In the early 1950s Lloyd Shapley proposed an ordinal and set-valued solution concept for zero-sum games called *weak saddle*. We show that all weak saddles of a given zero-sum game are interchangeable and equivalent. As a consequence, every such game possesses a unique set-based value.

### 1 Introduction

One of the earliest solution concepts considered in game theory are *saddle points*, combinations of actions such that no player can gain by deviating (see, e.g., von Neumann and Morgenstern, 1947). In two-player zero-sum games, every saddle point happens to coincide with the optimal outcome both players can guarantee in the worst case and thus enjoys a very strong normative foundation. Unfortunately, however, not every zero-sum game possesses a saddle point. In order to remedy this situation, Borel (1921) introduced *mixed*—i.e., randomized—strategies and von Neumann (1928) proved that every zero-sum game contains a mixed saddle point, or equilibrium, that moreover maintains two appealing properties of saddle points: *interchangeability* and *equivalence*. A zero-sum game may contain more than one equilibrium, but any combination of equilibrium strategies for either player forms an equilibrium (interchangeability), and all equilibria yield the same expected payoff (equivalence).

Mixed equilibria have been criticized for resting on demanding epistemic assumptions such as the expected utility axioms by von Neumann and Morgenstern (1947). See, for example, Luce and Raiffa (1957, pp. 74–76) and Fishburn (1978). A compelling critique by Aumann addresses the use of mixed strategies in one-shot games: “When randomized strategies are used in a strategic game, payoff must be replaced by expected payoff. Since the game is played only once, the law of large numbers does not apply, so it is not clear why a player would be interested specifically in the mathematical expectation of his payoff” (Aumann, 1987, p. 63).
Shapley (1953a,b) showed that the existence of saddle points can also be guaranteed by moving to minimal sets of actions rather than randomizations over them.\footnote{The main results of the 1953 reports later reappeared in revised form (Shapley, 1964).} Shapley defines a generalized saddle point (GSP) to be a tuple of subsets of actions for each player that satisfies a simple external stability condition: Every action not contained in a player’s subset is dominated by some action in the set, given that the remaining players choose actions from their respective sets. A GSP is minimal if it does not contain another GSP. Minimal GSPs, which Shapley calls saddles, also satisfy internal stability in the sense that no two actions within a set dominate each other, given that the remaining players choose actions from their respective sets. While Shapley was the first to conceive GSPs, he was not the only one. Apparently unaware of Shapley’s work, Samuelson (1992) uses the very related concept of a consistent pair to point out epistemic inconsistencies in the concept of iterated weak dominance. Also, weakly admissible sets as defined by McKelvey and Ordeshook (1976) in the context of spatial voting games and the minimal covering set as defined by Dutta (1988) in the context of majority tournaments are GSPs (Duggan and Le Breton, 1996a).\footnote{GSPs have also been considered in the context of general normal-form games (see, e.g., Duggan and Le Breton, 1996b, 2001; Brandt et al., 2009, 2011; Brandt and Brill, 2012).}

In this paper, we consider GSPs with respect to (very) weak dominance, i.e., an action dominates another action if it always yields at least as much utility no matter which action is selected by the other player. Shapley (1964, p. 10) notes that no general uniqueness result is available for this type of saddle. Later, uniqueness has been shown for restricted classes of zero-sum games, namely tournament games (Dutta, 1988) and confrontation games (Duggan and Le Breton, 1996a). We show that all saddles of a given zero-sum game are interchangeable and equivalent. This implies the above-mentioned uniqueness results and shows that every zero-sum game possesses a unique set-based value, which can be interpreted as an ordinal variant of the minimax theorem.

2 Preliminaries

A (finite) (two-player) zero-sum game is given by a matrix \( A = (a_{ij})_{i \in R, j \in C} \). The finite set \( R \) of rows represents the row player’s actions, and the finite set \( C \) of columns represents the column player’s actions. If the row player chooses action \( r \in R \), and the column player chooses action \( c \in C \), then the payoff (or utility) of the row player is given by the entry \( u_1(r, c) := a_{rc} \) of the matrix, while the payoff of the column player is given by the negative \( u_2(r, c) = -a_{rc} \). For nonempty subsets \( R' \subseteq R \) and \( C' \subseteq C \), \( A|_{R' \times C'} \) denotes the subgame in which the row player has action set \( R' \) and the column player has action set \( C' \).

An action \( r_1 \in R \) weakly dominates another action \( r_2 \in R \) with respect to a set \( C' \subseteq C \) of columns, denoted \( r_1 \succeq_{C'} r_2 \), if \( u_1(r_1, c) \geq u_1(r_2, c) \) for all \( c \in C' \).\footnote{What we call weak dominance here is sometimes also called very weak dominance (see, e.g., Leyton-Brown and Shoham, 2008).} Similarly, an action \( c_1 \in C \) weakly dominates another action \( c_2 \in C \) with respect to a set \( R' \subseteq R \) of rows,
denoted \( c_1 \leq_R c_2 \), if \( u_2(c_1, r) \geq u_2(c_2, r) \) (and thus \( u_1(c_1, r) \leq u_1(c_2, r) \)) for all \( r \in R' \). **Strict dominance** is defined analogously, with the weak inequalities replaced by strict inequalities.

Dominance relations can be extended to sets of actions as follows. A set \( R_1 \) of rows weakly (resp. strictly) dominates a set \( R_2 \) of rows with respect to \( C' \subseteq C \) if for every row \( r_2 \in R_2 \), there exists a row \( r_1 \in R_1 \) such that \( r_1 \) weakly (resp. strictly) dominates \( r_2 \) with respect to \( C' \). We denote this by \( R_1 \geq_C R_2 \) (resp. \( R_1 >_C R_2 \)). Dominance between sets of columns is defined analogously, and denoted \( C_1 \leq_R C_2 \) (for weak dominance) and \( C_1 \leq_R C_2 \) (for strict dominance).

We are now prepared to define **saddles**, which are based on the notion of a **generalized saddle point (GSP)** (Shapley, 1953a,b, 1964). Given a subset \( R' \subseteq R \) of rows and a subset \( C' \subseteq C \) of columns, the product \( R' \times C' \) is a **weak GSP** if \( R' \geq_{C'} R \setminus R' \) and \( C \leq_{R'} C \setminus C' \). Furthermore, the product \( R' \times C' \) is a **weak saddle** if it is a weak GSP and no proper subset of it is a weak GSP.\(^4\) Strict GSPs and strict saddles are defined analogously.

\[
A_1 = \begin{pmatrix}
2 & 1 & 0 & 1 & 2 \\
0 & 3 & 4 & 4 & 1 \\
0 & 2 & 2 & 1 & 2 \\
2 & 1 & 0 & 2 & 1
\end{pmatrix} \quad A_2 = \begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & -1 \\
0 & -1 & 1
\end{pmatrix} \quad A_3 = \begin{pmatrix}
2 & 2 & 1 & 3 & 2 \\
2 & 4 & 0 & 0 & 2 \\
1 & 3 & 3 & 4 & 1 \\
2 & 3 & 1 & 3 & 2 \\
1 & 0 & 2 & 2 & 0
\end{pmatrix}
\]

Figure 1: Three example zero-sum games. For each game, the rows are labeled \( r_1, r_2, \ldots \) and the columns are labeled \( c_1, c_2, \ldots \). The game \( A_1 \) contains one weak saddle: \( \{r_1, r_2\} \times \{c_1, c_2, c_3\} \). The game \( A_2 \) contains a saddle point \( \{r_1\} \times \{c_1\} \). This saddle point is the unique pure Nash equilibrium and the unique weak saddle of this game. Moreover, \( \frac{1}{2}r_2 + \frac{1}{2}r_3, \frac{1}{2}c_2 + \frac{1}{2}c_3 \) is a (mixed) Nash equilibrium of \( A_2 \). The game \( A_3 \) contains four weak saddles: \( \{r_1, r_3\} \times \{c_1, c_3\}, \{r_1, r_3\} \times \{c_3, c_5\}, \{r_3, r_4\} \times \{c_1, c_3\}, \) and \( \{r_3, r_4\} \times \{c_3, c_5\} \). For all three games, the product of all rows and all columns is the (unique) strict saddle.

In contrast to strict saddles, weak saddles are extensions of saddle points in the sense that every saddle point is a weak saddle. Since the product \( R \times C \) containing all actions is a trivial weak and strict GSP of any game, weak and strict saddles are guaranteed to exist. While strict saddles have been shown to be unique in zero-sum games (see Corollary 1), this is not the case for weak saddles. It is noteworthy that saddles generally cannot be

\(^4\)Weak saddles have been called **very weak saddles** by Brandt et al. (2011); see also Footnote 3. In some papers (e.g., Duggan and Le Breton, 1996a, 2001; Brandt et al., 2009, 2011), the dominance used for weak saddles requires at least one strict inequality. In the context of confrontation games (see Corollary 2), where weak saddles have usually been considered, both notions of weak saddles coincide. Shapley (1953a,b, 1964) defines weak saddles as we do here. It is easily seen that our theorem does not hold for weak saddles that require at least one strict inequality (see, for example, the restriction to the first two rows and columns of game \( A_2 \) in Figure 1).
found by the iterated elimination of (weakly or strictly) dominated actions. See Figure 1 for examples.

3 The Result

In this section, we prove that weak saddles in zero-sum games are interchangeable and equivalent. We begin with a lemma.

**Lemma 1.** Consider a zero-sum game \( A \) with row set \( R \) and column set \( C \). Let \( R_1 \subseteq R_2 \subseteq R \) and \( C_1 \subseteq C_2 \subseteq C \). Suppose that \( R_2 \times C_2 \) is a weak GSP. Then, \( R_1 \times C_1 \) is a weak GSP if and only if \( R_1 \times C_1 \) is a weak GSP in \( A|_{R_2 \times C_2} \).

**Proof.** The “only if” part follows straightforwardly from the definitions. For the “if” part, suppose that \( R_1 \times C_1 \) is a weak GSP in \( A|_{R_2 \times C_2} \). We will show that \( R_1 \geq C_1 \) \( R\backslash R_1 \); the argument for column domination is analogous. Consider an arbitrary row \( r \in R\backslash R_1 \). If \( r \in R_2 \backslash R_1 \), then since \( R_1 \times C_1 \) is a weak GSP in \( A|_{R_2 \times C_2} \), there exists a row \( r' \in R_1 \) such that \( r' \geq C_1 \) \( r \). Otherwise, we have \( r \in R\backslash R_2 \). Since \( R_2 \times C_2 \) is a weak GSP in \( A \), there exists a row \( r' \in R_2 \) such that \( r' \geq C_2 \) \( r \), and in particular \( r' \geq C_1 \) \( r \). But since \( R_1 \times C_1 \) is a weak GSP in \( A|_{R_2 \times C_2} \), there exists a row \( r'' \in R_1 \) such that \( r'' \geq C_1 \) \( r' \). It follows that \( r'' \geq C_1 \) \( r \), and hence \( R_1 \geq C_1 \) \( R\backslash R_1 \).

We are now ready to prove the main theorem.

**Theorem 1.** Let \( A \) be a zero-sum game with weak saddles \( R_1 \times C_1 \) and \( R_2 \times C_2 \). Then the following are true:

(i) \( R_1 \times C_2 \) and \( R_2 \times C_1 \) are also weak saddles (interchangeability).

(ii) The subgame \( A|_{R_2 \times C_2} \) can be derived from \( A|_{R_1 \times C_1} \) by permuting the rows and columns (equivalence). In particular, \( |R_1| = |R_2| \) and \( |C_1| = |C_2| \), and the multisets of entries of \( A|_{R_1 \times C_1} \) and \( A|_{R_2 \times C_2} \) are the same.

**Proof.** Suppose that \( R_1 \times C_1 \) and \( R_2 \times C_2 \) are weak saddles, with \( |R_1| = p_1, |R_2| = p_2, |C_1| = q_1 \), and \( |C_2| = q_2 \). Since \( R_1 \times C_1 \) is a weak GSP, for each row \( r_2 \in R_2 \) there exists a row \( r_1 \in R_1 \) such that \( r_1 \geq C_1 \ r_2 \). (If \( r_2 \in R_1 \cap R_2 \), then \( r_2 \geq C_1 \ r_2 \).) Hence there exists a function \( f_1 : [p_2] \rightarrow [p_1] \) such that \( f_1(i) \geq C_1 \ i \) for every row \( i \in R_2 \) and \( f_1(i) \in R_1 \), where \([n]\) denotes the set \( \{1, \ldots, n\} \). Similarly, there exist functions \( f_2 : [p_1] \rightarrow [p_2], g_1 : [q_2] \rightarrow [q_1], \) and \( g_2 : [q_1] \rightarrow [q_2] \) such that \( f_2(i) \geq C_2 \ i \) for every column \( i \in C_2 \) and \( f_2(i) \in R_2 \), \( g_1(i) \leq R_1 \ i \) for every column \( i \in C_2 \) and \( g_1(i) \in C_1 \), and \( g_2(i) \leq R_2 \ i \) for every column \( i \in C_1 \) and \( g_2(i) \in C_2 \).

Suppose that \( f_1, f_2, g_1, \) and \( g_2 \) are all bijections (which in particular implies that \( p_1 = p_2 \) and \( q_1 = q_2 \).) Then the rows in \( R_2 \times C_1 \) are dominated by the rows in \( R_1 \times C_1 \), one by one. Hence \( \text{sum}(R_1 \times C_1) \geq \text{sum}(R_2 \times C_1) \), where \( \text{sum}(R' \times C') \) denotes the sum of the entries of the submatrix \( A|_{R' \times C'} \). Similarly, we have \( \text{sum}(R_2 \times C_1) \geq \text{sum}(R_2 \times C_2), \text{sum}(R_2 \times C_2) \geq \text{sum}(R_1 \times C_2), \) and \( \text{sum}(R_1 \times C_2) \geq \text{sum}(R_1 \times C_1) \). It follows that equality
holds everywhere, and hence \( R_2 \times C_2 \) can be derived from \( R_1 \times C_1 \) by permuting the rows and columns. Moreover, \( R_1 \times C_2 \) and \( R_2 \times C_1 \) can also be derived from \( R_1 \times C_1 \) by permuting the rows and columns, and one can check that they are saddles in \( R \times C \) as well. Hence both (i) and (ii) hold in this case.

Suppose now that at least one of \( f_1, f_2, g_1, \) and \( g_2 \) is not a bijection. Then at least one of them is not a surjection. Indeed, if for instance \( p_1 < p_2 \), then \( f_2 \) is not a surjection. If \( p_1 = p_2 \) and \( q_1 = q_2 \), and any of the functions \( f_1, f_2, g_1, g_2 \) is not a bijection, then it is also not a surjection. Assume without loss of generality that \( f_1 \) is not a surjection. We will show that \( R_1 \times C_1 \) is not inclusion-minimal, i.e., there exists a proper subset \( R'_1 \times C'_1 \subset R_1 \times C_1 \) such that \( R'_1 \times C'_1 \) is a weak GSP. Since \( R_1 \times C_1 \) is a weak GSP, by Lemma 1 we only need to show that there exists a proper subset \( R'_1 \times C'_1 \subset R_1 \times C_1 \) such that \( R'_1 \times C'_1 \) is a weak GSP in \( A|_{R'_1 \times C'_1} \).

We index the entries of \( A|_{R_1 \times C_1} \) by \((x_{i,j})_{p_1 \times q_1}\), and the entries of \( A|_{R_2 \times C_2} \) by \((y_{i,j})_{p_2 \times q_2}\). We have \( x_{f_1(i),j} \geq y_{g_2(i),j} \) for all \((i,j) \in [p_2] \times [q_1] \) and \( x_{i,g_1(j)} \leq y_{g_2(i),j} \) for all \((i,j) \in [p_1] \times [q_2] \). Hence for all \( i \in [q_2] \) and all \( k \in [q_1] \) such that \( g_2(k) = j \), we have \( x_{f_1(i),k} \geq y_{i,j} \). We abuse notation and write \( x_{f_1(i),g_2^{-1}(j)} := \{ x_{f_1(i),k} \mid g_2(k) = j \} \), and for sets \( S, T \), the notation \( S \supseteq T \) means that \( s \in T \) for all \( s \in S \) and \( t \in T \). Similarly, we have that \( x_{f_2^{-1}(i),g_1(j)} \leq y_{i,j} \) for all \((i,j) \in [p_2] \times [q_2] \).

It follows that
\[ x_{f_1(i),g_2^{-1}(j)} \geq x_{f_2^{-1}(i),g_1(j)} \] (\( * \))
for all \((i,j) \in [p_2] \times [q_2] \). If \( f_2^{-1}(i) = \emptyset \) for some \( i \), the inequality is meaningless for that index \( i \) and we may remove the index from consideration. A similar statement holds for \( g_2 \). We may relabel the remaining indices as \( 1, \ldots, p_3 \) and \( 1, \ldots, q_3 \), respectively, so that \( f_2^{-1}(i) \neq \emptyset \) for all \( i \in [p_3] \) and \( g_2^{-1}(j) \neq \emptyset \) for all \( j \in [q_3] \). We have \( \bigcup_{i \in [p_3]} f_2^{-1}(i) = [p_1] \) and \( \bigcup_{j \in [q_3]} g_2^{-1}(j) = [q_1] \).

Consider the product \( S = \bigcup_{i \in [p_3]} f_1(i) \times \bigcup_{j \in [q_3]} g_1(j) \). Since \( f_1 \) is not surjective, we have that \( S \) is a proper subset of \( R_1 \times C_1 \). Hence it suffices to show that there exists a subset of \( S \) that is a weak GSP in \( A|_{R_1 \times C_1} \).

We define a directed graph \( G_R \) as follows. The nodes of \( G_R \) are given by \( 1, 2, \ldots, p_3 \). If \( f_1(i) \in f_2^{-1}(j) \), then we include a directed edge from \( j \) to \( i \). Let \( S_R \) be the set of nodes in \( G_R \) that belong to a directed cycle. (A self-loop counts as a directed cycle.) One can check that each node in \( G_R \) has exactly one incoming edge, and any node in \( G_R \) can be reached from a node in \( S_R \). Similarly, we define a directed graph \( G_C \) with nodes \( 1, 2, \ldots, q_3 \). If \( g_1(i) \in g_2^{-1}(j) \), then we include a directed edge from \( i \) to \( j \). One can check that each node in \( G_C \) has exactly one outgoing edge, and any node in \( G_C \) can reach a node in \( S_C \), where \( S_C \) is the set of nodes in \( G_C \) that belong to a directed cycle.

Suppose that \( G_R \) contains an edge \( i_2 \to i_1 \) and \( G_C \) contains an edge \( j_1 \to j_2 \). Then \( f_1(i_1) \in f_2^{-1}(i_2) \) and \( g_1(j_1) \in g_2^{-1}(j_2) \). From \( (\ast) \), we have \( x_{f_2^{-1}(i_1),g_1(j_1)} \leq x_{f_1(i_1),g_2^{-1}(j_2)} \) and \( x_{f_2^{-1}(i_2),g_1(j_1)} \leq x_{f_1(i_2),g_2^{-1}(j_1)} \). Since \( x_{f_1(i_1),g_1(j_1)} \) belongs to both \( x_{f_1(i_1),g_2^{-1}(j_2)} \) and
Shapley (1953a) has shown that every zero-sum game contains a unique minimax strategies. Let \( C \) be a weak saddle such that \( R \) is also a weak GSP and therefore contains a weak saddle. Let \( R' \) be a weak saddle such that \( R' \) is also a weak GSP and therefore contains a weak saddle. Theorem 1 implies that \( R' \) is also a weak saddle. Note that \( R' \) is a weak saddle, there must be \( r_2 \in R' \) such that \( r_1 \leq C_1' r_2 \). Since \( R \) is a strict saddle, there has to be \( r_3 \in R_1 \)

\[ x_{f_2^{-1}(i_2),g_1(j_1)} \], we have \( x_{f_2^{-1}(i_1),g_1(j_2)} \leq x_{f_1(i_1),g_1(j_1)} \leq x_{f_2(i_2),g_2^{-1}(j_1)}. \) In particular, we have

\[ x_{f_2^{-1}(i_1),g_1(j_2)} \leq x_{f_1(i_2),g_2^{-1}(j_1)}. \]

Suppose now that \( G_R \) contains edges \( i_n \to i_{n-1} \to \cdots \to i_1 \) and \( G_C \) contains edges \( j_2 \to \cdots \to j_n \). Then \( f_1(i_k) \in \alpha^{-1}(i_{k+1}) \) and \( g_1(j_k) \in \beta^{-1}(j_{k+1}) \) for all \( k \in [n-1] \). Applying the same argument as in the \( n=2 \) case repeatedly, we have \( x_{f_2^{-1}(i_1),g_1(j_n)} \leq x_{f_1(i_2),g_2^{-1}(j_1)}. \)

We claim that \( S' := \bigcup_{i \in S} f_1(i) \times \bigcup_{j \in S_C} g_1(j) \subseteq S \) is a weak GSP in \( A \mid R_1 \times C_1 \). To prove this claim, it suffices to consider row domination; column domination follows analogously.

For each node \( x \in S_C \), define \( c(x) \) to be the (unique) node in \( S_C \) such that the edge \( x \to c(x) \) exists. We must show that for any \( i \in [p_1] \), there exists \( j \in S_R \) such that \( x_{f_1(j),g_1(k)} \geq x_{i,g_1(k)} \) for all \( k \in S_C \). Since \( g_1(k) \in \beta^{-1}(c(k)) \), it suffices to show that for any \( i \in [p_1] \), there exists \( j \in S_R \) such that \( x_{f_1(j),g_2^{-1}(c(k))} \geq x_{i,g_1(k)} \) for all \( k \in S_C \). Moreover, since \( \bigcup_{i \in [p_3]} f_2^{-1}(i) = [p_1] \), we only need to show that for any \( i \in [p_3] \), there exists \( j \in S_R \) such that \( x_{f_1(j),g_2^{-1}(c(k))} \geq x_{f_2^{-1}(i),g_1(k)} \) for all \( k \in S_C \).

Let \( M \) denote the least common multiple of all the cycle lengths in \( S_C \). For any positive integer \( n \) and any node \( k \in S_C \), there exists a path of length \( nM-1 \) in \( S_C \) (and hence in \( G_C \)) that begins with \( c(k) \) and ends with \( k \). Since every node in \( G_R \) has one incoming edge, for large enough \( n' \) there exists a path of length \( n' \) in \( G_R \) that begins with some node \( j \in S_R \) and ends with \( i \). Taking \( n'' = nM-1 \) for large enough \( n \), there exists a path of length \( n'' \) in \( G_C \) that begins with \( c(k) \) and ends with \( k \), and a path of length \( n'' \) in \( G_R \) that begins with \( j \in S_R \) and ends with \( i \). It follows that \( x_{f_1(j),g_2^{-1}(c(k))} \geq x_{f_2^{-1}(i),g_1(k)} \), thus completing the proof.

### 4 Consequences and Remarks

Shapley (1953a) has shown that every zero-sum game contains a unique strict saddle. Shapley’s proof crucially relies on the minimax theorem (and the interchangeability of minimax strategies).\(^5\) Shapley’s result can be obtained as a corollary of Theorem 1 by leveraging the interchangeability of weak saddles.

#### Corollary 1 (Shapley, 1953a).

Every zero-sum game contains a unique strict saddle.

**Proof.** Let \( R_1 \times C_1 \) and \( R_2 \times C_2 \) be two strict GSPs. We first show that \( R_1 \cap R_2 = \emptyset \) and \( C_1 \cap C_2 = \emptyset \). Without loss of generality we may assume for contradiction that \( R_1 \cap R_2 = \emptyset \). Every strict GSP is also a weak GSP and therefore contains a weak saddle. Let \( R' \times C' \) be a weak saddle such that \( R'_1 \subseteq R_1 \) and \( C'_1 \subseteq C_1 \) and \( R'_2 \times C'_2 \) a weak saddle such that \( R'_2 \subseteq R_2 \) and \( C'_2 \subseteq C_2 \). Theorem 1 implies that \( R'_2 \times C'_1 \) is also a weak saddle. Note that \( R'_2 \cap R_1 = \emptyset \). Now let \( r_1 \) be an arbitrary row in \( R_1 \). Since \( R'_2 \times C'_1 \) is a weak saddle, there must be \( r_2 \in R'_2 \) such that \( r_1 \leq C'_1 r_2 \). Since \( R_1 \times C_1 \) is a strict saddle, there has to be \( r_3 \in R_1 \)

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\(^5\)Shapley notes that Hyman Bass proved this statement without making reference to the minimax theorem, but Bass’s report is unavailable.
such that $r_3 > c_1 r_2$ (and hence also $r_3 > c'_1 r_2$). The strict inequality implies that $r_3 \neq r_1$. Alternating these two arguments we obtain a sequence of rows $r_1, r_2, \ldots$ such that

$r_1 \leq c'_1 r_2 < c'_1 r_3 \leq c'_1 r_4 < \ldots$

All $r_i$ have to be distinct and finiteness of the game implies that we will eventually find a row $r_k$, which is witness to the fact that either $R'_2 \times C'_1$ is not a weak saddle or $R_1 \times C_1$ is not a strict saddle, both of which are contradictions.

We thus have that $R_1 \cap R_2 \neq \emptyset$ and $C_1 \cap C_2 \neq \emptyset$. An argument using a chain of strict inequalities (similar to the one above) easily shows that strict GSPs are closed under non-empty intersection, i.e., $(R_1 \cap R_2) \times (C_1 \cap C_2)$ is a strict GSP. Hence, none of the two original strict GSPs was minimal, a contradiction. 

A zero-sum game $A$ with row set $R$ and column set $C$ is symmetric if $R = C$ (with a slight abuse of notation) and the payoff matrix $A$ is skew-symmetric. In particular, all entries on the main diagonal of $A$ are zero. A confrontation game is a symmetric zero-sum game in which zeros appear only on the main diagonal. Strengthening a result by Dutta (1988), Duggan and Le Breton (1996a) have shown that every confrontation game contains a unique weak saddle.\footnote{By contrast, Nash equilibria are not unique in confrontation games (Le Breton, 2005).}

Corollary 2 (Duggan and Le Breton, 1996a). Every confrontation game contains a unique weak saddle.

Proof. Let $A$ be a confrontation game. A weak saddle $R' \times C'$ is called symmetric if $R' = C'$. Hence, a weak saddle $R' \times C'$ is symmetric if and only if every row and every column of $A|_{R' \times C'}$ contains exactly one zero.

Assume for contradiction that $A$ has two distinct weak saddles, $R_1 \times C_1$ and $R_2 \times C_2$. We consider the following two cases.

Case 1: At least one of the two weak saddles is not symmetric. Assume without loss of generality that $R_1 \times C_1$ is not symmetric. Since $A$ is skew-symmetric, $C_1 \times R_1$ is also a weak saddle. By the first part of Theorem 1, $R_1 \times R_1$ is a symmetric weak saddle. Hence, $R_1 \times R_1$ contains exactly one zero in each row and each column, while $R_1 \times C_1$ does not. But this contradicts the second part of Theorem 1.

Case 2: Both weak saddles are symmetric. Then $R_1 \times C_2$ is an asymmetric saddle, and we obtain a contradiction in the same way as in Case 1. 

We conclude the paper with a number of remarks.

Remark 1. If all payoffs of a game are pairwise distinct, strict dominance and weak dominance coincide. Thus, every such zero-sum game contains a unique weak saddle.

Remark 2. Duggan and Le Breton (2001) defined refinements of (weak and strict) saddles based on mixed dominance. Game $A_2$ in Figure 1 shows that Theorem 1 does not hold for mixed weak saddles.
Remark 3. If $R' \times C'$ is a weak saddle in a zero-sum game $A$, then every Nash equilibrium of $A|_{R' \times C'}$ is a Nash equilibrium of $A$. Therefore, every weak saddle contains the support of a Nash equilibrium. On the other hand, game $A_2$ in Figure 1 shows that there can be Nash equilibria whose support is disjoint from all weak saddles.

Remark 4. The subgame $A|_{R' \times C'}$ defined by a weak saddle $R' \times C'$ of a given zero sum game $A$ could be considered the “essence” of $A$. It follows from Theorem 1 that all such subgames are identical up to the permutation of rows and columns. Remark 3 implies that the value of $A|_{R' \times C'}$ is the same as that of the original game $A$.

Remark 5. A zero-sum game may contain an exponential number of weak saddles (Brandt and Brill, 2012). Thus, computing all weak saddles of a game is computationally intractable. The computational complexity of finding some weak saddle of a zero-sum game is an open problem.$^7$

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References


$^7$For two-player games that are not zero-sum, finding a weak saddle has been shown to be intractable (Brandt et al., 2011).


