on the closed interval  $[0, 1]$  (equipped with the standard topology). Let  $\succsim$  be the greater or equal relation and  $\hat{\succsim}$  be defined such that  $x \hat{\succsim} y$  if  $x \in (3/4, 1]$  and  $y \in [0, 1/4]$  and  $x \sim y$  otherwise. Both,  $\succsim$  and  $\hat{\succsim}$  are continuous and convex according to the weaker convexity assumption defined above. For  $\succsim$  this is clear. To see this for  $\hat{\succsim}$ , observe that for all  $x \in [0, 1]$ , either  $I(x) = [0, 3/4]$  and  $U(x) = (3/4, 1]$  or  $I(x) = [0, 1]$  or  $L(x) = [0, 1/4]$

The next lemma roughly corresponds to what is known as the *field expansion lemma* in traditional proofs of Arrow's theorem (see, e.g. Sen, 1986).<sup>15</sup> Let  $f: \mathcal{D}^N \rightarrow \mathcal{R}$  be an SWF,  $G, H \subseteq N$ , and  $a, b \in U$ . We say that  $(G, H)$  is decisive for  $a$  against  $b$ , denoted by  $a D_{G,H} b$ , if, for all  $R \in \mathcal{D}$ ,  $a \succ_i b$  for all  $i \in G$ ,  $a \sim_i b$  for all  $i \in H$ , and  $b \succ_i a$  for all  $i \in N \setminus (G \cup H)$  implies  $a \succ b$ . The completely indifferent preference relation on  $\Delta_X$  for some  $X \subseteq U$  is denoted by  $\sim^X = \Delta X \times \Delta X$ .

**Lemma 2.** *Let  $m \geq 3$ ,  $f$  be an Arrovian SWF on some domain  $\mathcal{D}$ ,  $G, H \subseteq N$ , and  $a, b \in U$ . Then  $a D_{G,H} b$  implies that  $D_{G,H} = U \times U$ .*

*Proof.* First we show that  $a D_{G,H} x$  and  $b D_{G,H} x$  for all  $x \in U \setminus \{a, b\}$ . To this end, let  $\tilde{\succ}_x \in \mathcal{D}$  such that  $a \succ_x b \succ_x x$  and  $a \succ_x x$  and consider the preference profile

$$R = (\underbrace{\tilde{\succ}_x, \dots, \tilde{\succ}_x}_G, \underbrace{\sim^U, \dots, \sim^U}_H, \tilde{\succ}_x^{-1}, \dots, \tilde{\succ}_x^{-1}).$$

Since  $\tilde{\succ}_x \cap \tilde{\succ}_x^{-1} = \sim_x$ , it follows from the Pareto indifference and Lemma 1 that  $\tilde{\succ} = f(R) \in \{\tilde{\succ}_x, \tilde{\succ}_x^{-1}, \sim^U\}$ . Since  $a D_{G,H} b$ ,  $\tilde{\succ} = \tilde{\succ}_x$  remains as the only possibility. Hence,  $a \succ x$  and  $b \succ x$ . By IIA, it follows that  $a D_{G,H} x$  and  $b D_{G,H} x$ .

Repeated application of the second statement implies that  $D_{G,H}$  is a complete relation. To show that  $D_{G,H}$  is symmetric, let  $x, y, z \in U$  such that  $x D_{G,H} y$ . The first statement implies that  $x D_{G,H} z$ . Two applications of the second statement yield  $z D_{G,H} y$  and  $y D_{G,H} x$ . Hence,  $D_{G,H} = U \times U$ .  $\square$

We first show that Arrovian aggregation is only possible on domains, in which preferences over outcomes are completely determined by preferences over pure outcomes.

**Lemma 3.** *Let  $f$  be an anonymous Arrovian SWF on some domain  $\mathcal{D}$ . Then,  $\tilde{\succ}|_A = \hat{\tilde{\succ}}|_A$  implies  $\tilde{\succ}|_{\Delta A} = \hat{\tilde{\succ}}|_{\Delta A}$  for all  $\tilde{\succ}, \hat{\tilde{\succ}} \in \mathcal{D}$  and  $A \in \mathcal{A}$ .*

*Proof.* Let  $\tilde{\succ}_0, \hat{\tilde{\succ}}_0 \in \mathcal{D}$  and  $A \in \mathcal{A}$  such that  $\tilde{\succ}_0|_A = \hat{\tilde{\succ}}_0|_A$ . Consider the preference profile

$$R = (\tilde{\succ}_0, \hat{\tilde{\succ}}_0^{-1}, \sim^U, \dots, \sim^U).$$

$R \in \mathcal{D}^N$  since  $\mathcal{D}$  satisfies our richness assumptions. Now let  $a, b \in U$  and define  $\bar{R} = R_{(12)}$  be like  $R$  except that the preferences of agents 1 and 2 are exchanged. Anonymity of  $f$  implies that  $\tilde{\succ} = f(\bar{R}) = f(R) = \tilde{\succ}$ . If  $a \succ b$ ,  $(1, N \setminus \{1, 2\})$  is decisive for  $a$  against  $b$ , Lemma 2 implies that  $(1, N \setminus \{1, 2\})$  is also decisive for  $b$  against  $a$ . Hence  $b \succ a$ , this contradicts  $\tilde{\succ} = \tilde{\succ}$ . Thus,  $a \sim b$ . Since  $\tilde{\succ}$  satisfies convexity, we get that  $\sim|_{\Delta A} = \sim^A$ . If  $\tilde{\succ}_0|_{\Delta A} \neq \hat{\tilde{\succ}}_0|_{\Delta A}$ , there exist  $p, q \in \Delta A$  such that  $p \succ_0 q$  and not  $q \hat{\succ}_0 p$ . Hence,  $p \hat{\succ}_0^{-1} q$ . The strict part of Pareto optimality of  $f$  implies that  $p \succ q$ . This contradicts  $\tilde{\succ}|_{\Delta A} = \sim^A$ . Hence,  $\tilde{\succ}_0|_{\Delta A} = \hat{\tilde{\succ}}_0|_{\Delta A}$ .  $\square$

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and  $I(x) = [1/4, 1]$ . In all cases,  $U(x)$  and  $L(x)$  are open and  $U(x)$ ,  $L(x)$ , and  $I(x)$  are convex.

<sup>15</sup>In contrast to Lemma 2, the consequence of the original field expansion lemma uses a stronger notion of decisiveness.

Next, we show that intensities of preferences between pure outcomes are identical.

**Lemma 4.** *Let  $f$  be an Arrovian SWF on some domain  $\mathcal{D}$ . Then, for all  $\succsim_0 \in \mathcal{D}$  and  $a, b, c \in U$  with  $a \succ_0 b$ ,*

(i)  $b \succ_0 c$  implies  $\phi_0(a, b) = \phi_0(b, c)$ ,

(ii)  $a \succ_0 c$  implies  $\phi_0(a, b) = \phi_0(a, c)$ , and

(iii)  $c \succ_0 b$  implies  $\phi_0(a, b) = \phi_0(c, b)$ .

*Proof.* Ad (i): continuity implies that  $b \sim_0 \lambda a + (1 - \lambda)c$  for some  $\lambda \in (0, 1)$ . Observe that  $\succsim_0^{(a,c)}|_{\{a,b,c\}} = \succsim_0^{-1}|_{\{a,b,c\}}$ . Lemma 3 implies that  $\succsim_0^{(a,c)}|_{\Delta(\{a,b,c\})} = \succsim_0^{-1}|_{\Delta(\{a,b,c\})}$ . Since  $\sim_0|_{\Delta(\{a,b,c\})} = \sim_0^{-1}|_{\Delta(\{a,b,c\})}$ , we have  $b \sim_0 (1 - \lambda)a + \lambda c$ . Convexity of  $\succsim_0$  then implies that  $b \sim_0 1/2 a + 1/2 c$ . This is equivalent to  $\phi_0(a, b) = \phi_0(b, c)$ .

Ad (ii): we distinguish two cases.

*Case 1* ( $b \sim_0 c$ ): let  $i, j \in N$  and consider the preference profile

$$R = (\succsim_0, (\succsim_0^{(bc)})^{-1}, \sim^U, \dots, \sim^U).$$

As in the proof of Lemma 3, we get that  $\succsim|_{\{a,b,c\}} = \Delta(\{a, b, c\}) \times \Delta(\{a, b, c\})$ . Without loss of generality, assume that  $\phi_0(a, b) = 1$  and  $\phi_0(a, c) = \lambda$  for some  $\lambda \in (0, 1]$ . Let  $p = 1/2 a + 1/2 b$  and  $q = 1/2 a + 1/2 c$ . Then  $\phi_1(p, q) = \phi_2(p, q) = 1/4(1 - \lambda)$ . If  $\lambda < 1$ , the strict part of Pareto optimality of  $f$  implies that  $p \succ q$ . This contradicts  $\succsim|_{\{a,b,c\}} = \Delta(\{a, b, c\}) \times \Delta(\{a, b, c\})$ . Hence,  $\lambda = 1$ .

*Case 2* ( $b \succ_0 c$ ): assume without loss of generality that  $\phi_0(a, b) = 1$ . By (i), we get  $\phi_0(a, b) = \phi_0(b, c) = 1$ . Our richness assumptions imply that there is  $\hat{\succsim}_0 \in \mathcal{D}$  with  $a \hat{\succsim}_0 b \hat{\succsim}_0 c$ ,  $a \hat{\succsim}_0 c$ , and  $c \hat{\succsim}_0 x$  for some  $x \in U$ . Lemma 3 implies that  $\phi_0|_{\{a,b,c\}} = \hat{\phi}_0|_{\{a,b,c\}}$ . Hence, it suffices to show that  $\hat{\phi}_0(a, c) = 1$ . By (i), we get that  $\hat{\phi}_0(a, c) = \hat{\phi}_0(c, x)$  and  $\hat{\phi}_0(b, c) = \hat{\phi}_0(c, x) = 1$ . Hence,  $\hat{\phi}_0(a, c) = 1$ .

Ad (iii): the proof is analogous to the proof of (ii).  $\square$

**Theorem 1.** *Let  $f$  be an anonymous Arrovian SWF on some domain  $\mathcal{D}$ . Then  $\mathcal{D} \subseteq \Phi^{PC}$ .*

*Proof.* Let  $\succsim_0 \in \mathcal{D}$  and  $a, b, c, d \in U$  such that  $a \succ_0 b$  and  $c \succ_0 d$ . We have to show that  $\phi_0(a, b) = \phi_0(c, d)$ . First assume there are  $x \in \{a, b\}$  and  $y \in \{c, d\}$  such that  $x \succ_0 y$  or  $y \succ_0 x$ . Then, Lemma 4 implies that  $\phi_0(a, b) = \phi_0(x, y) = \phi_0(c, d)$ . Otherwise,  $\sim_0|_{\{a,b,c,d\}} = \sim^{\{a,b,c,d\}}$  and  $\succsim_0|_{\{a,b,c,d\}} = \succsim_0^{(ac)(bd)}|_{\{a,b,c,d\}}$ . Lemma 3 implies that  $\succsim_0|_{\Delta(\{a,b,c,d\})} = \succsim_0^{(ac)(bd)}|_{\Delta(\{a,b,c,d\})}$ . Hence,  $\phi_0|_{\{a,b,c,d\}} = \phi_0^{(ac)(bd)}|_{\{a,b,c,d\}}$  and  $\phi_0(a, b) = \phi_0(c, d)$ .  $\square$



## B. Characterization of the Social Welfare Function

Theorem 1 has established that Arrovian aggregation is only possible if individual preferences are based on pairwise comparisons. In the remainder of this paper, we will—with slight abuse of notation—treat Arrovian SWFs as functions from  $\mathcal{D}^N$  to  $\Phi$  with  $\mathcal{D} \subseteq \Phi^{PC}$ . Since SSB utilities over outcomes are completely determined by SSB utilities over pure outcomes, we will write  $\phi_X$  instead of the more clumsy  $\phi_{\Delta(X)}$  for any SSB utility function  $\phi$  and subset of alternatives  $X \subseteq U$ .

We will only use Pareto optimality when necessary and make use of the weaker condition of Pareto indifference whenever this is possible.

The following lemmas show that for every preference profile  $R$  and all alternatives  $a$  and  $b$ ,  $\phi(a, b)$  only depends on the number of agents who prefer  $a$  to  $b$ , whenever  $R$  is from the domain of  $PC$ -preferences and  $\phi$  represents  $f(R)$ .

We first prove that, if an alternative is strictly Pareto dominated, then the intensities of collective preferences between each of the dominating alternatives and the dominated alternative are identical. (Using a symmetric argument, the same can be shown for profiles in which the Pareto dominance is reversed.)

**Lemma 5.** *Let  $f$  be an Arrovian SWF,  $a, b, c \in U$ , and  $R = (\phi_i)_{i \in N}$  such that  $\phi_i(a, c) = \phi_i(b, c) = 1$ . Then,  $\phi(a, c) = \phi(b, c)$  where  $\phi \equiv f(R)$ .*

*Proof.* The idea of the proof is to introduce a fourth alternative, which serves as a calibration device for the intensity of pairwise comparisons, and eventually disregard this alternative using IIA. To this end, let  $x \in U$  and consider a preference profile  $\hat{R} \in \mathcal{D}^N$  such that  $R|_{\{a,b,c\}} = \hat{R}|_{\{a,b,c\}}$  and

$$\hat{R}|_{\{a,b,c,x\}} = \underbrace{\left( \begin{array}{cccc} 0 & & 1 & 1 \\ & 0 & 1 & 1 \\ -1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{array} \right)}_N, \dots).$$

The values of  $\hat{\phi}_i(a, b)$  for all  $i \in N$  are irrelevant.<sup>16</sup> Let  $\hat{\phi} = f(\hat{R})$ . The Pareto indifference relation with respect to  $\hat{R}|_{\{a,c,x\}}$  is identical to  $\sim_1|_{\{a,c,x\}}$ . The analogous statement holds for the Pareto indifference relation with respect to  $\hat{R}|_{\{b,c,x\}}$ . Hence, the Pareto indifference, Lemma 1, and IIA imply that there are  $\alpha, \beta \in \mathbb{R}$  such that

$$\hat{\phi}|_{\{a,c,x\}} = \alpha \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix} \quad \text{and} \quad \hat{\phi}|_{\{b,c,x\}} = \beta \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}.$$

As a consequence,  $\alpha = \beta$  and  $\hat{\phi}(a, c) = \hat{\phi}(b, c)$ . Since  $R|_{\{a,b,c\}} = \hat{R}|_{\{a,b,c\}}$ , IIA implies that  $\phi|_{\{a,b,c\}} \equiv \hat{\phi}|_{\{a,b,c\}}$ . Hence,  $\phi(a, c) = \phi(b, c)$ .<sup>17</sup>  $\square$

<sup>16</sup>Also the values  $\hat{\phi}_i(x, z)$  for all  $z \in \{a, b, c\}$  are irrelevant as long as they are the same for all agents.

<sup>17</sup>Pareto dominance also implies that  $\phi(a, c), \phi(b, c) > 0$ .

Given preference profile  $R$ , let  $N_{ab} = \{i \in N : a \succ_i b\}$  be the set of agents who strictly prefer  $a$  over  $b$  and  $n_{ab} = |N_{ab}|$ . Also, let  $I_{ab} = N \setminus (N_{ab} \cup N_{ba})$  be the set of agents who are indifferent between  $a$  and  $b$ .

Lemma 6 shows that for a fixed preference profile,  $\phi(a, b)$  only depends on  $N_{ab}$  and  $I_{ab}$  (and not on the names of the alternatives).

**Lemma 6.** *Let  $f$  be an Arrovian SWF,  $a, b, c, d \in U$ ,  $R \in \mathcal{D}^N$ , and  $\pi = (a, c)(b, d) \in \Pi(U)$ . If  $\pi(R|_{\{a,b\}}) = R|_{\{c,d\}}$ , then  $\phi(a, b) = \phi(c, d)$  where  $\phi \equiv f(R)$ .*

*Proof.* We first prove the case when all of  $a, b, c, d$  are distinct. Let  $e \in U$  and consider a preference profile  $\hat{R} \in \mathcal{D}^N$  such that  $R|_{\{a,b,c,d\}} = \hat{R}|_{\{a,b,c,d\}}$  and  $\hat{\phi}_i(x, e) = 1$  for all  $x \in \{a, b, c, d\}$  and  $i \in N$ . Then, by Lemma 5, we can assume without loss of generality that  $\hat{\phi}(x, e) = \lambda \in \mathbb{R}$  for all  $x \in \{a, b, c, d\}$ . Now consider the preference profile

$$\hat{R}|_{\{a,b,c,d,e\}} = \left( \underbrace{\begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 & 1 \\ -1 & -1 & 0 & 1 & 1 \\ -1 & -1 & -1 & 0 & 1 \\ -1 & -1 & -1 & -1 & 0 \end{pmatrix}}_{N_{ab}}, \dots, \underbrace{\begin{pmatrix} 0 & -1 & -1 & -1 & 1 \\ 1 & 0 & -1 & -1 & 1 \\ 1 & 1 & 0 & -1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ -1 & -1 & -1 & -1 & 0 \end{pmatrix}}_{N_{ba}}, \dots, \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & -1 & 0 \end{pmatrix}, \dots \right).$$

Note that  $\hat{R}|_{\{a,b,e\}} = \hat{R}|_{\{a,b,e\}}$  and  $\hat{R}|_{\{c,d,e\}} = \hat{R}|_{\{c,d,e\}}$  because  $\pi(R|_{\{a,b\}}) = R|_{\{c,d\}}$  by assumption. Now, let  $\hat{\phi} \equiv f(\hat{R})$  and  $\check{\phi} \equiv f(\hat{R})$ . Since  $\hat{R}|_{\{a,b,e\}} = \hat{R}|_{\{a,b,e\}}$ , we have  $\hat{\phi}|_{\{a,b,e\}} \equiv \check{\phi}|_{\{a,b,e\}}$  by IIA. Moreover,  $\hat{R}|_{\{c,d,e\}} = \hat{R}|_{\{c,d,e\}}$  and IIA yield  $\hat{\phi}|_{\{c,d,e\}} = \check{\phi}|_{\{c,d,e\}}$ . Lemma 5 implies that  $\check{\phi}(x, e) = \lambda$  for some  $\lambda \in \mathbb{R}$  for all  $x \in \{a, b, c, d\}$ . Thus, for some  $\mu, \sigma \in \mathbb{R}$ ,  $\check{\phi}$  takes the form

$$\check{\phi}|_{\{a,b,c,d,e\}} = \begin{pmatrix} 0 & \mu & & & \lambda \\ -\mu & 0 & & & \lambda \\ & & 0 & \sigma & \lambda \\ & & -\sigma & 0 & \lambda \\ -\lambda & -\lambda & -\lambda & -\lambda & 0 \end{pmatrix}.$$

Now consider the preference profile

$$\bar{R}|_{\{a,b,c,d,e\}} = \left( \underbrace{\begin{pmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ -1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{pmatrix}}_{N_{ab}}, \dots, \underbrace{\begin{pmatrix} 0 & -1 & -1 & -1 \\ 1 & 0 & -1 & -1 \\ 1 & 1 & 0 & -1 \\ 1 & 1 & 1 & 0 \end{pmatrix}}_{N_{ba}}, \dots, 0, \dots \right)$$

and let  $\bar{\phi} \equiv f(\bar{R})$ . The values of  $\bar{\phi}_i(x, e)$  are irrelevant for all  $x \in \{a, b, c, d\}$ . Note that  $\bar{R}|_{\{a,b,c,d\}} = \hat{R}|_{\{a,b,c,d\}}$  and that  $\bar{R}$  only consists of one fixed preference relation, its inverse, and complete indifference. Hence, Pareto indifference and Lemma 1 imply that  $\bar{\phi} = \alpha \phi_i$  for some  $i \in N$  and  $\alpha \in \mathbb{R}$ . Since  $\hat{\phi}|_{\{a,b,c,d\}} = \bar{\phi}|_{\{a,b,c,d\}}$ , we get that  $\mu = \sigma$ .

The cases when  $a = c$  and  $b = d$  follow from repeated application of the above case. All other cases are symmetric to one of the covered cases.  $\square$

**Lemma 7.** *Let  $f$  be an Arrovian SWF,  $a, b, c, d \in U$ ,  $R, \hat{R} \in \mathcal{D}^N$ ,  $\phi \equiv f(R)$ , and  $\hat{\phi} \equiv f(\hat{R})$ . If  $R|_{\{a,b\}} = \hat{R}|_{\{a,b\}}$  and  $R|_{\{c,d\}} = \hat{R}|_{\{c,d\}}$ , then  $\phi(a, b) = \alpha \cdot \hat{\phi}(a, b)$  and  $\phi(c, d) = \alpha \cdot \hat{\phi}(c, d)$  for some  $\alpha > 0$ .*

*Proof.* Let  $e \in U \setminus \{a, b, c, d\}$  and  $R', \hat{R}' \in \mathcal{D}^N$  such that  $R'|_{\{a,b,c,d\}} = R|_{\{a,b,c,d\}}$ ,  $\hat{R}'|_{\{a,b,c,d\}} = \hat{R}|_{\{a,b,c,d\}}$ , and  $\phi'_i(x, e) = \hat{\phi}'_i(x, e) = 1$  for all  $x \in \{a, b, c, d\}$  and  $i \in N$ .  $\phi' = f(R')$  and  $\hat{\phi}' = f(\hat{R}')$  denote the corresponding collective SSB functions. Since  $f$  satisfies IIA, we have  $\phi|_{\{a,b,c,d\}} = \phi'|_{\{a,b,c,d\}}$  and  $\hat{\phi}|_{\{a,b,c,d\}} = \hat{\phi}'|_{\{a,b,c,d\}}$  without loss of generality. Lemma 5 implies that we can assume without loss of generality that  $\phi'$  and  $\hat{\phi}'$  take the following form for some  $\lambda, \mu, \hat{\mu}, \sigma, \hat{\sigma} \in \mathbb{R}$ .

$$\phi'|_{\{a,b,c,d,e\}} = \begin{pmatrix} 0 & \mu & & \lambda \\ -\mu & 0 & & \lambda \\ & & 0 & \sigma & \lambda \\ & & -\sigma & 0 & \lambda \\ -\lambda & -\lambda & -\lambda & -\lambda & 0 \end{pmatrix} \quad \hat{\phi}'|_{\{a,b,c,d,e\}} = \begin{pmatrix} 0 & \hat{\mu} & & \lambda \\ -\hat{\mu} & 0 & & \lambda \\ & & 0 & \hat{\sigma} & \lambda \\ & & -\hat{\sigma} & 0 & \lambda \\ -\lambda & -\lambda & -\lambda & -\lambda & 0 \end{pmatrix}$$

$R'|_{\{a,b,e\}} = \hat{R}'|_{\{a,b,e\}}$  and  $R'|_{\{c,d,e\}} = \hat{R}'|_{\{c,d,e\}}$  by the way they were constructed from  $R$  and  $\hat{R}$ . Since  $f$  satisfies IIA, we get that  $\phi'|_{\{a,b,e\}} \equiv \hat{\phi}'|_{\{a,b,e\}}$  and  $\phi'|_{\{c,d,e\}} \equiv \hat{\phi}'|_{\{c,d,e\}}$ . In particular, this means that  $\mu = \hat{\mu}$  and  $\sigma = \hat{\sigma}$ . The scalar  $\alpha$  disappears by choosing suitable SSB functions representing the collective preferences without loss of generality.  $\square$

Lemma 7 shows that  $\phi(a, b)$  only depends on  $N_{ab}$  and  $I_{ab}$  and not on  $a, b$  and  $R$ . Hence, there is a function  $g: 2^N \times 2^N \rightarrow \mathbb{R}$  such that  $g(N_{ab}, I_{ab}) = \phi(a, b)$  for all  $a, b \in U$  and  $R \in \mathcal{D}^N$ . We now leverage Pareto indifference to show that  $\phi$  is a linear combination of the  $\phi_i$ 's. Hence,  $f$  is affine utilitarian.

**Lemma 8.** *Let  $f$  be an Arrovian SWF. Then, there are  $w_1, \dots, w_n \in \mathbb{R}$  such that  $f(R) \equiv \sum_{i \in N} w_i \phi_i$  for all  $R \in \mathcal{D}^N$ .*

*Proof.* For all  $G \subseteq N$ , let  $w_G = 1/2(g(N, \emptyset) + g(G, \emptyset))$ . For convenience, we write  $w_i$  for  $w_{\{i\}}$ . Since  $\phi(x, y) = g(N_{xy}, I_{xy})$  for all  $x, y \in U$ , it suffices to show that

$$g(N_{xy}, I_{xy}) = \sum_{i \in N} w_i \phi_i(x, y) = \sum_{i \in N_{xy}} w_i - \sum_{i \in N_{yx}} w_i, \quad (1)$$

for all  $x, y \in U$ . To this end, we will first show that  $w_G + w_{\hat{G}} = w_{G \cup \hat{G}}$  for all  $G, \hat{G} \subseteq N$  with  $G \cap \hat{G} = \emptyset$ . Let  $G, \hat{G}$  as above,  $a, b, c, x, y \in U$ , and consider the following preference profile  $R \in \mathcal{D}^N$  with  $\phi \equiv f(R)$ .

$$R|_{\{a,b,c,x,y\}} = \left( \underbrace{\begin{pmatrix} 0 & & -1 & 1 \\ & 0 & -1 & 1 \\ & & 0 & -1 & 1 \\ 1 & 1 & 1 & 0 \\ -1 & -1 & -1 & & 0 \end{pmatrix}}_G, \dots, \underbrace{\begin{pmatrix} 0 & & -1 & -1 \\ & 0 & 1 & 1 \\ & & 0 & 1 & -1 \\ 1 & -1 & -1 & 0 \\ 1 & -1 & 1 & & 0 \end{pmatrix}}_{\hat{G}}, \dots, \begin{pmatrix} 0 & & -1 & 1 \\ & 0 & -1 & -1 \\ & & 0 & 1 & 1 \\ 1 & 1 & -1 & 0 \\ -1 & 1 & -1 & & 0 \end{pmatrix}, \dots \right)$$

We have that, for  $p = 1/2x + 1/2y$  and  $q = 1/3a + 1/3b + 1/3c$ ,  $\phi_i(p, q) = 0$  for all  $i \in N$ . Pareto indifference implies that  $\phi(p, q) = 0$ . Let  $\mu = g(G, \emptyset)$ ,  $\hat{\mu} = g(\hat{G}, \emptyset)$ , and  $\sigma = g(G \cup \hat{G}, \emptyset)$ . By definition of  $w$ ,

$$w_G + w_{\hat{G}} = w_{G \cup \hat{G}}$$

is equivalent to

$$(g(N, \emptyset) + g(G, \emptyset)) + (g(N, \emptyset) + g(\hat{G}, \emptyset)) = g(N, \emptyset) + g(G \cup \hat{G}, \emptyset).$$

Hence, we have to show that  $\mu + \hat{\mu} + g(N, \emptyset) = \sigma$ . By definition of  $g$ , we get that  $\phi$  takes the following form.

$$\phi|_{\{a,b,c,x,y\}} = \begin{pmatrix} 0 & & -g(N, \emptyset) & -\hat{\mu} \\ & 0 & \hat{\mu} & \sigma \\ & & 0 & -\mu & -\hat{\mu} \\ g(N, \emptyset) & -\hat{\mu} & \mu & 0 \\ \hat{\mu} & -\sigma & \hat{\mu} & & 0 \end{pmatrix}$$

From  $\phi(p, q) = 0$ , it follows that  $1/6(\mu + \hat{\mu} + g(N, \emptyset) - \sigma) = 0$ . This proves the desired relationship.

Now we can rewrite (1) to

$$g(N_{xy}, I_{xy}) = w(N_{xy}) - w(N_{yx}). \quad (2)$$

By definition of  $w$ , this is equivalent to

$$2g(N_{xy}, I_{xy}) = g(N_{xy}, \emptyset) - g(N_{yx}, \emptyset). \quad (3)$$

To prove (3), let  $a, b, x, y \in U$  and consider the following preference profile  $\hat{R} \in \mathcal{D}^N$  with  $\hat{\phi} \equiv f(\hat{R})$ .

$$\hat{R}|_{\{a,b,x,y\}} = \left( \underbrace{\begin{pmatrix} 0 & 1 & 1 \\ & 0 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 1 & & 0 \end{pmatrix}}_G, \dots, \underbrace{\begin{pmatrix} 0 & -1 & 1 \\ & 0 & -1 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & & 0 \end{pmatrix}}_{\hat{G}}, \dots, \begin{pmatrix} 0 & -1 & -1 \\ & 0 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & -1 & & 0 \end{pmatrix}, \dots \right)$$

Observe that, for  $p = 1/3x + 2/3y$  and  $q = 1/2a + 1/2b$ ,  $\hat{\phi}_i(p, q) = 0$  for all  $i \in N$ . Pareto indifference implies that  $\hat{\phi}(p, q) = 0$ . With the same definitions as before and  $\epsilon = g(G, \hat{G})$ ,  $\hat{\phi}$  takes the following form.

$$\hat{\phi}|_{\{a,b,x,y\}} = \begin{pmatrix} 0 & \mu & \sigma \\ & 0 & -\sigma & -\epsilon \\ -\mu & \sigma & 0 \\ -\sigma & \epsilon & & 0 \end{pmatrix}$$

From  $\hat{\phi}(p, q) = 0$ , we get that  $1/6(-\mu + \sigma - 2\sigma + 2\epsilon) = 0$ . Hence,  $2\epsilon = \mu + \sigma$ . This is equivalent to

$$2g(G, \hat{G}) = g(G, \emptyset) + g(G \cup \hat{G}, \emptyset) = g(G, \emptyset) - g(N \setminus (G \cup \hat{G}), \emptyset),$$

where the last equality follows from skew-symmetry of  $\hat{\phi}$  and the definition of  $g$ . This proves (3).  $\square$

Finally, the strict part of Pareto optimality implies that individual weights have to be strictly positive.

**Theorem 2.** *Let  $f$  be an Arrovian SWF. Then, there are  $w_1, \dots, w_n \in \mathbb{R}_{>0}$  such that  $f(R) \equiv \sum_{i \in N} w_i \phi_i$  for all  $R \in \mathcal{D}^N$ .*

*Proof.* From Lemma 8 we know that there are  $w_1, \dots, w_n \in \mathbb{R}$  such that, for all  $R \in \mathcal{R}^N$ ,  $\phi = \sum_{i \in N} w_i \phi_i \equiv f(R)$ . Assume for contradiction that  $w_i \leq 0$  for some  $i \in N$ . Let  $G$  be

the set of agents such that  $w_i \leq 0$  and consider a preference profile  $R \in \mathcal{D}^N$  with  $a, b, c \in U$  such that

$$R|_{\{a,b,c\}} = \underbrace{\left( \begin{pmatrix} 0 & & 1 \\ & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & & 1 \\ & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}, \dots \right)}_G.$$

Then, for  $p = 1/2 a + 1/2 b$ , we have that  $\phi_i(p, c) > 0$  for all  $i \in G$  and  $\phi_i(p, c) = 0$  for all  $i \in N \setminus G$ . Pareto optimality of  $f$  implies that  $\phi(p, c) > 0$ . However, we have

$$\phi(p, c) = \sum_{i \in G} w_i \phi_i(p, c) + \sum_{i \in N \setminus G} \underbrace{w_i \phi_i(p, c)}_{=0} = \sum_{i \in G} \underbrace{w_i \phi_i(p, c)}_{\leq 0} \leq 0,$$

which is a contradiction. □