Arrovian Aggregation of Convex Preferences and Pairwise Utilitarianism

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We consider social welfare functions that satisfy Arrow's classic axioms of *independence of irrelevant alternatives* and *Pareto optimality* when individual and collective preferences are continuous and convex. These assumptions are sufficient for the existence of maximal elements and the choice consistency of functions that return these elements. We provide characterizations of both the domains of preferences and the social welfare functions that allow for Arrovian aggregation. The domains allow for arbitrary preferences over pure outcomes, which in turn completely determine an agent's preferences over all remaining outcomes. The only Arrovian social welfare functions on these domains constitute an intriguing combination of utilitarianism and pairwiseness. When also assuming anonymity, Arrow's impossibility turns into a complete characterization of a unique desirable social welfare function.

1 Introduction

A central concept in welfare economics are social welfare functions (SWFs) in the tradition of Arrow, i.e., functions that map a collection of individual preference relations over some set of alternatives to a social preference relation over the alternatives. Arrow's seminal theorem states that every SWF that satisfies Pareto optimality and independence of irrelevant alternatives is dictatorial (Arrow, 1951). This sweeping impossibility significantly strengthened an observation by Condorcet (1785) and sent shockwaves throughout economics as well as political philosophy and political theory (see, e.g., Maskin and Sen, 2014). A large body of subsequent work has studied whether more positive results can be obtained by modifying implicit assumptions on the domain of admissible preferences, both individually and collectively. These approaches can be roughly divided into two categories.

For one, as pioneered by Sen (1969), the assumption of *collective* transitivity has been weakened to quasi-transitivity, acyclicity, path independence or similar conditions. Al-though this does allow for non-dictatorial aggregation functions that meet Arrow's criteria, these functions turned out to be highly objectionable, usually on grounds of involving a weak kind of dictatorship or violating other conditions deemed to be indispensable for reasonable preference aggregation (for an overview of the extensive literature, see Kelly, 1978; Sen, 1977, 1986; Schwartz, 1986; Campbell and Kelly, 2002). Particularly noteworthy are results about acyclic collective preference relations (e.g., Mas-Colell and Sonnenschein, 1972; Brown, 1975; Blau and Deb, 1977; Banks, 1995) because acyclicity is necessary and sufficient for the existence of maximal elements when there is a finite number of alternatives. Sen (1995) concludes that "the arbitrariness of power of which Arrow's case of dictatorship is an extreme example, lingers in one form or another even when transitivity is dropped, so long as *some* regularity is demanded (such as the absence of cycles)."

Another stream of research has analyzed the implications of imposing structure on the *individual* preferences. This has resulted in a number of positive results for restricted domains, such as dichotomous or single-peaked preferences, which allow for attractive SWFs (e.g., Black, 1948; Arrow, 1951; Sen and Pattanaik, 1969; Ehlers and Storcken, 2008). Many economic domains are concerned with an infinite set of outcomes, which satisfies structural restrictions such as compactness and convexity. Preferences over these outcomes are typically assumed to satisfy some form of continuity and convexity, which roughly imply that preferences are robust with respect to minimal changes in outcomes and with respect to convex combinations of outcomes. Various results have shown that Arrow's impossibility remains intact under these assumptions (e.g., Kalai et al., 1979; Border, 1983; Bordes and Le Breton, 1989, 1990a,b; Campbell, 1989; Redekop, 1995). Le Breton and Weymark (2011) provide an overview and conclude that "economic domain restrictions do not provide a satisfactory way of avoiding Arrovian social choice impossibilities, except when the set of alternatives is one-dimensional and preferences are single-peaked."

The point of departure for the present approach is the observation that, to the best of our knowledge, all impossibilities require some form of transitivity (e.g., acyclicity), even though no such assumption is necessary to guarantee the existence of maximal elements in continuous and convex domains. Sonnenschein (1971) has shown that all continuous and convex preference relations admit a maximal element in every non-empty, compact, and convex set of outcomes. Moreover, returning maximal elements under the given conditions satisfies standard properties of choice consistency introduced by Sen (1969, 1971). Hence, there is little justification for demanding transitivity, which has come under independent attack in normative and descriptive decision theory (see, e.g., May, 1954; Fishburn, 1970; Bar-Hillel and Margalit, 1988; Fishburn, 1991; Anand, 1993, 2009).[1] As Anand (2009) writes, "once considered a cornerstone of rational choice theory, the status of transitivity has been dramatically reevaluated by economists and philosophers in recent years."

We show that, not only does Arrow's theorem cease to hold on convex domains when

[1]For example, the preference reversal phenomenon, where lottery p is preferred to lottery q but the certainty equivalent of p is less preferred than the certainty equivalent of q, shows experimental failures of transitivity (see, e.g., Grether and Plott, 1979).

dispensing with transitivity, but, moreover, Arrow's axioms and some weak technical assumptions narrow down the choice of a suitable SWF to an intriguing combination of pairwiseness and utilitarianism.

More precisely, we consider a convex set of outcomes consisting of all probability measures with finite support on some abstract set of alternatives. Examples of outcome sets are shares of divisible public goods, lotteries, time shares, monetary shares, etc. Individual and collective preference relations over these outcomes are assumed to satisfy continuity, convexity, and symmetry. We then show that there is a unique inclusion-maximal Cartesian domain of preference profiles that allows for Arrovian aggregation while satisfying minimal richness conditions. This domain allows for arbitrary preferences over pure outcomes, which in turn completely determine an agent's preferences over all remaining outcomes. When interpreting outcomes as lotteries, this preference extension has a particularly simple and intuitive explanation: an agent prefers one lottery to another if and only if the former is more likely to return a more preferred pure outcome. Incidentally, this preference extension, which constitutes a central special case of skew-symmetric bilinear (SSB) utility functions as introduced by Fishburn (1982), has been backed by recent experimental evidence (see Blavatskyy, 2006).

We then prove that the only Arrovian SWFs on this domain are affine welfare maximizers for the underlying SSB utility functions. As a consequence, there is a unique anonymous Arrovian SWF, which compares outcomes by the sign of the bilinear form given by the pairwise majority margins. The resulting collective preference relation over *pure* outcomes coincides with majority rule and the corresponding choice function is therefore consistent with Condorcet's principle of always returning a pure outcome that is majority-preferred to every other pure outcome.[2] This relation is naturally extended to mixed outcomes such that, by the Minimax Theorem, every compact and convex set of outcomes admits a collectively most preferred outcome.

Our results challenge the traditional—transitive—way of thinking about preferences, which has been largely influenced by the pervasiveness of scores and grades. A compelling opinion on transitivity is expressed in the following apt quote by decision theorist Peter C. Fishburn: "Transitivity is obviously a great practical convenience and a nice thing to have for mathematical purposes, but long ago this author ceased to understand why it should be a cornerstone of normative decision theory. [...] The presence of intransitive preferences complicates matters [...] however, it is not cause enough to reject intransitivity. An analogous rejection of non-Euclidean geometry in physics would have kept the familiar and simpler Newtonian mechanics in place, but that was not to be. Indeed, intransitivity challenges us to consider more flexible models that retain as much simplicity and elegance as circumstances allow. It challenges old ways of analyzing decisions and suggests new possibilities" (Fishburn, 1991).

[2] It is therefore in line with Dasgupta and Maskin (2008) who, also based on Arrow's axioms, have forcefully argued in favor of majority rule in domains where Condorcet winners are guaranteed to exist. Our arguments extend to unrestricted preferences over pure outcomes.

2 Related Work

A special case of our setting, which has been particularly well studied, concerns sets of outcomes that consist of all lotteries over some finite set of alternatives and individual preferences over lotteries that satisfy the *von Neumann-Morgenstern axioms*, i.e., preferences over lotteries that can be represented by assigning cardinal utilities to alternatives such that lotteries are compared based on expected utility. Samuelson (1967) conjectured that Arrow's impossibility still holds under these assumptions and Kalai and Schmeidler (1977) showed that this is indeed the case when there are at least four alternatives. There are various versions of this statement which differ in modeling assumptions and whether SWFs aggregate cardinal utilities or the preference relations represented by these utilities (Sen, 1970; Hylland, 1980; Dhillon and Mertens, 1997; d'Aspremont and Gevers, 2002). The one closest to the framework of this paper is Theorem 4.3 by d'Aspremont and Gevers (2002).

Our results apply to Arrovian aggregation of preferences over lotteries under much loosened assumptions about preferences over lotteries. In particular, the axioms we presume entail that preferences over lotteries can be represented by *skew-symmetric bilinear (SSB) utility functions*, which assign a utility value to each pair of lotteries. One lottery is preferred to another lottery if the SSB utility for this pair is positive. SSB utility theory is a generalization of linear expected utility theory due to von Neumann and Morgenstern (1947), which does not require the controversial independence axiom and transitivity (see, e.g., Fishburn, 1982, 1984b, 1988). Independence requires that if lottery p is preferred to lottery q, then a coin toss between p and a third lottery r is preferred to a coin toss between q and r (with the same coin used in both cases). There is experimental evidence that independence is systematically violated by human decision makers. The Allais Paradox (Allais, 1953) is perhaps the most famous example. Detailed reviews of such violations, including those reported by Kahnemann and Tversky (1979), have been provided by Machina (1983, 1989) and McClennen (1988).[3]

Our characterization of Arrovian SWFs is related to Harsanyi's *Social Aggregation Theorem* (Harsanyi, 1955), which shows that, for von Neumann-Morgenstern preferences over lotteries, weighted welfare maximization already follows from Pareto indifference. However, Harsanyi's theorem is a statement about a single preference profile considered in isolation. The weights given to the agents may depend on their preferences. This can be prevented by adding axioms that connect the collective preferences across different profiles. The SWF that derives the collective preferences by adding up the normalized utility representations is known as *relative utilitarianism* (Dhillon, 1998; Dhillon and Mertens, 1999; Börgers and Choo, 2015, 2017). It was characterized by Dhillon and Mertens (1999) using essentially independence of redundant alternatives (a weakening of independence of irrelevant alternatives) and monotonicity (a weakening of a Pareto-type axiom). As shown by Fishburn and Gehrlein (1987) and further explored by Turunen-Red and Weymark

[3]Fishburn and Wakker (1995) give an interesting historical perspective on the independence axiom.

(1999), aggregating SSB utility functions is fundamentally different from aggregating von Neumann-Morgenstern utility functions in that Harsanyi's Pareto axiom does not imply weighted welfare maximization. As we show in this paper, this can be rectified by considering a multi-profile framework and independence of irrelevant alternatives.

The probabilistic voting rule that returns the maximal elements of the unique anonymous Arrovian SWF is known as *maximal lotteries* (Fishburn, 1984a) and was recently axiomatized using two consistency conditions (Brandl et al., 2016b). Interestingly, this function and some variations of it were repeatedly reinvented (see, e.g., Kreweras, 1965; Felsenthal and Machover, 1992; Laffond et al., 1993; Rivest and Shen, 2010).

3 Preliminaries

Let U be a non-empty universal set of alternatives. By Δ we denote the set of all probability measures with finite support on U. We require all one-element sets to be measurable. Hence, for $p \in \Delta$, $p(a) = 0$ for all but finitely many $a \in U$. For $X \subseteq U$, let Δ_X be the set of probability measures in Δ with support in X, i.e., $\Delta_X = \{p \in \Delta \colon p(X) = 1\}$. We will refer to elements of Δ as outcomes and one-point measures in Δ as pure outcomes. Furthermore, let $\succ$ be an asymmetric binary relation over Δ, which is interpreted as the preference relation of an agent. Given two outcomes $p, q \in \Delta$, we write $p \sim q$ when neither $p \succ q$ nor $q \succ p$, and $p \succsim q$ if $p \succ q$ or $p \sim q$. For $p \in \Delta$, let $U(p) = \{q \in \Delta \colon q \succ p\}$ and $L(p) = \{q \in \Delta \colon p \succ q\}$ be the strict upper and strict lower contour set of p; $I(p) = \{q \in \Delta \colon p \sim q\}$ denotes the indifference set of p. For $X \subseteq \Delta$, $\succ|_X = \{(p, q) \in \succ \colon p, q \in X\}$ is the preference relation $\succ$ restricted to outcomes in X.

We will consider preference relations that are continuous, i.e., small changes in probabilities do not result in a reversal of preference. One of several possibilities to define continuity is the Archimedean axiom, which requires that, for any given outcome p, the convex hull of a more preferred outcome and a less preferred outcome also contains an equally preferred outcome. A preference relation $\succ$ is continuous if, for all $p, q, r \in \Delta$,

$$p \succ q \succ r \text{ implies } \lambda p + (1 - \lambda) r \sim q \text{ for some } \lambda \in (0, 1). \qquad \text{(Continuity)}$$

Another standard assumption is that preferences are convex. We will use convexity as defined by Fishburn (1982). $\succ$ is convex if, for all $p, q, r \in \Delta$ and $\lambda \in (0, 1)$,

$$\begin{aligned} &p \succ q \text{ and } p \succsim r \text{ imply } p \succ \lambda q + (1 - \lambda) r, \\ &q \succ p \text{ and } r \succsim p \text{ imply } \lambda q + (1 - \lambda) r \succ p, \text{ and} \\ &p \sim q \text{ and } p \sim r \text{ imply } p \sim \lambda q + (1 - \lambda) r. \end{aligned} \qquad \text{(Convexity)}$$

Equivalently, one could require that the indifference set for an outcome p is a hyperplane through p; the upper and lower contour sets are the corresponding half spaces. Note that convexity implies that upper contour sets, lower contour sets, and indifference sets are convex. Moreover, upper contour and lower contour sets are either open or empty and indifference sets are closed.

The existence of maximal elements is usually quoted as the main reason for insisting on transitivity of preference relations. It was shown by Sonnenschein (1971) that continuity and convexity are already sufficient for the existence of maximal elements, even when preferences are intransitive (see also Bergstrom, 1992; Llinares, 1998).[4]

Proposition 1. *(Sonnenschein, 1971) If $\succ$ is a continuous and convex preference relation, then $\max_\succ(X) \neq \emptyset$ for all non-empty, compact, and convex sets $X \subseteq \Delta$.*

Sen (1969, 1971) has shown that two intuitive choice consistency conditions are equivalent to choosing maximal elements according to an acyclic relation. These conditions are known as *Sen's α* (or *contraction*) and *Sen's γ* (or *expansion*). We will show that choosing maximal elements according to convex relation implies contraction and expansion, even in the absence of acyclicity. For some set of outcomes X, the convex hull $\text{conv}(X)$ is the smallest convex set that contains X. A choice function maps every feasible (i.e., compact and convex) set out outcomes to a subset thereof. Contraction requires that if some outcome is chosen from some feasible set, then it is also chosen from any feasible subset that it is contained in. A choice function f satisfies contraction if for all compact and convex $X, Y \subseteq \Delta$,

$$Y \subseteq X \text{ implies } f(X) \cap Y \subseteq f(Y). \tag{Contraction}$$

Expansion prescribes that an outcome that is chosen from two feasible sets, should also be chosen for the convex hull of their union. f satisfies expansion if for all compact and convex $X, Y \subseteq \Delta$,

$$f(X) \cap f(Y) \subseteq f(\text{conv}(X \cup Y)). \tag{Expansion}$$

Our goal is to link the above choice consistency conditions to choosing the maximal elements of some relation. We show that if a choice function chooses the maximal elements of a convex relation, then it satisfies contraction and expansion. The converse holds if convexity is weakened to only require that the weak upper and lower contour sets are convex. Convexity is only needed for the expansion part.

Proposition 2. *Let $\succ$ be a convex preference relation. Then $\max_\succ$ satisfies contraction and expansion for compact and convex subsets of Δ.*

Proof. First we show contraction. Let $X, Y \subseteq \Delta$ be compact and convex with $Y \subseteq X$ and $p \in \max_\succ(X) \cap Y$. This implies that $p \in \max_\succ(Y)$.

Second we show expansion. Let $X, Y \subseteq \Delta$ be compact and convex and $p \in \max_\succ X \cap \max_\succ Y$. Then, $p \succsim q$ for all $q \in X \cup Y$. Since $\succ$ satisfies convexity, we have $p \succsim q$ for all $q \in \text{conv}(X \cup Y)$. Thus, $p \in \max_\succ(\text{conv}(X \cup Y))$. □

Convexity implies that indifference curves are straight lines. The symmetry condition introduced by Fishburn (1982) prescribes that either all indifference curves are parallel or

[4] Sonnenschein (1971) only required lower contour sets to be convex.

meet at one point (which may be outside of Δ). For all $p, q, r \in \Delta$ and $\lambda \in (0,1)$,

$$\begin{gathered}\text{if } p \succ q \succ r,\ p \succ r, \text{ and } q \sim 1/2\, p + 1/2\, r, \text{ then}\\ [\lambda p + (1-\lambda) r \sim 1/2\, p + 1/2\, q \text{ if and only if } \lambda r + (1-\lambda) p \sim 1/2\, r + 1/2\, q].\end{gathered} \tag{Symmetry}$$

Fishburn (1984b) justifies this axiom by stating that "the degree to which p is preferred to q is equal in absolute magnitude but opposite in sign to the degree to which q is preferred p." He continues by writing that he is "a bit uncertain as to whether this should be regarded more as a convention than a testable hypothesis – much like the asymmetry axiom [...], which can almost be thought of as a definitional characteristic of strict preference."

By $\mathcal{R}$ we denote the set of all continuous, convex, and symmetric preference relations. Despite the richness of $\mathcal{R}$, preference relations therein admit a particularly nice representation. It was shown by Fishburn (1982) that if $\succ \in \mathcal{R}$, then there is a skew-symmetric and bilinear (SSB) utility function $\phi\colon \Delta \times \Delta \to \mathbb{R}$ such that, for all $p, q \in \Delta$,[5]

$$p \succ q \text{ if and only if } \phi(p,q) > 0.$$

Moreover, ϕ is unique up to scalar multiplication. We denote by Φ the set of all SSB functions on $\Delta \times \Delta$. For outcomes with finite support, $\phi(p,q)$ can be written as a convex combination of the values of ϕ for one-point measures (Fishburn, 1984b). For this purpose, we identify every alternative $a \in \mathcal{U}$ with the one-point measure that puts probability 1 on a. Then, for all $p, q \in \Delta$,

$$\phi(p,q) = \sum_{a,b \in \mathcal{U}} p(a) q(b) \phi(a,b).$$

We will oftentimes represent SSB functions restricted to Δ_X for finite X as skew-symmetric matrices in $\mathbb{R}^{X \times X}$.

When requiring transitivity on top of continuity, convexity, and symmetry, the four axioms characterize *weighted linear (WL)* utility functions as introduced by Chew (1983).[6] When additionally requiring independence, then ϕ is separable, i.e., $\phi(p,q) = u(p) - u(q)$, where u is a linear von Neumann-Morgenstern utility function representing $\succ$. For independently distributed outcomes (as considered in this paper), SSB utility theory coincides with regret theory as introduced by Loomes and Sugden (1982) (see also Loomes and Sugden, 1987; Blavatskyy, 2006).

Through the representation of $\succ \in \mathcal{R}$ restricted to a finite $X \subseteq U$ by a skew-symmetric matrix, it becomes apparent that the Minimax Theorem implies the existence of maximal elements of $\succ$ on Δ_X. This was noted by Fishburn (1984b, Theorem 4) and already follows from Proposition 1. Fishburn (1984b) goes on to show that choosing maximal elements of

[5] A function ϕ is skew-symmetric if $\phi(p,q) = -\phi(q,p)$ for all $p, q \in \Delta$. ϕ is bilinear if it is linear in both arguments.

[6] A WL function is characterized by a linear utility function and a non-vanishing weight function. The utility of an outcome is the utility derived by the linear utility function weighted according to the weight function. Thus, WL functions are more general than linear utility functions, as every linear utility function is equivalent to a WL function with constant weight function. See also Fishburn (1983).

$\succ$ from feasible sets satisfies contraction and expansion, which follows from Proposition 2 because relations in $\mathcal{R}$ satisfy convexity by definition.[7]

4 Social welfare functions

For the remainder of the paper we deal with the problem of aggregating the preferences of multiple agents into a collective preference relation. The set of agents is $N = \{1, \ldots, n\}$ for some $n \geq 2$. The preference relations of agents belong to some *domain* $\mathcal{D} \subseteq \mathcal{R}$. A function from the set of agents to the domain $R \in \mathcal{D}^N$ is a preference profile. We will write preference profiles as tuples with indices in N. A *social welfare function (SWF)* $f \colon \mathcal{D}^N \to \mathcal{R}$ maps a preference profile to a collective preference relation.

Arrow's impossibility theorem shows that the only SWFs that satisfy two desirable properties, Pareto optimality and independence of irrelevant alternatives, are dictatorial functions. Pareto optimality prescribes that a unanimous preference of one outcome over another in the individual preferences should be reflected in the collective preference. An SWF f is *Pareto optimal* if, for all $p, q \in \Delta$, $R \in \mathcal{D}^N$, and $f(R) = \succ$,

$$\begin{aligned}&p \succsim_i q \text{ for all } i \in N \text{ implies } p \succsim q, \text{ and}\\&\text{if additionally } p \succ_i q \text{ for some } i \in N \text{ then } p \succ q.\end{aligned} \tag{Pareto optimality}$$

Independence of irrelevant alternatives requires that collective preferences over some feasible set of outcomes should only depend on the individual preferences over this set (and not on the preferences over outcomes outside this set). Together with transitivity, it is the driving force in Arrow's theorem and has much stronger implications than apparent at first sight. In our framework, we will assume that feasible sets are based on the availability of alternatives and are therefore of the form Δ_X for $X \subseteq U$. Formally, we say that an SWF f satisfies *independence of irrelevant alternatives (IIA)* if, for all $R, \hat{R} \in \mathcal{D}^N$ and $X \subseteq U$,

$$R|_{\Delta_X} = \hat{R}|_{\Delta_X} \text{ implies } f(R)|_{\Delta_X} = f(\hat{R})|_{\Delta_X}. \tag{IIA}$$

Any SWF that satisfies Pareto optimality and IIA will be called an *Arrovian* SWF. Arrow has shown that, when no structure is imposed on preference relations and feasible sets, every Arrovian SWF is dictatorial, i.e., the preference relation of one fixed agent is a sub-relation of the collective preference relation (formally, there is $i \in N$ such that for all $R \in \mathcal{R}^N$, $\succ_i \subseteq f(R)$). Dictatorships are examples of SWFs that are extremely biased towards one agent. In many applications, *any* differentiation between agents is unacceptable and all agents should be treated equally. This property is known as anonymity. We denote by Π_N the set of all permutations on N. For $\pi \in \Pi_N$ and a preference profile $R \in \mathcal{D}^N$, $R^\pi = R \circ \pi$ is the preference profile where agents are renamed according to π. Then, an SWF f is *anonymous* if for all $R \in \mathcal{D}^N$ and $\pi \in \Pi_N$,

$$f(R) = f(R^\pi). \tag{Anonymity}$$

[7]Fishburn (1984b) defines expansion without taking the convex closure of the union of two feasible sets.

Anonymity is clearly a stronger requirement than non-dictatorship.

In order to prove our characterization, we need to assume that any domain $\mathcal{D} \subseteq \mathcal{R}$ satisfies certain richness conditions. We denote the completely indifferent preference relation on Δ_X for some $X \subseteq U$ by $\sim^X = \Delta X \times \Delta X$ and require that $\sim^U \in \mathcal{D}$. This allows agents to express complete indifference. Second, we require that the domain is neutral in the sense that it is not biased towards certain alternatives. For $\pi \in \Pi_U$ and $p \in \Delta$, let $p^\pi \in \Delta$ such that $p^\pi(\pi(a)) = p(a)$ for all $a \in U$. Then, for $\succ \in \mathcal{R}$, we define $\succ^\pi$ such that $p^\pi \succ^\pi q^\pi$ if and only if $p \succ q$ for all $p, q \in \Delta$. It is assumed that $\succ \in \mathcal{D}$ if and only if $\succ^\pi \in \mathcal{D}$ for all $\succ \in \mathcal{D}$ and $\pi \in \Pi_U$. It should also be possible for agents to declare completely opposed preferences. For $\succ \in \mathcal{D}$, $\succ^{-1}$ is the inverse of $\succ$, i.e., $p \succ^{-1} q$ if and only if $q \succ p$ for all $p, q \in \Delta$. Then $\succ \in \mathcal{D}$ implies $\succ^{-1} \in \mathcal{D}$ for all $\succ \in \mathcal{R}$. Note that this condition is not implied by the previous neutrality condition because not only the preferences over alternatives, but also the preferences over outcomes are inverted. Finally, we demand that $\mathcal{D}$ contains a preference relation with (at least) four linearly ordered alternatives.[8]

5 Characterization of the Domain

We characterize the largest domain $\mathcal{D} \subseteq \mathcal{R}$ for which an anonymous Arrovian SWF exists. It turns out that preferences within this domain have been studied before and have a natural interpretation. $\phi \in \Phi$ is based on *pairwise comparisons* if $\phi(a, b) \in \{-1, 0, 1\}$ for all $a, b \in U$. By $\Phi^{PC} \subset \Phi$ we denote the set of SSB functions that are based on pairwise comparisons. Hence, if $\succ$ is based on pairwise comparisons and outcomes are interpreted as lotteries, one outcome is preferred to another if and only if the former is more likely to return a more preferred pure outcome. These preferences over outcomes are quite natural and can be seen as the canonical SSB representation consistent with a given ordinal preference relation over alternatives (see Blavatskyy (2006) for an axiomatic characterization and Aziz et al. (2015, 2016); Brandl et al. (2016a) for an investigation of efficiency, strategyproofness, and related properties with respect to such preference relations). Figure 1 illustrates preferences based on pairwise comparisons for three transitively ordered alternatives.

The proof starts by showing that continuous and convex preference relations are completely determined by their symmetric part up to orientation.

Lemma 1. *Let $\succ, \hat{\succ}$ be continuous and convex preference relations such that $\sim \subseteq \hat{\sim}$. Then, $\hat{\succ} \in \{\succ, \succ^{-1}, \emptyset\}$.*

Proof. We first show an auxiliary statement: if $\succ$ is continuous and convex and $p \in \Delta$ such that $I(p)$ contains an open set, then $I(p) = \Delta$. Assume for contradiction that $I(p) \neq \Delta$ or, equivalently, $U(p) \cup L(p) \neq \emptyset$ and let $q \in I(p)$ such that a neighborhood of q is contained in $I(p)$. Consider the case when $U(p) \neq \emptyset$ and let $r \in U(p)$. Then convexity implies that

[8]To derive the conclusion of Theorem 1, a weaker condition suffices: if $\succ \in \mathcal{D}$ with $a \succ b \succ c$ and $a \succ c$ for some $a, b, c \in U$, then there is some $\succ' \in \mathcal{D}$ with $a \succ' b \succ' c \succ' x$ and $a \succ' c$ for some $x \in U$.

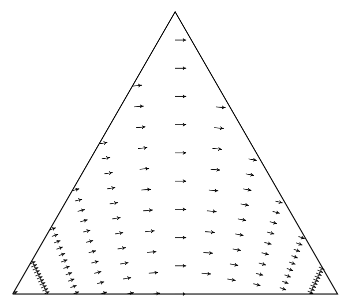

Figure 1: Illustration of the preferences derived from pairwise comparison when $a \succ b \succ c$ and $a \succ c$. The arrows represent normal vectors to the indifference curves which separate the corresponding upper and lower contour set (pointing towards the lower contour set).

$(1-\lambda)q + \lambda r \in U(p)$ for all $\lambda \in (0,1)$. This contradicts the fact that a neighborhood of q is contained in $I(p)$. The case $L(p) \neq \emptyset$ is symmetric.

Now let $\succ, \hat{\succ}$ be continuous and convex such that $\sim \subseteq \hat{\sim}$. Let $p \in \Delta$. By assumption, we have $I(p) \subseteq \hat{I}(p)$. Moreover, Δ is the disjoint union of $I(p)$, $U(p)$, $L(p)$ and $\hat{I}(p)$, $\hat{U}(p)$, $\hat{L}(p)$, respectively. This implies that $\hat{U}(p) \cup \hat{L}(p) \subseteq U(p) \cup L(p)$. Assume for contradiction that $\hat{U}(p) \cap U(p) \neq \emptyset$ and $\hat{U}(p) \cap L(p) \neq \emptyset$. Let $q \in \hat{U}(p) \cap U(p)$ and $r \in \hat{U}(p) \cap L(p)$. Continuity implies that $\mathrm{conv}(\{q,r\}) \cap I(p) \neq \emptyset$. Convexity of $\hat{\succ}$ implies that $\mathrm{conv}(\{q,r\}) \subseteq \hat{U}(p)$. Hence, $\emptyset \neq \mathrm{conv}(\{q,r\}) \cap I(p) \subseteq \hat{U}(p)$, which contradicts $I(p) \subseteq \hat{I}(p)$. Hence, $\hat{U}(p) \subseteq U(p)$ or $\hat{U}(p) \subseteq L(p)$. Similarly, $\hat{L}(p) \subseteq L(p)$ or $\hat{L}(p) \subseteq U(p)$. If $\hat{U}(p) \cup \hat{L}(p) \subseteq U(p)$ or $\hat{U}(p) \cup \hat{L}(p) \subseteq L(p)$, then $\hat{I}(p)$ contains either $L(p)$ or $U(p)$. If $L(p) \subseteq \hat{I}(p)$ and $L(p)$ in non-empty or $U(p) \subseteq \hat{I}(p)$ and $U(p)$ in non-empty, then $\hat{I}(p)$ contains an open set. Similarly, if $\hat{U}(p) \neq \emptyset$. Hence, $\hat{I}(p) = \Delta$. Thus, for all $p \in \Delta$ with $L(p) \neq \emptyset$ and $U(p) \neq \emptyset$, either $\hat{U}(p) \subseteq U(p)$ and $\hat{L}(p) \subseteq L(p)$ or $\hat{U}(p) \subseteq L(p)$ and $\hat{L}(p) \subseteq U(p)$. In the first case, we say that $\hat{\succ}$ is oriented positively at p, in the latter case oriented negatively.

Let $\Delta^+ = \{p \in \Delta \colon U(p) = \emptyset \text{ and } L(p) \neq \emptyset\}$ and $\Delta^- = \{p \in \Delta \colon L(p) = \emptyset \text{ and } U(p) \neq \emptyset\}$. Note that both Δ^+ and Δ^- are convex. Moreover, for all $p \in \Delta^+$, $\Delta^+ \subseteq I(p)$ and, for all $p \in \Delta^-$, $\Delta^- \subseteq I(p)$. Since $I(p) \neq \Delta$ for all $p \in \Delta^+ \cup \Delta^-$, neither of Δ^+ and Δ^- contains an open set. Then also $\Delta^+ \cup \Delta^-$ does not contain an open set. Let $\Delta^* = \Delta \setminus (\Delta^+ \cup \Delta^-)$. Note that Δ^* contains an open set.

Let $p, q \in \Delta^*$ such that $q \in \hat{U}(p)$ and $\hat{\succ}$ is oriented positively at p. Assume for contradiction that $\hat{\succ}$ is oriented negatively at q, i.e., $\hat{U}(q) \subseteq L(q)$ and $\hat{L}(q) \subseteq U(q)$. $q \in \hat{U}(p)$ implies $q \in U(p)$, which is equivalent to $p \in L(q)$. Also $q \in \hat{U}(p)$ is equivalent to $p \in \hat{L}(q)$ and then $p \in U(q)$. Since $L(q)$ and $U(q)$ are disjoint, this is a contradiction. Hence $\hat{U}(q) \subseteq U(q)$ and $\hat{L}(q) \subseteq L(q)$, i.e., $\hat{\succ}$ is oriented positively at q. Similarly, if $p, q \in \Delta^*$ such that $q \in \hat{U}(p)$ and $\hat{\succ}$ is oriented negatively at p, then $\hat{\succ}$ is oriented negatively at q.

If $\hat{\succ} = \emptyset$, the statement of the lemma holds trivially. If $\hat{\succ} \neq \emptyset$, there is $p \in \Delta^*$ such that $\hat{\succ}$ is oriented positively or negatively at p (but not both), i.e., $\hat{I}(p) \neq \Delta$. Assume that $\hat{\succ}$ is oriented positively at p. Let $q \in \Delta^*$. If there is $r \in \Delta^* \setminus (\hat{I}(p) \cup \hat{I}(q))$, then two applications of what we have proven above yield that $\hat{\succ}$ is oriented positively at q. If $\Delta^* \subseteq \hat{I}(p) \cup \hat{I}(q)$, then either $\hat{I}(p)$ or $\hat{I}(q)$ contains an open set. Hence either $\hat{I}(p) = \Delta$ or $\hat{I}(q) = \Delta$. Since $\hat{I}(p) \neq \Delta$ by assumption, we have that $\hat{I}(q) = \Delta$. Then $\hat{\succ}$ is trivially oriented positively (and negatively) at q. Together we get that $\hat{\succ}$ is oriented positively at all $p \in \Delta^*$.

By $\succ_p$ we denote the restriction of a relation to those comparisons involving p, i.e., $\succ_p = \succ \cap (\{p\} \times \Delta \cup \Delta \times \{p\})$. By continuity and convexity of $\hat{\succ}$, $\hat{I}(p)$ is the hyperplane separating $\hat{U}(p)$ and $\hat{L}(p)$. Similarly, $I(p)$ is the hyperplane separating $U(p)$ and $L(p)$. Since $I(p) \subseteq \hat{I}(p)$, it follows that $I(p) = \hat{I}(p)$. Since $\hat{\succ}$ is oriented positively at p, we get that $\hat{\succ}_p = \succ_p$. Now let $q \in \Delta^*$ be arbitrary. If $\hat{I}(q) \neq \Delta$, we have that $\hat{I}(p) \cup \hat{I}(q) \neq \Delta$, as this would imply that either $\hat{I}(p)$ or $\hat{I}(q)$ contains an open set in which case $\hat{I}(p) = \Delta$ or $\hat{I}(q) = \Delta$, respectively. Hence, there is $r \in \Delta^* \setminus (\hat{I}(p) \cup \hat{I}(q))$. Since $r \in \Delta^* \setminus \hat{I}(p) = \Delta^* \setminus I(p)$, it follows that $\hat{\succ}_r = \succ_r$. Also, since $r \in \Delta^* \setminus \hat{I}(q)$, i.e., $q \in \Delta^* \setminus \hat{I}(r) = \Delta^* \setminus I(r)$, it follows that $\hat{\succ}_q = \succ_q$. Now consider the case when $\hat{I}(q) = \Delta$. If $I(q) = \Delta$, then $\hat{\succ}_q = \succ_q$ follows trivially. In case $I(q) \neq \Delta$, let $S = \{p' \in \Delta \colon \hat{I}(p') = \Delta\}$. If S contains an open set, let $r \in \Delta$ such that $\hat{I}(r) \neq \Delta$. r exists, since $\hat{\succ} \neq \emptyset$. Since $S \subseteq \hat{I}(r)$, $\hat{I}(r)$ contains an open set, which contradicts $\hat{I}(r) \neq \Delta$. Hence, S does not contain an open set. Since $I(q) \neq \Delta$, it does not contain an open set either, which implies that $(\Delta^* \setminus S) \cap (\Delta^* \setminus I(q)) \neq \emptyset$. Let $r \in (\Delta^* \setminus S) \cap (\Delta^* \setminus I(q))$. Since $\hat{I}(r) \neq \Delta$ it follows from a previous case that $\hat{\succ}_r = \succ_r$. Then $r \in \Delta^* \setminus I(q)$ implies $r \in \Delta^* \setminus \hat{I}(q)$ contradicting $\hat{I}(q) = \Delta$, which means that the current case cannot occur.

Lastly, consider $q \in \Delta \setminus \Delta^*$. If $q \in \Delta^+$, then $U(q) = \emptyset$ and from before we know that $\hat{U}(q) \cup \hat{L}(q) \subseteq L(q)$. If $\hat{U}(q) \neq \emptyset$, then $\hat{U}(q)$ is open and hence intersects with Δ^*. For $r \in \hat{U}(q) \cap \Delta^*$ we know that $\hat{\succ}_r = \succ_r$. This means that $q \succ r$ and $r \succ q$, which is a contradiction. Hence $\hat{U}(q) = \emptyset$, which means that $\hat{\succ}$ is oriented positively at q. From before it follows that $\hat{\succ}_q = \succ_q$.

Together, we have that $\hat{\succ}_q = \succ_q$ for all $q \in \Delta$, i.e, $\hat{\succ} = \succ$. If $\hat{\succ}$ is oriented negatively at p, we get $\hat{\succ} = \succ^{-1}$ by an analogous argument.

$\square$

Lemma 1 is a generalization of Theorem 2 by Fishburn and Gehrlein (1987). The proof only requires continuity and convexity, but not symmetry, of $\mathcal{R}$.[9]

The next lemma roughly corresponds to what is known as the *field expansion lemma*

[9]Lemma 1 does not hold if convexity is weakened to the assumption that $U(p)$, $L(p)$, and $I(p)$ need to be convex for all $p \in \Delta$. To see this, consider the following preference relations on the closed interval $[0,1]$ (equipped with the standard topology). Let $\succsim$ be the greater or equal relation and $\hat{\succsim}$ be defined such that $x \mathrel{\hat{\succ}} y$ if $x \in (3/4, 1]$ and $y \in [0, 1/4)$ and $x \mathrel{\hat{\sim}} y$ otherwise. Both, $\succsim$ and $\hat{\succsim}$ are continuous and convex according to the weaker convexity assumption defined above. For $\succsim$ this is clear. To see this for $\hat{\succsim}$, observe that for all $x \in [0,1]$, either $I(x) = [0, 3/4]$ and $U(x) = (3/4, 1]$ or $I(x) = [0,1]$ or $L(x) = [0, 1/4)$ and $I(x) = [1/4, 1]$. In all cases, $U(x)$ and $L(x)$ are open and $U(x)$, $L(x)$, and $I(x)$ are convex.

in traditional proofs of Arrow's theorem (see, e.g. Sen, 1986).[10] Let $f\colon \mathcal{D}^N \to \mathcal{R}$ be an SWF, $G, H \subseteq N$, and $a, b \in U$. We say that (G, H) is decisive for a against b, denoted by $a\ D_{G,H}\ b$, if, for all $R \in \mathcal{D}$, $a \succ_i b$ for all $i \in G$, $a \sim_i b$ for all $i \in H$, and $b \succ_i a$ for all $i \in N \setminus (G \cup H)$ implies $a \succ b$.

Lemma 2. *Let $m \geq 3$, f be an Arrovian SWF on some domain $\mathcal{D}$, $G, H \subseteq N$, and $a, b \in U$. Then $a\ D_{G,H}\ b$ implies that $D_{G,H} = U \times U$.*

Proof. First we show that $a\ D_{G,H}\ x$ and $b\ D_{G,H}\ x$ for all $x \in U \setminus \{a, b\}$. To this end, let $\succsim_x \in \mathcal{D}$ such that $a \succ_x b \succ_x x$ and $a \succ_x x$ and consider the preference profile

$$R = (\underbrace{\succsim_x, \dots, \succsim_x}_{G}, \underbrace{\sim^U, \dots, \sim^U}_{H}, \succsim_x^{-1}, \dots, \succsim_x^{-1}).$$

Since $\succsim_x \cap \succsim_x^{-1} = \sim_x$, it follows from the indifference part of Pareto optimality and Lemma 1 that $\succsim = f(R) \in \{\succsim_x, \succsim_x^{-1}, \sim^U\}$. Since $a\ D_{G,H}\ b$, $\succsim = \succsim_x$ remains as the only possibility. Hence, $a \succ x$ and $b \succ x$. By IIA, it follows that $a\ D_{G,H}\ x$ and $b\ D_{G,H}\ x$.

Repeated application of the second statement implies that $D_{G,H}$ is a complete relation. To show that $D_{G,H}$ is symmetric, let $x, y, z \in U$ such that $x\ D_{G,H}\ y$. The first statement implies that $x\ D_{G,H}\ z$. Two applications of the second statement yield $z\ D_{G,H}\ y$ and $y\ D_{G,H}\ x$. Hence, $D_{G,H} = U \times U$. □

We first show that Arrovian aggregation is only possible on domains, in which preferences over outcomes are completely determined by preferences over pure outcomes.

Lemma 3. *Let f be an anonymous Arrovian SWF on some domain $\mathcal{D}$. Then, $\succsim|_A = \hat{\succsim}|_A$ implies $\succsim|_{\Delta A} = \hat{\succsim}|_{\Delta A}$ for all $\succsim, \hat{\succsim} \in \mathcal{D}$ and $A \in \mathcal{A}$.*

Proof. Let $\succsim_0, \hat{\succsim}_0 \in \mathcal{D}$ and $A \in \mathcal{A}$ such that $\succsim_0|_A = \hat{\succsim}_0|_A$. Consider the preference profile

$$R = (\succsim_0, \hat{\succsim}_0^{-1}, \sim^U, \dots, \sim^U).$$

$R \in \mathcal{D}^N$ since $\mathcal{D}$ satisfies our richness assumptions. Now let $a, b \in U$ and define $\bar{R} = R_{(12)}$ be like R except that the preferences of agents 1 and 2 are exchanged. Anonymity of f implies that $\bar{\succsim} = f(\bar{R}) = f(R) = \succsim$. If $a \succ b$, $(1, N \setminus \{1, 2\})$ is decisive for a against b, Lemma 2 implies that $(1, N \setminus \{1, 2\})$ is also decisive for b against a. Hence $b\ \bar{\succ}\ a$, this contradicts $\bar{\succsim} = \succsim$. Thus, $a \sim b$. Since $\succsim$ satisfies convexity, we get that $\sim|_{\Delta A} = \sim^A$. If $\succsim_0|_{\Delta A} \neq \hat{\succsim}_0|_{\Delta A}$, there exist $p, q \in \Delta A$ such that $p \succ_0 q$ and not $q \hat{\succ}_0 p$. Hence, $p\ \hat{\succsim}_0^{-1}\ q$. The strict part of Pareto optimality of f implies that $p \succ q$. This contradicts $\succsim|_{\Delta A} = \sim^A$. Hence, $\succsim_0|_{\Delta A} = \hat{\succsim}_0|_{\Delta A}$. □

Next, we show that intensities of preferences between pure outcomes are identical.

[10] In contrast to Lemma 2, the consequence of the original field expansion lemma uses a stronger notion of decisiveness.

Lemma 4. *Let f be an Arrovian SWF on some domain $\mathcal{D}$. Then, for all $\succsim_0 \in \mathcal{D}$ and $a, b, c \in U$ with $a \succ_0 b$,*

(i) $b \succ_0 c$ implies $\phi_0(a,b) = \phi_0(b,c)$,

(ii) $a \succ_0 c$ implies $\phi_0(a,b) = \phi_0(a,c)$, and

(iii) $c \succ_0 b$ implies $\phi_0(a,b) = \phi_0(c,b)$.

Proof. Ad (i): continuity implies that $b \sim_0 \lambda a + (1-\lambda)c$ for some $\lambda \in (0,1)$. Observe that $\succsim_0^{(a,c)}|_{\{a,b,c\}} = \succsim_0^{-1}|_{\{a,b,c\}}$. Lemma 3 implies that $\succsim_0^{(a,c)}|_{\Delta(\{a,b,c\})} = \succsim_0^{-1}|_{\Delta(\{a,b,c\})}$. Since $\sim_{0\Delta(\{a,b,c\})} = \sim_{0\Delta(\{a,b,c\})}^{-1}$, we have $b \sim_0 (1-\lambda)a + \lambda c$. Convexity of $\succsim_0$ then implies that $b \sim_0 {}^1\!/\!_2\, a + {}^1\!/\!_2\, c$. This is equivalent to $\phi_0(a,b) = \phi_0(b,c)$.

Ad (ii): we distinguish two cases.

Case 1 ($b \sim_0 c$): let $i, j \in N$ and consider the preference profile

$$R = (\succsim_0, (\succsim_0^{(bc)})^{-1}, \sim^U, \ldots, \sim^U).$$

As in the proof of Lemma 3, we get that $\succsim|_{\{a,b,c\}} = \Delta(\{a,b,c\}) \times \Delta(\{a,b,c\})$. Without loss of generality, assume that $\phi_0(a,b) = 1$ and $\phi_0(a,c) = \lambda$ for some $\lambda \in (0,1]$. Let $p = {}^1\!/\!_2\, a + {}^1\!/\!_2\, b$ and $q = {}^1\!/\!_2\, a + {}^1\!/\!_2\, c$. Then $\phi_1(p,q) = \phi_2(p,q) = {}^1\!/\!_4(1-\lambda)$. If $\lambda < 1$, the strict part of Pareto optimality of f implies that $p \succ q$. This contradicts $\succsim|_{\{a,b,c\}} = \Delta(\{a,b,c\}) \times \Delta(\{a,b,c\})$. Hence, $\lambda = 1$.

Case 2 ($b \succ_0 c$): assume without loss of generality that $\phi_0(a,b) = 1$. By (i), we get $\phi_0(a,b) = \phi_0(b,c) = 1$. Our richness assumptions imply that there is $\hat{\succsim}_0 \in \mathcal{D}$ with $a \mathrel{\hat{\succsim}_0} b \mathrel{\hat{\succsim}_0} c$, $a \mathrel{\hat{\succsim}_0} c$, and $c \mathrel{\hat{\succsim}_0} x$ for some $x \in U$. Lemma 3 implies that $\phi_0|_{\{a,b,c\}} = \hat{\phi}_0|_{\{a,b,c\}}$. Hence, if suffices to show that $\hat{\phi}_0(a,c) = 1$. By (i), we get that $\hat{\phi}_0(a,c) = \hat{\phi}_0(c,x)$ and $\hat{\phi}_0(b,c) = \hat{\phi}_0(c,x) = 1$. Hence, $\hat{\phi}_0(a,c) = 1$.

Ad (iii): the proof is analogous to the proof of (ii). ☐

Theorem 1. *Let f be an anonymous Arrovian SWF on some domain $\mathcal{D}$. Then $\mathcal{D} \subseteq \Phi^{PC}$.*

Proof. Let $\succsim_0 \in \mathcal{D}$ and $a, b, c, d \in U$ such that $a \succ_0 b$ and $c \succ_0 d$. We have to show that $\phi_0(a,b) = \phi_0(c,d)$. First assume there are $x \in \{a,b\}$ and $y \in \{c,d\}$ such that $x \succ_0 y$ or $y \succ_0 x$. Then, Lemma 4 implies that $\phi_0(a,b) = \phi_0(x,y) = \phi_0(c,d)$. Otherwise, $\sim_0 |_{\{a,b,c,d\}} = \sim^{\{a,b,c,d\}}$ and $\succsim_0|_{\{a,b,c,d\}} = \succsim_0{}^{(ac)(bd)}|_{\{a,b,c,d\}}$. Lemma 3 implies that $\succsim_0|_{\Delta(\{a,b,c,d\})} = \succsim_0{}^{(ac)(bd)}|_{\Delta(\{a,b,c,d\})}$. Hence, $\phi_0|_{\{a,b,c,d\}} = \phi_0^{(ac)(bd)}|_{\{a,b,c,d\}}$ and $\phi_0(a,b) = \phi_0(c,d)$. ☐

6 Characterization of the Social Welfare Function

Theorem 1 has established that Arrovian aggregation is only possible if individual preferences are based on pairwise comparisons. In the remainder of this paper, we will—with slight abuse of notation—treat Arrovian SWFs as functions from $\mathcal{D}^N$ to Φ with $\mathcal{D} \subseteq \Phi^{PC}$.

The following lemmas show that for every preference profile R and all alternatives x and y, $\phi(a,b)$ only depends on the number of agents who prefer a to b, whenever R is from the domain of PC-preferences and ϕ represents $f(R)$.

Given preference profile R, let $N_{xy} = \{i \in N \colon x \succ_i y\}$ be the set of agents who strictly prefer x over y and $n_{xy} = |N_{xy}|$. Also, let $I_{xy} = N \setminus (N_{xy} \cup N_{yx})$ be the set of agents who are indifferent between x and y.

Lemmas 5 and 6 show that for a fixed preference profile, $\phi(x,y)$ only depends on N_{xy} and I_{xy} (and not on the names of the alternatives).

Lemma 5. *Let f be an Arrovian SWF, $a,b,c \in U$, and $R = (\phi_i)_{i\in N}$ such that*

(i) $\phi_i(a,b) = \phi_i(a,c) \geq 0$ for all $i \in N$, or

(ii) $\phi_i(a,b) = \phi_i(a,c) \leq 0$ for all $i \in N$.

Then, $\phi(a,b) = \phi(a,c)$ where $\phi = f(R)$.

Proof. We only show case (i). Case (ii) can be proved by a symmetric argument. Let $d \in U$ and consider a preference profile $\hat{R} \in \mathcal{D}^N$ such that $R|_{\{a,b,c\}} = \hat{R}|_{\{a,b,c\}}$ and

$$\hat{R}|_{\{a,b,c,d\}} = (\underbrace{\begin{pmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & & 1 \\ -1 & & 0 & 1 \\ -1 & -1 & -1 & 0 \end{pmatrix}, \dots,}_{N_{ab}} 0, \dots).$$

The values of $\phi_i(b,c)$ are irrelevant for all $i \in N_{ab}$. Let $\hat{\phi} = f(\hat{R})$. Since $\hat{\phi}_i(1/2\,a + 1/2\,d, b) = 0$ for all $i \in N$, the indifference part of Pareto optimality implies that $\hat{\phi}_i(1/2\,a + 1/2\,d, b) = 0$. Hence, $\hat{\phi}(a,b) = \hat{\phi}(b,d)$. Similarly, we get $\hat{\phi}(a,c) = \hat{\phi}(c,d)$. $\hat{\phi}$ takes the following form for some $\lambda, \mu, \sigma \in \mathbb{R}$.

$$\hat{\phi}|_{\{a,b,c,d\}} = \begin{pmatrix} 0 & \mu & \sigma & \lambda \\ -\mu & 0 & & \mu \\ -\sigma & & 0 & \sigma \\ -\lambda & -\mu & -\sigma & 0 \end{pmatrix}$$

The indifference part of Pareto optimality and Lemma 1 imply that $\hat{\phi}|_{\{a,b,d\}} \in \{\pm\phi_i|_{\{a,b,d\}}, 0\}$ for some $i \in N$. This implies that $\hat{\phi}(a,b) = \hat{\phi}(a,d)$, i.e., $\mu = \lambda$. Similarly, we get $\sigma = \lambda$. In particular, $\hat{\phi}(a,b) = \hat{\phi}(a,c)$. Since $R|_{\{a,b,c\}} = \hat{R}|_{\{a,b,c\}}$, IIA implies that $\phi|_{\{a,b,c\}} = \hat{\phi}|_{\{a,b,c\}}$. Hence, $\phi(a,b) = \phi(a,c)$. □

Lemma 6. *Let f be an Arrovian SWF, $a,b,c,d \in U$, $R \in \mathcal{D}^N$, and $\pi = (a,c)(b,d) \in \Pi(U)$. If $\pi(R|_{\{a,c\}}) = R|_{\{b,d\}}$, then $\phi(a,c) = \phi(b,d)$ where $\phi = f(R)$.*

Proof. We first prove the case when all of a, b, c, d are distinct. Let $e \in U$ and consider a preference profile $\hat{R} \in \mathcal{D}^N$ such that $R|_{\{a,b,c,d\}} = \hat{R}|_{\{a,b,c,d\}}$ and $\hat{\phi}_i(x, e)$ for all $x \in \{a, b, c, d\}$ and $i \in N$. Then, by Lemma 5, we can assume without loss of generality that $\hat{\phi}(x, e) = \lambda \in \mathbb{R}$ for all $x \in \{a, b, c, d\}$. Now consider the preference profile

$$\mathring{R}|_{\{a,b,c,d,e\}} = (\underbrace{\begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 & 1 \\ -1 & -1 & 0 & 1 & 1 \\ -1 & -1 & -1 & 0 & 1 \\ -1 & -1 & -1 & -1 & 0 \end{pmatrix}, \ldots,}_{N_{ac}} \underbrace{\begin{pmatrix} 0 & -1 & -1 & -1 & 1 \\ 1 & 0 & -1 & -1 & 1 \\ 1 & 1 & 0 & -1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ -1 & -1 & -1 & -1 & 0 \end{pmatrix}, \ldots,}_{N_{ca}}$$
$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & -1 & 0 \end{pmatrix}, \ldots).$$

Note that $\hat{R}|_{\{a,c,e\}} = \mathring{R}|_{\{a,c,e\}}$ and $\hat{R}|_{\{b,d,e\}} = \mathring{R}|_{\{b,d,e\}}$ because $\sigma(R|_{\{a,c\}}) = R|_{\{b,d\}}$ by assumption. Let $\hat{\phi} = f(\hat{R})$ and $\mathring{\phi} = f(\mathring{R})$. Since $\hat{R}|_{\Delta\{a,c,e\}} = \mathring{R}|_{\Delta\{a,c,e\}}$, we have $\hat{\phi}|_{\{a,c,e\}} = \mathring{\phi}|_{\{a,c,e\}}$ by IIA. Moreover, $\hat{R}^{\pi}|_{\Delta\{b,d,e\}} = \mathring{R}|_{\Delta\{b,d,e\}}$ and IIA yield $\hat{\phi}|_{\{b,d,e\}} = \mathring{\phi}|_{\{b,d,e\}}$. Lemma 5 implies that $\mathring{\phi}(x, e) = \lambda$ for all $x \in \{a, b, c, d\}$ without loss of generality. Thus, for some $\mu, \sigma > 0$, $\mathring{\phi}$ takes the form

$$\mathring{\phi}|_{\{a,b,c,d,e\}} = \begin{pmatrix} 0 & & \mu & & \lambda \\ & 0 & & \sigma & \lambda \\ -\mu & & 0 & & \lambda \\ & -\sigma & & 0 & \lambda \\ -\lambda & -\lambda & -\lambda & -\lambda & 0 \end{pmatrix}.$$

Now consider the preference profile

$$\mathring{R}|_{\{a,b,c,d,e\}} = (\underbrace{\begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 & 1 \\ -1 & -1 & 0 & 1 & 1 \\ -1 & -1 & -1 & 0 & 1 \\ -1 & -1 & -1 & -1 & 0 \end{pmatrix}, \ldots,}_{N_{ac}} \underbrace{\begin{pmatrix} 0 & -1 & -1 & -1 & -1 \\ 1 & 0 & -1 & -1 & -1 \\ 1 & 1 & 0 & -1 & -1 \\ 1 & 1 & 1 & 0 & -1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}, \ldots,}_{N_{ca}} 0, \ldots)$$

and let $\bar{\phi} = f(\bar{R})$. The indifference part of Pareto optimality of f and Lemma 1 imply that $\bar{\phi} \in \{\pm\bar{\phi}_i, 0\}$ for some $i \in N$. Since $\mathring{\phi}|_{\{a,b,c,d\}} = \bar{\phi}|_{\{a,b,c,d\}}$, we get that $\mu = \sigma$.

The cases when $a = b$ or $b = c$ follow from repeated application of the above case. All other cases are symmetric to one of these cases. $\square$

Lemma 7. *Let f be an Arrovian SWF, $a,b,c,d \in U$, $R, \hat{R} \in \mathcal{D}^N$, $\phi = f(R)$, and $\hat{\phi} = f(\hat{R})$. If $R|_{\{a,c\}} = \hat{R}|_{\{a,c\}}$ and $R|_{\{b,d\}} = \hat{R}|_{\{b,d\}}$, then $\phi(a,c) = \alpha \cdot \hat{\phi}(a,c)$ and $\phi(b,d) = \alpha \cdot \hat{\phi}(b,d)$ for some $\alpha > 0$.*

Proof. Let $e \in U \setminus \{a,b,c,d\}$. Let $R', \hat{R}' \in \mathcal{D}^N$ such that $R'|_{\{a,b,c,d\}} = R|_{\{a,b,c,d\}}$, $\hat{R}'|_{\{a,b,c,d\}} = \hat{R}|_{\{a,b,c,d\}}$, $x \succ_i' e$, and $x \hat{\succ}_i' e$ for all $x \in \{a,b,c,d\}$ and $i \in N$. $\phi' = f(R')$ and $\hat{\phi}' = f(\hat{R}')$ denote the corresponding collective SSB functions. Since f satisfies IIA, we have $\phi|_{\{a,b,c,d\}} = \phi'|_{\{a,b,c,d\}}$ and $\hat{\phi}|_{\{a,b,c,d\}} = \hat{\phi}'|_{\{a,b,c,d\}}$ without loss of generality. Lemma 5 implies that we can assume without loss of generality that ϕ' and $\hat{\phi}'$ take the following form for some $\lambda, \mu, \hat{\mu}, \sigma, \hat{\sigma} \in \mathbb{R}$.

$$\phi'|_{\{a,b,c,d,e\}} = \begin{pmatrix} 0 & & \mu & & \lambda \\ & 0 & & \sigma & \lambda \\ -\mu & & 0 & & \lambda \\ & -\sigma & & 0 & \lambda \\ -\lambda & -\lambda & -\lambda & -\lambda & 0 \end{pmatrix} \qquad \hat{\phi}'|_{\{a,b,c,d,e\}} = \begin{pmatrix} 0 & & \hat{\mu} & & \lambda \\ & 0 & & \hat{\sigma} & \lambda \\ -\hat{\mu} & & 0 & & \lambda \\ & -\hat{\sigma} & & 0 & \lambda \\ -\lambda & -\lambda & -\lambda & -\lambda & 0 \end{pmatrix}$$

Recall that $R'|_{\{a,c,e\}} = \hat{R}'|_{\{a,c,e\}}$ and $R'|_{\{b,d,e\}} = \hat{R}'|_{\{b,d,e\}}$ by the way they were constructed from R and $\hat{R}$. Since f satisfies IIA, we get that $\phi'|_{\{a,c,e\}} = \hat{\phi}'|_{\{a,c,e\}}$ and $\phi'|_{\{b,d,e\}} = \hat{\phi}'|_{\{b,d,e\}}$. In particular, this means that $\mu = \hat{\mu}$ and $\sigma = \hat{\sigma}$. The scalar α disappears by choosing suitable SSB functions representing the collective preferences without loss of generality. $\square$

Lemma 7 implies that $\phi(a,b)$ only depends on N_{ab} and I_{ab} and not on a,b and R. Hence, there is a function $g\colon 2^N \times 2^N \to \mathbb{R}$ such that $g(N_{ab}, I_{ab}) = \phi(a,b)$ for all $a,b \in U$ and $R \in \mathcal{D}^N$. We now leverage the indifference part of Pareto optimality to show that ϕ is a linear combination of the ϕ_i's. Hence, f is affine welfare maximizing.

Lemma 8. *Let f be an Arrovian SWF. Then, there are $w_1, \dots, w_n \in \mathbb{R}$ such that $f(R) = \sum_{i \in N} w_i \phi_i$ for all $R \in \mathcal{D}^N$.*

Proof. For all $G \subseteq N$, let $w_G = {}^1\!/_2\,(1 + g(G, \emptyset))$. For convenience, we write w_i for $w_{\{i\}}$. Since $\phi(x,y) = g(N_{xy}, I_{xy})$ for all $x,y \in U$, it suffices to show that

$$g(N_{xy}, I_{xy}) = \sum_{i \in N_{xy}} w_i - \sum_{i \in N_{yx}} w_i, \tag{1}$$

for all $x,y \in U$. To this end, we will first show that $w_G + w_{\hat{G}} = w_{G \cup \hat{G}}$ for all $G, \hat{G} \subseteq N$ with $G \cap \hat{G} = \emptyset$. Let $G, \hat{G}$ as above, $a,b,c,x,y \in U$, and consider the following preference profile $R \in \mathcal{D}^N$ with $\phi = f(R)$.

$$R|_{\{a,b,c,x,y\}} = (\underbrace{\begin{pmatrix} 0 & 1 & 1 & -1 & 1 \\ -1 & 0 & 1 & -1 & 1 \\ -1 & -1 & 0 & -1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ -1 & -1 & -1 & -1 & 0 \end{pmatrix}, \dots,}_{G} \underbrace{\begin{pmatrix} 0 & -1 & -1 & -1 & -1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & -1 & 0 & 1 & -1 \\ 1 & -1 & -1 & 0 & -1 \\ 1 & -1 & 1 & 1 & 0 \end{pmatrix}, \dots,}_{\hat{G}}$$

$$\begin{pmatrix} 0 & 1 & -1 & -1 & 1 \\ -1 & 0 & -1 & -1 & -1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & -1 & 0 & 1 \\ -1 & 1 & -1 & -1 & 0 \end{pmatrix}, \dots)$$

We have that, for $p = 1/2\, x + 1/2\, y$ and $q = 1/3\, a + 1/3\, b + 1/3\, c$, $\phi_i(p,q) = 0$ for all $i \in N$. The indifference part of Pareto optimality implies that $\phi(p,q) = 0$. Let $\mu = g(G, \emptyset)$, $\hat{\mu} = g(\hat{G}, \emptyset)$, and $\sigma = g(G \cup \hat{G}, \emptyset)$. Since, by definition of w,

$$w_G + w_{\hat{G}} = w_{G \cup \hat{G}} \quad \text{is equivalent to} \quad (1 + g(G, \emptyset)) + (1 + g(\hat{G}, \emptyset)) = 1 + g(G \cup \hat{G}, \emptyset),$$

we have to show that $\mu + \hat{\mu} + 1 = \sigma$. By definition of g, we get that ϕ takes the following form.

$$\phi|_{\{a,b,c,x,y\}} = \begin{pmatrix} 0 & & & -1 & -\hat{\mu} \\ & 0 & & \hat{\mu} & \sigma \\ & & 0 & -\mu & -\hat{\mu} \\ 1 & -\hat{\mu} & \mu & 0 & \\ \hat{\mu} & -\sigma & \hat{\mu} & & 0 \end{pmatrix}$$

From $\phi(p,q) = 0$, it follows that $1/6\,(\mu + \hat{\mu} + 1 - \sigma) = 0$. This proves the desired relationship.

Now we can rewrite (1) to

$$g(N_{xy}, I_{xy}) = w(N_{xy}) - w(N_{yx}). \tag{2}$$

By definition of w, this is equivalent to

$$2g(N_{xy}, I_{xy}) = g(N_{xy}, \emptyset) - g(N_{yx}, \emptyset). \tag{3}$$

To show this, let $a, b, x, y \in U$ and consider the following preference profile $\hat{R} \in \mathcal{D}^N$ with $\hat{\phi} = f(\hat{R})$.

$$\hat{R}|_{\{a,b,x,y\}} = (\underbrace{\begin{pmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & -1 & -1 \\ -1 & 1 & 0 & 1 \\ -1 & 1 & -1 & 0 \end{pmatrix}, \dots,}_{G} \underbrace{\begin{pmatrix} 0 & 1 & -1 & 1 \\ -1 & 0 & -1 & 0 \\ 1 & 1 & 0 & 1 \\ -1 & 0 & -1 & 0 \end{pmatrix}, \dots,}_{\hat{G}} \begin{pmatrix} 0 & -1 & -1 & -1 \\ 1 & 0 & 1 & 1 \\ 1 & -1 & 0 & -1 \\ 1 & -1 & 1 & 0 \end{pmatrix}, \dots)$$

Observe that, for $p = {}^1\!/\!{}_3\, x + {}^2\!/\!{}_3\, y$ and $q = {}^1\!/\!{}_2\, a + {}^1\!/\!{}_2\, b$, $\hat{\phi}_i(p,q) = 0$ for all $i \in N$. The indifference part of Pareto optimality implies that $\hat{\phi}(p,q) = 0$. With the same definitions as before and $\epsilon = g(G,\hat{G})$, $\hat{\phi}$ takes the following form.

$$\hat{\phi}|_{\{a,b,x,y\}} = \begin{pmatrix} 0 & & \mu & \sigma \\ & 0 & -\sigma & -\epsilon \\ -\mu & \sigma & 0 & \\ -\sigma & \epsilon & & 0 \end{pmatrix}$$

From $\hat{\phi}(p,q) = 0$, we get that ${}^1\!/\!{}_6\,(-\mu + \sigma - 2\sigma + 2\epsilon) = 0$. Hence, $2\epsilon = \mu + \sigma$. This is equivalent to

$$2g(G,\hat{G}) = g(G,\emptyset) + g(G \cup \hat{G},\emptyset) = g(G,\emptyset) - g(N \setminus (G \cup \hat{G}),\emptyset),$$

where the last equality follows from skew-symmetry of $\hat{\phi}$ and the definition of g. This proves (3). $\square$

Finally, the strict part of Pareto optimality implies that individual weights have to be strictly positive.

Theorem 2. *Let f be an Arrovian SWF. Then, there are $w_1,\dots,w_n \in \mathbb{R}_{>0}$ such that $f(R) = \sum_{i\in N} w_i\phi_i$ for all $R \in \mathcal{D}^N$.*

Proof. From Lemma 8 we know that there are $w_1,\dots,w_n \in \mathbb{R}$ such that, for all $R \in \mathcal{R}^N$, $\phi = f(R) = \sum_{i\in N} w_i\phi_i$. Assume for contradiction that $w_i \leq 0$ for some $i \in N$. Let G be the set of agents such that $w_i \leq 0$ and consider a preference profile $R \in \mathcal{D}^N$ with $a,b,c \in U$ such that

$$R|_{\{a,b,c\}} = (\underbrace{\begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix},\dots,}_{G} \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix},\dots).$$

Then, for $p = {}^1\!/\!{}_2\, a + {}^1\!/\!{}_2\, b$, we have that $\phi_i(p,c) > 0$ for all $i \in G$ and $\phi_i(p,c) = 0$ for all $i \in N \setminus G$. Pareto optimality of f implies that $\phi(p,c) > 0$. However, we have

$$\phi(p,c) = \sum_{i\in G} w_i\phi_i(p,c) + \sum_{i\in N\setminus G} w_i \underbrace{\phi_i(p,c)}_{=0} = \sum_{i\in G} \underbrace{w_i\phi_i(p,c)}_{\leq 0} \leq 0,$$

which is a contradiction. $\square$

The following corollary is a direct consequence of Theorem 2.

Corollary 1. *Let f be an anonymous Arrovian SWF. Then, $f(R) = \sum_{i\in N} \phi_i$ for all $R \in \mathcal{D}$.*

For illustrative purposes, Figure 2 shows the collective preference relation of the unique anonymous Arrovian SWF for Condorcet's classic example of cyclical majorities.

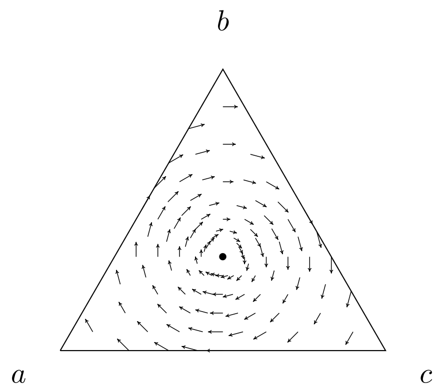

Figure 2: Illustration of collective preferences for Condorcet's paradox. There are three agents with the following transitive preferences over pure outcomes: $a \succ_1 b \succ_1 c$, $b \succ_2 c \succ_2 a$, and $c \succ_3 a \succ_3 b$. Note that the majority relation is cyclic, since there is a majority for a over b, b over c, and c over a. The figure depicts the collective preference relation derived from the unique anonymous Arrovian SWF. The arrows represent normal vectors to the indifference curves which separate the corresponding upper and lower contour set (pointing towards the lower contour set). The unique collectively most preferred outcome is $1/3\, a + 1/3\, b + 1/3\, c$.

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