An Ordinal Minimax Theorem

Felix Brandt
TU München
Germany

Markus Brill
Duke University
USA

Warut Suksompong
Stanford University
USA

In the early 1950s Lloyd Shapley proposed an ordinal and set-valued solution concept for zero-sum games called *weak saddle*. We show that all weak saddles of a given zero-sum game are interchangeable and equivalent. As a consequence, every such game possesses a unique set-based value.

JEL Classifications Code: C72

1 Introduction

One of the earliest solution concepts considered in game theory are *saddle points*, combinations of actions such that no player can gain by deviating (see, e.g., von Neumann and Morgenstern, 1947). In two-player zero-sum games, every saddle point happens to coincide with the optimal outcome both players can guarantee in the worst case and thus enjoys a very strong normative foundation. Unfortunately, however, saddle points are not guaranteed to exist. This situation can be rectified by the introduction of *mixed*—i.e., randomized—strategies, as first proposed by Borel (1921). Von Neumann (1928) proved that every zero-sum game contains a mixed saddle point, or equilibrium. While equilibria need not be unique, they maintain two appealing properties of saddle points: *interchangeability* (any combination of equilibrium strategies for either player forms an equilibrium) and *equivalence* (all equilibria yield the same expected payoff).

Mixed equilibria have been criticized for resting on demanding epistemic assumptions such as the expected utility axioms by von Neumann and Morgenstern (1947). See, for example, Luce and Raiffa (1957, pp. 74–76) and Fishburn (1978). As Aumann puts it: “When randomized strategies are used in a strategic game, payoff must be replaced by expected payoff. Since the game is played only once, the law of large numbers does not apply, so it is not clear why a player would be interested specifically in the mathematical expectation of his payoff” (Aumann, 1987, p. 63).

Shapley (1953a,b) showed that the existence of saddle points can also be guaranteed by moving to *minimal sets* of actions rather than randomizations over them.[1] Shapley defines a *generalized saddle point (GSP)* to be a tuple of subsets of actions for each player that satisfies a simple external stability condition: Every action not contained in a player's subset is dominated by some action in the set, given that the other player chooses actions from his set. A GSP is minimal if it does not contain another GSP. Minimal GSPs, which Shapley calls *saddles*, also satisfy internal stability in the sense that no two actions within a set dominate each other, given that the other player chooses actions from his set. While Shapley was the first to conceive GSPs, he was not the only one. Apparently unaware of Shapley's work, Samuelson (1992) uses the very related concept of a *consistent pair* to point out epistemic inconsistencies in the concept of iterated weak dominance. Also, *weakly admissible sets* as defined by McKelvey and Ordeshook (1976) in the context of spatial voting games and the *minimal covering set* as defined by Dutta (1988) in the context of majority tournaments are GSPs (Duggan and Le Breton, 1996a).[2]

In this paper, we consider GSPs with respect to weak dominance. An action weakly dominates another action if it always yields at least as much utility. Shapley (1964, p. 10) notes that no general uniqueness result is available for this type of saddle. Later, uniqueness has been shown for restricted classes of zero-sum games, namely tournament games (Dutta, 1988) and confrontation games (Duggan and Le Breton, 1996a). We show that all weak saddles of a given zero-sum game are interchangeable and equivalent. This implies the above-mentioned uniqueness results and shows that every zero-sum game possesses a unique set-based value. Our result can be interpreted as an ordinal variant of the minimax theorem.

2 Preliminaries

A finite two-player zero-sum game is given by a matrix $A = (a_{i,j})_{i\in R, j\in C}$. The finite set R of rows represents the row player's actions, and the finite set C of columns represents the column player's actions. If the row player chooses action $r \in R$, and the column player chooses action $c \in C$, then the *payoff* (or *utility*) of the row player is given by the entry $a_{r,c}$ of the matrix, while the payoff of the column player is given by $-a_{r,c}$. For nonempty subsets $R' \subseteq R$ and $C' \subseteq C$, $A|_{R'\times C'}$ denotes the *subgame* in which the row player has action set R' and the column player has action set C'.

An action $r_1 \in R$ *weakly dominates* another action $r_2 \in R$ with respect to a set $C' \subseteq C$ of columns, denoted $r_1 \geq_{C'} r_2$, if $a_{r_1,c} \geq a_{r_2,c}$ for all $c \in C'$.[3] Similarly, an action $c_1 \in C$ *weakly dominates* another action $c_2 \in C$ with respect to a set $R' \subseteq R$ of rows, denoted $c_1 \leq_{R'} c_2$, if $-a_{r,c_1} \geq -a_{r,c_2}$ (and thus $a_{r,c_1} \leq a_{r,c_2}$) for all $r \in R'$. *Strict dominance* is defined analogously, with the weak inequalities replaced by strict inequalities.

[1] The main results of the 1953 reports later reappeared in revised form (Shapley, 1964).

[2] GSPs have also been considered in the context of general normal-form games (see, e.g., Duggan and Le Breton, 1996b; Brandt et al., 2009, 2011; Brandt and Brill, 2012)

[3] What we call weak dominance here is sometimes also called *very weak dominance* (see, e.g., Leyton-Brown and Shoham, 2008).

Dominance relations can be extended to sets of actions as follows. A set R_1 of rows *weakly* (resp. *strictly*) *dominates* a set R_2 of rows with respect to $C' \subseteq C$ if for every row $r_2 \in R_2$, there exists a row $r_1 \in R_1$ such that r_1 weakly (resp. strictly) dominates r_2 with respect to C'. We denote this by $R_1 \geq_{C'} R_2$ (resp. $R_1 >_{C'} R_2$). Dominance between sets of columns is defined analogously, and denoted $C_1 \leq_{R'} C_2$ (for weak dominance) and $C_1 <_{R'} C_2$ (for strict dominance).

We are now prepared to define *saddles*, which are based on the notion of a *generalized saddle point (GSP)* (Shapley, 1953a,b, 1964). Given a subset $R' \subseteq R$ of rows and a subset $C' \subseteq C$ of columns, the product $R' \times C'$ is a *weak GSP* if $R' \geq_{C'} R \backslash R'$ and $C' \leq_{R'} C \backslash C'$. Furthermore, the product $R' \times C'$ is a *weak saddle* if it is a weak GSP and no proper subset of it is a weak GSP.[4] Strict GSPs and strict saddles are defined analogously.

$$A_1 = \begin{pmatrix} 2 & 1 & 0 & 1 & 2 \\ 0 & 3 & 4 & 4 & 1 \\ 0 & 2 & 2 & 1 & 2 \\ 2 & 1 & 0 & 2 & 1 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \qquad A_3 = \begin{pmatrix} 2 & 2 & 1 & 3 & 2 \\ 2 & 4 & 0 & 0 & 2 \\ 1 & 3 & 3 & 4 & 1 \\ 2 & 3 & 1 & 3 & 2 \\ 1 & 0 & 2 & 2 & 0 \end{pmatrix}$$

Figure 1: Three example zero-sum games. For each game, the rows and columns are labeled $r_1, r_2, \ldots$ and $c_1, c_2, \ldots$, respectively. The game A_1 contains one weak saddle: $\{r_1, r_2\} \times \{c_1, c_2, c_3\}$. The game A_2 contains a saddle point $\{r_1\} \times \{c_1\}$. This saddle point is the unique pure Nash equilibrium and the unique weak saddle of this game. Moreover, $(\frac{1}{2}r_2 + \frac{1}{2}r_3, \frac{1}{2}c_2 + \frac{1}{2}c_3)$ is a (mixed) Nash equilibrium of A_2. The game A_3 contains four weak saddles: $\{r_1, r_3\} \times \{c_1, c_3\}$, $\{r_1, r_3\} \times \{c_3, c_5\}$, $\{r_3, r_4\} \times \{c_1, c_3\}$, and $\{r_3, r_4\} \times \{c_3, c_5\}$. For all three games, the product of all rows and all columns is the unique strict saddle.

In contrast to strict saddles, weak saddles are extensions of saddle points in the sense that every saddle point constitutes a weak saddle. Since the product $R \times C$ containing all actions is a trivial weak and strict GSP of any game, weak and strict saddles are guaranteed to exist. While strict saddles have been shown to be unique in zero-sum games (see Corollary 1), this is not the case for weak saddles. It is noteworthy that saddles generally *cannot* be found by the iterated elimination of (weakly or strictly) dominated actions.[5] See Figure 1 for examples.

[4]Weak saddles have been called *very weak saddles* by Brandt et al. (2011); see also Footnote 3. In some papers (e.g., Duggan and Le Breton, 1996a, 2001; Brandt et al., 2009, 2011), the dominance used for weak saddles requires at least one strict inequality. In the context of confrontation games (see Corollary 2), where weak saddles have usually been considered, both notions of weak saddles coincide. Shapley (1953a,b, 1964) defines weak saddles as we do here. It is easily seen that our theorem does not hold for weak saddles that require at least one strict inequality (see, for example, the restriction to the first two rows and columns of game A_2 in Figure 1).

[5]While the subgames generated by iteratively eliminating dominated strategies are GSPs, these GSPs

3 The Result

In this section, we prove that weak saddles in zero-sum games are interchangeable and equivalent. We begin with a lemma.

Lemma 1. *Consider a zero-sum game A with row set R and column set C. Let $R_1 \subseteq R_2 \subseteq R$ and $C_1 \subseteq C_2 \subseteq C$. Suppose that $R_2 \times C_2$ is a weak GSP. Then, $R_1 \times C_1$ is a weak GSP if and only if $R_1 \times C_1$ is a weak GSP in $A|_{R_2 \times C_2}$.*

Proof. The "only if" part follows straightforwardly from the definitions. For the "if" part, suppose that $R_1 \times C_1$ is a weak GSP in $A|_{R_2 \times C_2}$. We will show that $R_1 \geq_{C_1} R \backslash R_1$; the argument for column domination is analogous. Consider an arbitrary row $r \in R \backslash R_1$. If $r \in R_2 \backslash R_1$, then since $R_1 \times C_1$ is a weak GSP in $A|_{R_2 \times C_2}$, there exists a row $r' \in R_1$ such that $r' \geq_{C_1} r$. Otherwise, we have $r \in R \backslash R_2$. Since $R_2 \times C_2$ is a weak GSP in A, there exists a row $r' \in R_2$ such that $r' \geq_{C_2} r$, and in particular $r' \geq_{C_1} r$. But since $R_1 \times C_1$ is a weak GSP in $A|_{R_2 \times C_2}$, there exists a row $r'' \in R_1$ such that $r'' \geq_{C_1} r'$. It follows that $r'' \geq_{C_1} r$, and hence $R_1 \geq_{C_1} R \backslash R_1$. □

We are now ready to prove the main theorem.

Theorem 1. *Let A be a zero-sum game with weak saddles $R_1 \times C_1$ and $R_2 \times C_2$. Then the following statements are true.*

(i) *$R_1 \times C_2$ and $R_2 \times C_1$ are also weak saddles (interchangeability).*

(ii) *The subgame $A|_{R_2 \times C_2}$ can be derived from $A|_{R_1 \times C_1}$ by permuting the rows and columns (equivalence). In particular, $|R_1| = |R_2|$ and $|C_1| = |C_2|$, and the multisets of entries of $A|_{R_1 \times C_1}$ and $A|_{R_2 \times C_2}$ are the same.*

Proof. The general proof structure is as follows. We first show both statements for the case in which every row of one saddle dominates some row of the other saddle with respect to the columns of the first saddle (and similarly for the columns). We then consider the more difficult case in which a saddle contains an "idle" action that does not dominate any action of the other saddle. It turns out that this violates minimality of the saddle and is therefore not possible.

Suppose that $R_1 \times C_1$ and $R_2 \times C_2$ are weak saddles, with $|R_1| = p_1, |R_2| = p_2, |C_1| = q_1$, and $|C_2| = q_2$. We label the saddle actions in R_1 with $1, \ldots, p_1$, and similarly for the actions in R_2, C_1, and C_2. Since $R_1 \times C_1$ is a weak GSP, for each row $r_2 \in R_2$ there exists a row $r_1 \in R_1$ such that $r_1 \geq_{C_1} r_2$. (If $r_2 \in R_1 \cap R_2$, then $r_2 \geq_{C_1} r_2$.) Hence there exists a function $f_1 : [p_2] \to [p_1]$ such that $f_1(i) \in R_1$ and $f_1(i) \geq_{C_1} i$ for every row $i \in R_2$, where $[n]$ denotes the set $\{1, \ldots, n\}$. Similarly, there exist functions $f_2 : [p_1] \to [p_2]$, $g_1 : [q_2] \to [q_1]$, and $g_2 : [q_1] \to [q_2]$ such that $f_2(i) \in R_2$ and $f_2(i) \geq_{C_2} i$ for every row $i \in R_1$, $g_1(j) \in C_1$

need not be minimal.

and $g_1(j) \leq_{R_1} j$ for every column $j \in C_2$, and $g_2(j) \in C_2$ and $g_2(j) \leq_{R_2} j$ for every column $j \in C_1$.

Suppose that f_1, f_2, g_1, and g_2 are all bijections (which in particular implies that $p_1 = p_2$ and $q_1 = q_2$). Then the rows in $R_2 \times C_1$ are dominated by the rows in $R_1 \times C_1$, one by one. Hence $\sum_{i \in R_1, j \in C_1} a_{ij} \geq \sum_{i \in R_2, j \in C_1} a_{ij}$. By putting together these inequalities for $R_1 \times C_1$, $R_2 \times C_1$, $R_2 \times C_2$, and $R_1 \times C_2$, we get

$$\sum_{i \in R_1, j \in C_1} a_{ij} \geq \sum_{i \in R_2, j \in C_1} a_{ij} \geq \sum_{i \in R_2, j \in C_2} a_{ij} \geq \sum_{i \in R_1, j \in C_2} a_{ij} \geq \sum_{i \in R_1, j \in C_1} a_{ij}.$$

It follows that equality holds everywhere and that rows are only dominated by identical rows and that columns are only dominated by identical columns. As a consequence, $R_2 \times C_2$ can be derived from $R_1 \times C_1$ by permuting the rows and columns. Moreover, $R_1 \times C_2$ and $R_2 \times C_1$ can also be derived from $R_1 \times C_1$ by permuting the rows and columns, and one can check that they are saddles in $R \times C$ as well. Hence both (i) and (ii) hold in this case.

Suppose now that at least one of f_1, f_2, g_1, and g_2 is not a bijection. Then at least one of them is not a surjection. Indeed, if for instance $p_1 < p_2$, then f_2 is not a surjection. If $p_1 = p_2$ and $q_1 = q_2$, and any of the functions f_1, f_2, g_1, g_2 is not a bijection, then it is also not a surjection. Assume without loss of generality that f_1 is not a surjection, i.e., $R_1 \times C_1$ contains an idle row that does not dominate any row in R_2 with respect to C_1. We will show that $R_1 \times C_1$ is not inclusion-minimal, i.e., there exists a proper subset $R_1' \times C_1' \subset R_1 \times C_1$ such that $R_1' \times C_1'$ is a weak GSP. But since $R_1 \times C_1$ is a weak GSP, by Lemma 1 we only need to show that there exists a proper subset $R_1' \times C_1' \subset R_1 \times C_1$ such that $R_1' \times C_1'$ is a weak GSP in $A|_{R_1 \times C_1}$.

We index the entries of $A|_{R_1 \times C_1}$ by $(x_{i,j})_{i \in [p_1], j \in [q_1]}$, and the entries of $A|_{R_2 \times C_2}$ by $(y_{i,j})_{i \in [p_2], j \in [q_2]}$. We have $x_{f_1(i),j} \geq y_{i,g_2(j)}$ for all $(i,j) \in [p_2] \times [q_1]$ and $x_{i,g_1(j)} \leq y_{f_2(i),j}$ for all $(i,j) \in [p_1] \times [q_2]$. Hence for all $j \in [q_2]$ and all $k \in [q_1]$ such that $g_2(k) = j$, we have $x_{f_1(i),k} \geq y_{i,j}$ for all $i \in [p_2]$. Define $x_{f_1(i),g_2^{-1}(j)} := \{x_{f_1(i),k} \mid g_2(k) = j\}$, and for two sets S, T, write $S \geq T$ if and only if $s \geq t$ for all $s \in S$ and $t \in T$. Using this notation, $x_{f_1(i),g_2^{-1}(j)} \geq y_{i,j}$ for all $(i,j) \in [p_2] \times [q_2]$. Similarly, we have that $x_{f_2^{-1}(i),g_1(j)} \leq y_{i,j}$ for all $(i,j) \in [p_2] \times [q_2]$. See Figure 2 for an illustration. It follows that

$$x_{f_1(i),g_2^{-1}(j)} \geq x_{f_2^{-1}(i),g_1(j)} \tag{$*$}$$

for all $(i,j) \in [p_2] \times [q_2]$. This inequality will be leveraged later in the proof. If $f_2^{-1}(i) = \emptyset$ for some i, the inequality is meaningless for that index i and we may remove the index from consideration. A similar statement holds for g_2. We may relabel the remaining indices as $1, \ldots, p_3$ and $1, \ldots, q_3$, respectively, so that $[p_3] = \mathrm{Im}(f_2)$ and $[q_3] = \mathrm{Im}(g_2)$, where $\mathrm{Im}(f)$ is the image of a function f. We have $\bigcup_{i \in [p_3]} f_2^{-1}(i) = [p_1]$ and $\bigcup_{j \in [q_3]} g_2^{-1}(j) = [q_1]$.

Consider the product $S = \mathrm{Im}(f_1 \circ f_2) \times \mathrm{Im}(g_1 \circ g_2)$. Since f_1 is not surjective, we have that S is a proper subset of $R_1 \times C_1$. Hence it suffices to show that there exists a subset of S that is a weak GSP in $A|_{R_1 \times C_1}$.

We now define a directed graph $\mathcal{G}_R$ as follows. The nodes of $\mathcal{G}_R$ are given by $1, 2, \ldots, p_3$. For each i, we include a directed edge from $f_2(f_1(i))$ to i. Let S_R be the set of nodes in

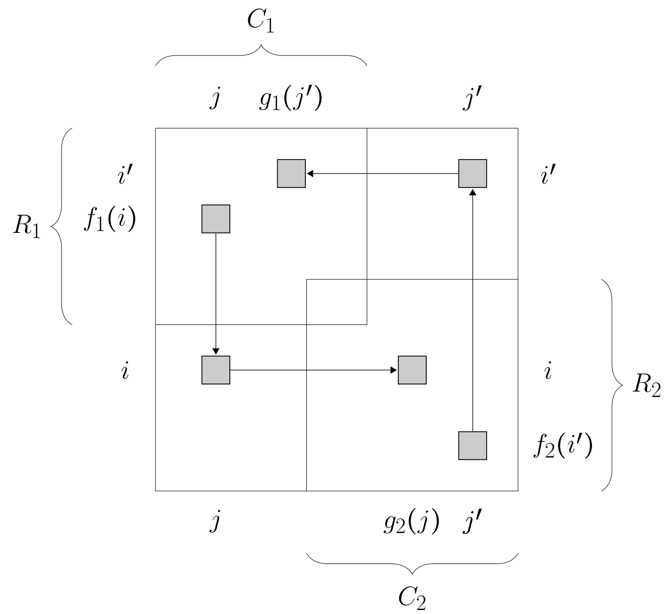

Figure 2: Construction in the proof of Theorem 1. An arrow from one entry to another indicates that the former is greater than or equal to the latter.

$\mathcal{G}_R$ that belong to a directed cycle. (A self-loop counts as a directed cycle.) Each node in $\mathcal{G}_R$ has exactly one incoming edge, and any node in $\mathcal{G}_R$ can be reached from a node in S_R. Similarly, we define a directed graph $\mathcal{G}_C$ with nodes $1, 2, \ldots, q_3$. For each i, we include a directed edge from i to $g_2(g_1(i))$. Each node in $\mathcal{G}_C$ has exactly one outgoing edge, and any node in $\mathcal{G}_C$ can reach a node in S_C, where S_C is the set of nodes in $\mathcal{G}_C$ that belong to a directed cycle.

Suppose that $\mathcal{G}_R$ contains an edge $i_2 \to i_1$ and $\mathcal{G}_C$ contains an edge $j_1 \to j_2$. Then $f_1(i_1) \in f_2^{-1}(i_2)$ and $g_1(j_1) \in g_2^{-1}(j_2)$. From $(*)$, we have $x_{f_2^{-1}(i_1),g_1(j_2)} \le x_{f_1(i_1),g_2^{-1}(j_2)}$ and $x_{f_2^{-1}(i_2),g_1(j_1)} \le x_{f_1(i_2),g_2^{-1}(j_1)}$. Since $x_{f_1(i_1),g_1(j_1)}$ belongs to both $x_{f_1(i_1),g_2^{-1}(j_2)}$ and $x_{f_2^{-1}(i_2),g_1(j_1)}$, we have $x_{f_2^{-1}(i_1),g_1(j_2)} \le x_{f_1(i_1),g_1(j_1)} \le x_{f_1(i_2),g_2^{-1}(j_1)}$. In particular, we have

$$x_{f_2^{-1}(i_1),g_1(j_2)} \le x_{f_1(i_2),g_2^{-1}(j_1)}.$$

Suppose now that $\mathcal{G}_R$ contains edges $i_n \to i_{n-1} \to \cdots \to i_1$ and $\mathcal{G}_C$ contains edges $j_1 \to j_2 \to \cdots \to j_n$. Then $f_1(i_k) \in f_2^{-1}(i_{k+1})$ and $g_1(j_k) \in g_2^{-1}(j_{k+1})$ for all $k \in [n-1]$. Applying the same argument as in the $n = 2$ case repeatedly, we have $x_{f_2^{-1}(i_1),g_1(j_n)} \le x_{f_1(i_n),g_2^{-1}(j_1)}$.

We claim that $S' := \bigcup_{i \in S_R} f_1(i) \times \bigcup_{j \in S_C} g_1(j) \subseteq S$ is a weak GSP in $A|_{R_1 \times C_1}$. To prove this claim, it suffices to consider row domination; column domination follows analogously.

For each node $x \in S_C$, define $c(x) = g_2(g_1(x))$, i.e., c maps a node to its successor in graph $\mathcal{G}_C$. We must show that for any $i \in [p_1]$, there exists $j \in S_R$ such that $x_{f_1(j),g_1(k)} \ge x_{i,g_1(k)}$ for all $k \in S_C$. Since $g_1(k) \in g_2^{-1}(c(k))$, it suffices to show that for any $i \in [p_1]$, there exists

$j \in S_R$ such that $x_{f_1(j),g_2^{-1}(c(k))} \geq x_{i,g_1(k)}$ for all $k \in S_C$. Moreover, since $\bigcup_{i\in[p_3]} f_2^{-1}(i) = [p_1]$, we only need to show that for any $i \in [p_3]$, there exists $j \in S_R$ such that $x_{f_1(j),g_2^{-1}(c(k))} \geq x_{f_2^{-1}(i),g_1(k)}$ for all $k \in S_C$.

Let M denote the least common multiple of all the cycle lengths in S_C. For any positive integer n and any node $k \in S_C$, there exists a path of length $nM - 1$ in S_C (and hence in $\mathcal{G}_C$) that begins with $c(k)$ and ends with k. Since every node in $\mathcal{G}_R$ has one incoming edge, for large enough n' there exists a path of length n' in $\mathcal{G}_R$ that begins with some node $j \in S_R$ and ends with i. Taking $n'' = nM - 1$ for large enough n, there exists a path of length n'' in $\mathcal{G}_C$ that begins with $c(k)$ and ends with k, and a path of length n'' in $\mathcal{G}_R$ that begins with $j \in S_R$ and ends with i. It follows that $x_{f_1(j),g_2^{-1}(c(k))} \geq x_{f_2^{-1}(i),g_1(k)}$. Thus S' is a weak GSP, contradicting the minimality of $R_1 \times C_1$. Hence the case in which at least one of f_1, f_2, g_1 and g_2 is not a bijection cannot occur, and the proof is complete. □

Every weak saddle $R' \times C'$ of a given zero-sum game A defines a subgame $A|_{R'\times C'}$. It follows from Theorem 1 that all such subgames are identical up to the permutation of rows and columns. As a consequence, $A|_{R'\times C'}$ could be considered the "essence" (or set-based value) of A. Moreover, every Nash equilibrium of $A|_{R'\times C'}$ is a Nash equilibrium of A. Therefore, every weak saddle contains the support of a Nash equilibrium and the (von Neumann) value of $A|_{R'\times C'}$ is the same as that of the original game A.[6]

4 Consequences and Remarks

Shapley (1953a) has shown that every zero-sum game contains a unique *strict* saddle. Shapley's proof crucially relies on the minimax theorem (and the interchangeability of minimax strategies).[7] Shapley's result can be obtained as a corollary of Theorem 1 by leveraging the interchangeability of weak saddles.

Corollary 1 (Shapley, 1953a)**.** *Every zero-sum game contains a unique strict saddle.*

Proof. Let $R_1 \times C_1$ and $R_2 \times C_2$ be two distinct strict saddles. We first show that $R_1 \cap R_2 \neq \varnothing$ and $C_1 \cap C_2 \neq \varnothing$. Without loss of generality we may assume for contradiction that $R_1 \cap R_2 = \varnothing$. Every strict GSP is also a weak GSP and therefore contains a weak saddle. Let $R_1' \times C_1'$ be a weak saddle such that $R_1' \subseteq R_1$ and $C_1' \subseteq C_1$ and let $R_2' \times C_2'$ be a weak saddle such that $R_2' \subseteq R_2$ and $C_2' \subseteq C_2$. Theorem 1 implies that $R_2' \times C_1'$ is also a weak saddle. Note that $R_2' \cap R_1 = \varnothing$. Now let r_1 be an arbitrary row in R_1. Since $R_2' \times C_1'$ is a weak saddle, there must be $r_2 \in R_2'$ such that $r_1 \leq_{C_1'} r_2$. Since $R_1 \times C_1$ is a strict saddle, there has to be $r_3 \in R_1$ such that $r_3 >_{C_1} r_2$ (and hence also $r_3 >_{C_1'} r_2$). The strict inequality implies that

[6] However, game A_2 in Figure 1 shows that there can also be Nash equilibria whose support is disjoint from all weak saddles.

[7] Shapley notes that Hyman Bass proved this statement without making reference to the minimax theorem, but Bass's report is unavailable.

$r_3 \neq r_1$. Alternating these two arguments, we obtain a sequence of rows r_1, r_2, ... such that

$$r_1 \leq_{C'_1} r_2 <_{C'_1} r_3 \leq_{C'_1} r_4 <_{C'_1} \cdots$$

All r_i have to be distinct and finiteness of the game implies that we will eventually find a row r_k, which is witness to the fact that either $R'_2 \times C'_1$ is not a weak saddle or $R_1 \times C_1$ is not a strict saddle, both of which are contradictions.

We thus have that $R_1 \cap R_2 \neq \varnothing$ and $C_1 \cap C_2 \neq \varnothing$. An argument using a chain of strict inequalities (similar to the one above) easily shows that strict GSPs are closed under non-empty intersection, i.e., $(R_1 \cap R_2) \times (C_1 \cap C_2)$ is a strict GSP. Hence, at least one of the two original strict GSPs was not minimal, a contradiction. □

A zero-sum game A with row set R and column set C is *symmetric* if $R = C$ (with a slight abuse of notation) and the payoff matrix A is skew-symmetric. In particular, all entries on the main diagonal of A are zero. A *confrontation game* is a symmetric zero-sum game in which zeroes appear *only* on the main diagonal. Strengthening a result by Dutta (1988), Duggan and Le Breton (1996a) have shown that every confrontation game contains a unique weak saddle.[8] This now also follows as a consequence of Theorem 1.

Corollary 2 (Duggan and Le Breton, 1996a)**.** *Every confrontation game contains a unique weak saddle.*

Proof. Let A be a confrontation game. A weak saddle $R' \times C'$ is called *symmetric* if $R' = C'$. Hence, a weak saddle $R' \times C'$ is symmetric if and only if every row and every column of $A|_{R' \times C'}$ contains exactly one zero.

Assume for contradiction that A has two distinct weak saddles, $R_1 \times C_1$ and $R_2 \times C_2$. We consider the following two cases.

Case 1: At least one of the two weak saddles is not symmetric. Assume without loss of generality that $R_1 \times C_1$ is not symmetric. Since A is skew-symmetric, $C_1 \times R_1$ is also a weak saddle. By the first part of Theorem 1, $R_1 \times R_1$ is a symmetric weak saddle. Hence, $R_1 \times R_1$ contains exactly one zero in each row and each column, while $R_1 \times C_1$ does not. This contradicts the second part of Theorem 1.

Case 2: Both weak saddles are symmetric. Then $R_1 \times C_2$ is an asymmetric saddle, and we obtain a contradiction in the same way as in Case 1. □

We conclude the paper with three remarks.

Remark 1. *If all payoffs of a game are pairwise distinct, strict dominance and weak dominance coincide. Thus, every such zero-sum game contains a unique weak saddle.*

Remark 2. *Duggan and Le Breton (2001) defined refinements of (weak and strict) saddles based on* mixed *dominance. Game A_2 in Figure 1 shows that Theorem 1 does* not *hold for mixed weak saddles.*

[8] By contrast, Nash equilibria are not unique in confrontation games (Le Breton, 2005).

Remark 3. *A zero-sum game may contain an exponential number of weak saddles (Brandt and Brill, 2012). Thus, computing* all *weak saddles of a game is not feasible in polynomial time. The computational complexity of finding* some *weak saddle of a zero-sum game is open.*[9]

Acknowledgements

This material is based on work supported by Deutsche Forschungsgemeinschaft under grants BR 2312/7-2 and BR 2312/10-1, by a Feodor Lynen research fellowship of the Alexander von Humboldt Foundation, and by the MIT-Germany program. The authors thank Vincent Conitzer and Paul Harrenstein for helpful discussions.

References

R. J. Aumann. Game theory. In J. Eatwell, M. Milgate, and P. Newman, editors, *The New Palgrave: A Dictionary of Economics*, volume 2, pages 460–482. MacMillan, 1987.

E. Borel. La théorie du jeu et les équations intégrales à noyau symétrique. *Comptes Rendus de l'Académie des Sciences*, 173:1304–1308, 1921.

F. Brandt and M. Brill. Computing dominance-based solution concepts. In *Proceedings of the 13th ACM Conference on Electronic Commerce (ACM-EC)*, page 233. ACM Press, 2012.

F. Brandt, M. Brill, F. Fischer, and P. Harrenstein. Computational aspects of Shapley's saddles. In *Proceedings of the 8th International Conference on Autonomous Agents and Multi-Agent Systems (AAMAS)*, pages 209–216. IFAAMAS, 2009.

F. Brandt, M. Brill, F. Fischer, and J. Hoffmann. The computational complexity of weak saddles. *Theory of Computing Systems*, 49(1):139–161, 2011.

J. Duggan and M. Le Breton. Dutta's minimal covering set and Shapley's saddles. *Journal of Economic Theory*, 70(1):257–265, 1996a.

J. Duggan and M. Le Breton. Dominance-based solutions for strategic form games. Mimeo, 1996b.

J. Duggan and M. Le Breton. Mixed refinements of Shapley's saddles and weak tournaments. *Social Choice and Welfare*, 18(1):65–78, 2001.

B. Dutta. Covering sets and a new Condorcet choice correspondence. *Journal of Economic Theory*, 44(1):63–80, 1988.

[9] For two-player games that are not zero-sum, finding a weak saddle has been shown to be NP-hard (Brandt et al., 2011).

P. C. Fishburn. Non-cooperative stochastic dominance games. *International Journal of Game Theory*, 7(1):51–61, 1978.

M. Le Breton. On the uniqueness of equilibrium in symmetric two-player zero-sum games with integer payoffs. *Économie publique*, 17(2):187–195, 2005.

K. Leyton-Brown and Y. Shoham. *Essentials of Game Theory: A Concise, Multidisciplinary Introduction*. Morgan & Claypool, 2008.

R. D. Luce and H. Raiffa. *Games and Decisions: Introduction and Critical Survey*. John Wiley & Sons Inc., 1957.

R. D. McKelvey and P. C. Ordeshook. Symmetric spatial games without majority rule equilibria. *The American Political Science Review*, 70(4):1172–1184, 1976.

L. Samuelson. Dominated strategies and common knowledge. *Games and Economic Behavior*, 4(2):284–313, 1992.

L. Shapley. Order matrices. I. Technical Report RM-1142, The RAND Corporation, 1953a.

L. Shapley. Order matrices. II. Technical Report RM-1145, The RAND Corporation, 1953b.

L. Shapley. Some topics in two-person games. In M. Dresher, L. S. Shapley, and A. W. Tucker, editors, *Advances in Game Theory*, volume 52 of *Annals of Mathematics Studies*, pages 1–29. Princeton University Press, 1964.

J. von Neumann. Zur Theorie der Gesellschaftspiele. *Mathematische Annalen*, 100(1): 295–320, 1928.

J. von Neumann and O. Morgenstern. *Theory of Games and Economic Behavior*. Princeton University Press, 2nd edition, 1947.