

A Note on the McKelvey Uncovered Set and Pareto Optimality

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Abstract We consider the notion of Pareto optimality under the assumption that only the pairwise majority relation is known and show that the set of necessarily Pareto optimal alternatives coincides with the McKelvey uncovered set. As a consequence, the McKelvey uncovered set constitutes the coarsest Pareto optimal majoritarian social choice function. Moreover, every majority relation is induced by a preference profile in which the uncovered alternatives precisely coincide with the Pareto optimal ones. We furthermore discuss the structure of the McKelvey covering relation and the McKelvey uncovered set.

Keywords Pareto optimality · McKelvey uncovered set · majoritarian social choice

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1 Introduction

Let A be a finite set of m alternatives. The preferences of an agent i over these alternatives are represented by a complete, transitive, and antisymmetric preference relation $R_i \subseteq A \times A$.¹ The interpretation of $(x, y) \in R_i$, usually denoted by $x R_i y$, is that agent i values alternative x at least as much as alternative y . In accordance with conventional notation, we write P_i for the strict part of R_i , i.e., $x P_i y$ if $x R_i y$ but not $y R_i x$. Since R_i is antisymmetric, $x P_i y$ iff $x R_i y$ and $x \neq y$. A preference profile R is a finite vector of pairs which associate each agent i with its corresponding preference relation R_i . For a given preference profile, N_R denotes the set of agents represented in R . In particular, we do not assume a fixed number of agents. For convenience, we furthermore define $n_R(x, y) := |\{i \in N_R \mid x R_i y\}|$. The relations R_{Maj} and R_{Par} of a given preference profile R are then given by

$$\begin{aligned} x R_{\text{Maj}} y &\text{ iff } n_R(x, y) \geq n_R(y, x), \text{ and} \\ x R_{\text{Par}} y &\text{ iff } n_R(x, y) = |N_R|. \end{aligned}$$

By convention, again P_{Maj} denotes the strict part of R_{Maj} and P_{Par} the strict part of R_{Par} . Thus, $x P_{\text{Par}} y$ iff $x R_i y$ for all i and $x P_j y$ for some j . The relation R_{Maj} will be called the *majority relation* and P_{Par} the *Pareto relation* of R , respectively. Note that, by definition, R_{Maj} is complete whereas the Pareto relation P_{Par} is transitive and antisymmetric.

We say that a profile R is *consistent* with the majority relation R'_{Maj} of another profile R' if $R_{\text{Maj}} = R'_{\text{Maj}}$. In this case, R is also said to be consistent with R' .

An alternative $x \in A$ is called a *Condorcet winner* of a given (majority) relation R_{Maj} if it strictly dominates all other alternatives, i.e., if $x P_{\text{Maj}} y$ for all $y \in A \setminus \{x\}$.

Given a preference profile R , an alternative x is said to be *Pareto optimal* whenever there is no alternative y with $y P_{\text{Par}} x$, i.e., if x is a maximal element of R_{Par} .²

A *social choice function (SCF)* f associates with every preference profile R over a set A of alternatives a nonempty subset $f(R)$ of A . An SCF f is called a *refinement* of another SCF g if $f(R) \subseteq g(R)$ for all preference profiles R . In short, we write $f \subseteq g$ in this case and also say that g is *coarser* than f . An example of an SCF is the *Pareto set PO*, which selects the alternatives that are Pareto optimal, i.e.,

$$PO(R) = \{x \in A \mid y P_{\text{Par}} x \text{ for no } y \in A\}$$

¹ Antisymmetry is not required for any of our results to hold. In fact, Theorem 1 is even stronger when also assuming antisymmetric individual preferences (since this only increases the difficulty of constructing a suitable preference profile).

² Some authors call this *strong* Pareto optimality. In contrast, an alternative x would be *weakly* Pareto optimal if there is no alternative y with $y P_i x$ for all $i \in N_R$. In the case of antisymmetric preferences, the two notions coincide.

An SCF f is called *Pareto optimal* if it is a refinement of the Pareto set, i.e., if $f \subseteq PO$.

We will restrict attention to so-called *majoritarian* SCFs, whose outcomes only depend on the majority relation, i.e., to SCFs f such that, for all preference profiles R, R' ,

$$R_{\text{Maj}} = R'_{\text{Maj}} \text{ implies } f(R) = f(R').$$

An interesting class of majoritarian SCFs are defined using certain refinements of the majority relation called *covering relations*. For a given covering relation, the *uncovered set* contains those alternatives that are not covered by any other alternative. For a comprehensive overview of the theory of covering relations and uncovered sets, see Duggan (2013). A range of varying definitions of the covering relation exist, all of which coincide when restricted to antisymmetric majority relations. We will be concerned with what Duggan refers to as *McKelvey covering* (Bordes, 1983; McKelvey, 1986).

In order to define the McKelvey covering relation, we need to introduce the notions of *strict* and *weak dominators* of a given alternative. The strict dominators $P_{\text{Maj}}(x)$ of an alternative $x \in A$ are defined by the set of all alternatives $y \in A$ that are strictly majority preferred to x , i.e., $P_{\text{Maj}}(x) := \{y \in A \mid y P_{\text{Maj}} x\}$. Analogously, the weak dominators $R_{\text{Maj}}(x)$ of an alternative $x \in A$ are defined as the set of all alternatives $y \in A$ that are weakly majority preferred to x , i.e., $R_{\text{Maj}}(x) := \{y \in A \mid y R_{\text{Maj}} x\}$.

Let $C_{R_{\text{Maj}}}$ denote the (*McKelvey*) *covering relation*, i.e., for any pair of alternatives x, y , the relation $x C_{R_{\text{Maj}}} y$ holds iff each of the following three conditions is satisfied:

- (i) $x P_{\text{Maj}} y$,
- (ii) $P_{\text{Maj}}(x) \subseteq P_{\text{Maj}}(y)$, and
- (iii) $R_{\text{Maj}}(x) \subseteq R_{\text{Maj}}(y)$.

As can easily be seen, $C_{R_{\text{Maj}}}$ is transitive and asymmetric. The (*McKelvey*) *uncovered set* UC is then defined as

$$UC(R) := \{y \in A \mid x C_{R_{\text{Maj}}} y \text{ for no } x \in A\}.$$

Alternative, but equivalent, definitions of the McKelvey uncovered set were used by Dutta and Laslier (1999), Peris and Subiza (1999), and Brandt and Fischer (2008).

It is also well-known (see, e.g., Duggan, 2013) that the McKelvey uncovered set can be characterized as the set of alternatives that dominate every other alternative in at most two steps (of which at most one may be a tie). Formally, the McKelvey uncovered set then consists of all alternatives $x \in A$ such that for all $y \in A \setminus \{x\}$ at least one of the following three conditions is satisfied:

- (i) $x R_{\text{Maj}} y$,
- (ii) there is a $z \in A$ such that $x R_{\text{Maj}} z P_{\text{Maj}} y$, or
- (iii) there is a $z \in A$ such that $x P_{\text{Maj}} z R_{\text{Maj}} y$.

For brevity, we will omit any reference to McKelvey in the following and just write “covering” and “uncovered set.”

2 The Structure of the McKelvey Covering Relation

In this section we consider the structural properties of the *covering relation* and observe that any transitive and asymmetric relation can be obtained as the covering relation of some preference profile.

Proposition 1 *Let $Q \subseteq A \times A$ be a binary relation. Then the following are equivalent:*

- (i) Q is transitive and asymmetric.
- (ii) There exists a preference profile R such that $C_{R_{\text{Maj}}} = Q$.

Proof The implication from (ii) to (i) is immediate, as for every preference profile R the covering relation $C_{R_{\text{Maj}}}$ is transitive and asymmetric. For the other direction, assume that Q is transitive and asymmetric and let R be a preference profile with $P_{\text{Maj}} = Q$. By virtue of McGarvey’s Theorem (McGarvey, 1953) we know such a preference profile exists. It then remains to observe that $(x, y) \in C_{R_{\text{Maj}}}$ iff $(x, y) \in Q$. The “if”-direction is immediate. For the “only if”-direction, consider an arbitrary edge $(x, y) \in Q$. By construction of R , also $x P_{\text{Maj}} y$ and, by transitivity of $Q = P_{\text{Maj}}$, we obtain $P_{\text{Maj}}(x) \subseteq P_{\text{Maj}}(y)$. To see that also $R_{\text{Maj}}(x) \subseteq R_{\text{Maj}}(y)$, consider an arbitrary $z \in R_{\text{Maj}}(x)$ and assume for contradiction that $y P_{\text{Maj}} z$. Again by transitivity of P_{Maj} , then $x P_{\text{Maj}} z$, a contradiction. It follows that $z \in R_{\text{Maj}}(y)$. \square

Interestingly, the implication from (i) to (ii) does not hold if R_{Maj} is required to be antisymmetric (e.g., when the number of agents is odd). We are not aware of a non-trivial characterization of potential covering relations for this case.³

A result analogous to Proposition 1 was shown by Dushnik and Miller (1941) for the Pareto relation. They proved that for any transitive and asymmetric relation $Q \subseteq A \times A$ there exists a preference profile R such that $P_{\text{Par}} = Q$. Note that this does not imply Proposition 1 as the majority relation outside Q may be very different in the preference profile R instantiating Q as the Pareto relation, and a preference profile R' instantiating Q as the covering relation. In our proof, in particular, all edges outside Q are required to be majority ties.

Regarding the internal structure of the *uncovered set*, it was already shown by Moulin (1986) that any complete binary relation without a (non-trivial) Condorcet winner is the majority relation between uncovered alternatives for some preference profile. While Moulin (1986) proved this result for tournaments only, the argument can easily be adapted to cover our setting, in which majority ties are allowed.

³ Consider, for instance, the simple case of $A = \{a, b, c\}$ and $Q = \{(a, b), (a, c)\}$, which is easily seen not to be the covering relation for any preference profile.

3 The McKelvey Uncovered Set and the Pareto Set are Majority-equivalent

In this section, we consider the relationship between the uncovered set and the Pareto set for profiles that yield the same majority relation (i.e., *consistent* preference profiles).

Our main result shows that for every preference profile R , we can find another preference profile R' consistent with R_{Maj} such that the uncovered set of R and the Pareto set of R' coincide. This has a number of consequences. First, if we assume that only the majority relation is known, the uncovered set not only coincides with the Pareto optimal alternatives for some consistent preference profile, but also consists of precisely those alternatives that are Pareto optimal for *every* consistent preference profile. Moreover, there exists a consistent profile in which *all* covered alternatives are Pareto dominated. Secondly, the theorem implies that the uncovered set can be characterized as the coarsest majoritarian Pareto optimal SCF.

Theorem 1 *For every preference profile R , there is another preference profile R' with $R_{\text{Maj}} = R'_{\text{Maj}}$ such that*

$$UC(R) = PO(R').$$

Proof Consider an arbitrary preference profile R . Then, for every alternative $y \notin UC(R)$ there is some $x \in UC(R)$ such that $x C_{R_{\text{Maj}}} y$. Thus, let $UC(R) = \{x_1, \dots, x_\ell\}$ and associate with every $x_i \in UC(R)$ a (possibly empty) set $f(x_k) \subseteq \{x' \in A \mid x C_{R_{\text{Maj}}} x'\}$ such that

$$f(x_k) \cap f(x_{k'}) = \emptyset \text{ for } 1 \leq k < k' \leq \ell \text{ and } f(x_1) \cup \dots \cup f(x_\ell) = A \setminus UC(R).$$

Define the relation $F \subseteq C_{R_{\text{Maj}}}$ such that, for all alternatives x and y ,

$$x F y \quad \text{iff} \quad x \in UC(R) \text{ and } y \in f(x).$$

We construct a preference profile R' such that $R'_{\text{Maj}} = R_{\text{Maj}}$ and $R'_{\text{Par}} = F$. It can readily be appreciated that then $UC(R) = PO(R')$, as desired.

For notational convenience we denote $f(x_k)$ by X_k . For any subset Y of alternatives, by \vec{Y} and \overleftarrow{Y} we denote an enumeration of Y and its inverse, respectively, i.e., if $\vec{Y} = y_1, \dots, y_k$ then $\overleftarrow{Y} = y_k, \dots, y_1$.

First, we generate the relation F . To this end, we introduce two agents, i_F and j_F with preferences given by the following two sequences:

$$\begin{aligned} i_F: x_1, \vec{X}_1, \dots, x_\ell, \vec{X}_\ell, \\ j_F: x_\ell, \overleftarrow{X}_\ell, \dots, x_1, \overleftarrow{X}_1. \end{aligned}$$

Letting $R^F = (R_{i_F}, R_{j_F})$, we thus have $R^F_{\text{Par}} = F$.

Furthermore, for every pair (v, w) in R_{Maj} that is not contained in F we also introduce two additional agents i_{vw} and j_{vw} . We distinguish five cases and denote by R^{vw} the profile $(R_{i_{vw}}, R_{j_{vw}})$. Without loss of generality and for notational convenience, we will usually assume that $v^+ = x_1$ and $w^+ = x_2$.

Case 1 There is a $v^+ \in UC(R)$ such that $v, w \in f(v^+)$. Let $V = f(v^+) \setminus \{v, w\}$. For an illustration see Figure 1a. Then, define the preferences of i_{vw} and j_{vw} by the following lists:

$$\begin{aligned} i_{vw}: v^+, v, w, \vec{V}, x_2, \vec{X}_2, \dots, x_\ell, \vec{X}_\ell, \\ j_{vw}: x_\ell, \vec{X}_\ell, \dots, x_2, \vec{X}_2, v^+, \vec{V}, v, w. \end{aligned}$$

Observe that in this case, we have $P_{\mathbf{Maj}}^{vw} = F \cup \{(v, w)\}$.

Case 2 There are $v^+, w^+ \in UC(R)$ with $v \in f(v^+)$ and $w \in f(w^+)$. Let $V = f(v^+) \setminus \{v\}$ and $W = f(w^+) \setminus \{w\}$. Observe that in this case we have that $v^+ C_{R_{\mathbf{Maj}}} v$ for R and, hence, also $v^+ P_{\mathbf{Maj}} w$. For an illustration see Figure 1b. In this case define the preferences of i_{vw} and j_{vw} as

$$\begin{aligned} i_{vw}: v^+, \vec{V}, v, w^+, w, \vec{W}, x_3, \vec{X}_3, \dots, x_\ell, \vec{X}_\ell, \\ j_{vw}: x_\ell, \vec{X}_\ell, \dots, x_3, \vec{X}_3, w^+, \vec{W}, v^+, v, w, \vec{V}. \end{aligned}$$

It thus follows that $P_{\mathbf{Maj}}^{vw} = F \cup \{(v, w), (v^+, w)\}$.

Case 3 There are $v^+, w^+ \in UC(R)$ with $v = v^+$ and $w = w^+$. Let $V = f(v^+)$ and $W = f(w^+)$. Observe that in this case we have for R that $w^+ C_{R_{\mathbf{Maj}}} x$ and, hence, also $v^+ P_{\mathbf{Maj}} x$ for all $x \in W$. The situation is depicted in Figure 1c. Now define

$$\begin{aligned} i_{vw}: v^+, \vec{V}, w^+, \vec{W}, x_3, \vec{X}_3, \dots, x_\ell, \vec{X}_\ell, \\ j_{vw}: x_\ell, \vec{X}_\ell, \dots, x_3, \vec{X}_3, v^+, w^+, \vec{W}, \vec{V}. \end{aligned}$$

Observe that now $P_{\mathbf{Maj}}^{vw} = F \cup \{(v^+, w^+)\} \cup \{(v^+, x) \mid x \in W\}$.

Case 4 There are $v^+, w^+ \in UC(R)$ with $v = v^+$ and $w \in f(w^+)$. Let $V = f(v^+) \setminus \{v\}$ and $W = f(w^+) \setminus \{w\}$. The situation is depicted in Figure 1d. In this case define the preferences of i_{vw} and j_{vw} as follows:

$$\begin{aligned} i_{vw}: v^+, \vec{V}, w^+, w, \vec{W}, x_3, \vec{X}_3, \dots, x_\ell, \vec{X}_\ell, \\ j_{vw}: x_\ell, \vec{X}_\ell, \dots, x_3, \vec{X}_3, w^+, \vec{W}, v^+, w, \vec{V}. \end{aligned}$$

Accordingly, we also have $P_{\mathbf{Maj}}^{vw} = F \cup \{(v^+, w)\}$.

Case 5 There are $v^+, w^+ \in UC(R)$ with $v \in f(v^+)$ and $w = w^+$. Let $V = f(v^+) \setminus \{v\}$ and $W = f(w^+)$. Observe that in this case we have for R that $v^+ C_{R_{\mathbf{Maj}}} v$ and $w^+ C_{R_{\mathbf{Maj}}} x$ for all $x \in W$. As a consequence, also $v^+ P_{\mathbf{Maj}} w^+$, and thus $v^+ P_{\mathbf{Maj}} x$ and $v P_{\mathbf{Maj}} x$ for all $x \in W$. The situation is depicted in Figure 1e. In this case define:

$$\begin{aligned} i_{vw}: v^+, \vec{V}, v, w^+, \vec{W}, x_3, \vec{X}_3, \dots, x_\ell, \vec{X}_\ell, \\ j_{vw}: x_\ell, \vec{X}_\ell, \dots, x_3, \vec{X}_3, v^+, v, w^+, \vec{W}, \vec{V}. \end{aligned}$$

Hence, $P_{\mathbf{Maj}}^{vw} = F \cup (\{v^+, v\} \times (W \cup \{w^+\}))$.

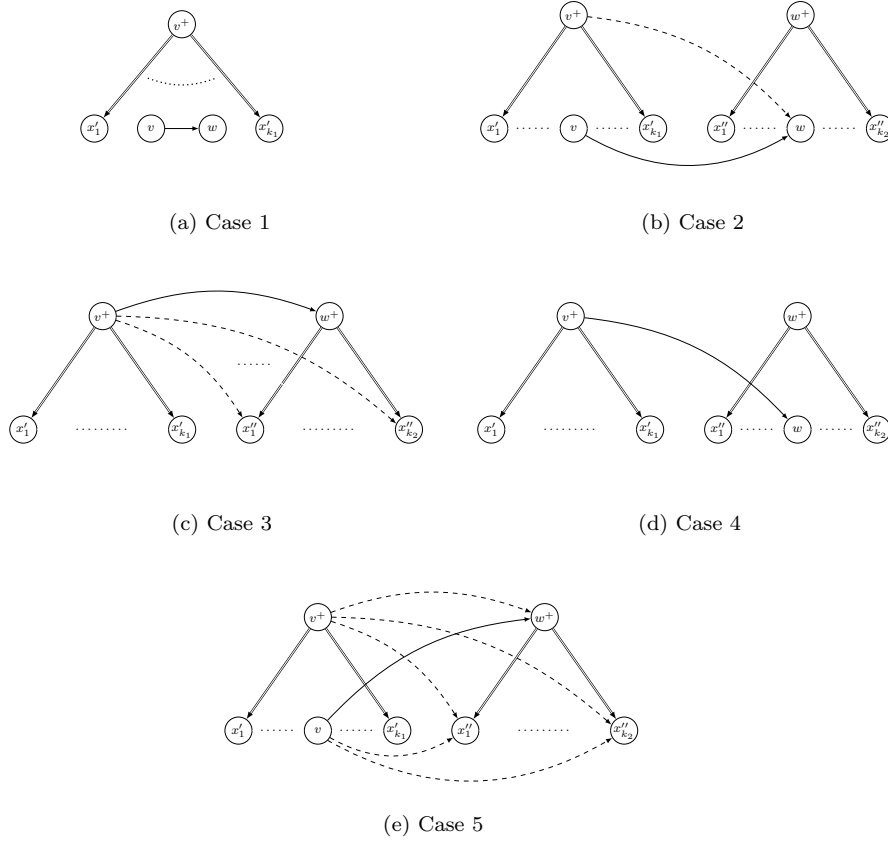


Fig. 1: The five cases distinguished in the proof of Theorem 1. The relation F is indicated by double arrows.

Let $R_{\text{Maj}} \setminus F$ be given by $\{(v_1, w_1), \dots, (v_p, w_p)\}$ and consider the profile

$$R' = (R_1, \dots, R_{2p+2}) = (R_{i_F}, R_{j_F}, R_{i_{v_1 w_1}}, \dots, R_{i_{v_p w_p}}).$$

Some reflection reveals that $R'_{\text{Maj}} = R_{\text{Maj}}$. Also observe that $F \subseteq R_i$ for all $1 \leq i \leq 2p + 2$. Moreover, $x P_{i_F} y$ iff $y P_{j_F} x$, for all $(x, y) \notin F$. Hence, $R'_{\text{Par}} = F$. Since, $x \notin UC(R)$ iff $y F x$ for some $y \in UC(R)$, it follows that $PO(R') = UC(R)$, which concludes the proof. \square

For the next corollary we additionally need the (easy-to-prove) fact that the uncovered set UC is Pareto optimal, which to the best of our knowledge was first mentioned by Bordes (1983). We formalize it in the following lemma for the sake of completeness. Note that this result relies on the definition of the majority relation via *simple majority rule* (see, e.g., Gaertner, 2009, p. 39). If

majorities are defined via (the less common) *absolute* majority rule, then the uncovered set only satisfies *weak* Pareto optimality (Duggan, 2013).

Lemma 1 (Bordes, 1983) *The Pareto relation is a subrelation of the McKelvey covering relation. Hence, $UC \subseteq PO$.*

Proof Let R be a preference profile and $x, y \in A$ alternatives such that $x P_{\text{Par}} y$. To show that then also $x C_{R_{\text{Maj}}} y$, first suppose $z P_{\text{Maj}} x$ for some alternative $z \in A$. Since individual preferences are assumed to be transitive it follows that $z P_{\text{Maj}} y$. The case of $z R_{\text{Maj}} x$ (implying $z R_{\text{Maj}} y$) is analogous and $x P_{\text{Maj}} y$ is an immediate consequence of $x P_{\text{Par}} y$. \square

The following corollary, which follows from Theorem 1 and Lemma 1, provides a characterization of the McKelvey uncovered set via Pareto optimality.⁴

Corollary 1 *A majoritarian SCF f is Pareto optimal iff $f \subseteq UC$. Consequently, the McKelvey uncovered set is the coarsest Pareto optimal majoritarian SCF.*

Proof Lemma 1 establishes that the McKelvey uncovered set UC (and any refinement of it) is Pareto optimal. To prove the other direction by contraposition, consider an arbitrary majoritarian SCF f such that $f \not\subseteq UC$. Then, there is a profile R and an alternative a such that $a \in f(R)$ and $a \notin UC(R)$. By Theorem 1, some preference profile R' exists such that $R_{\text{Maj}} = R'_{\text{Maj}}$ and $UC(R) = PO(R')$. Having assumed f to be majoritarian, $f(R') = f(R)$ and hence $a \in f(R')$. It also follows that $a \notin PO(R')$ and we may conclude that f is not Pareto optimal. \square

4 Potential Extensions

We consider two natural extensions of Theorem 1 and show that neither of them holds.

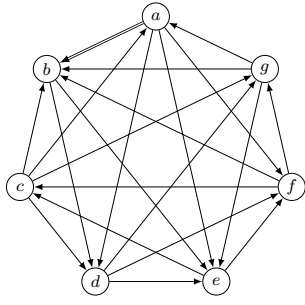
4.1 Constant number of agents

The proof of Theorem 1 crucially depends on the assumption of a variable electorate since R' usually has a different number of agents than R . The fact that the same result cannot be achieved with a constant number of agents is exhibited by the following minimal, computer-generated example.⁵ The majority relation R_{Maj} depicted in Figure 2 contains a unique covering edge $a C_{R_{\text{Maj}}} b$ and can be realized by 3 agents.⁶

⁴ Corollary 1 also entails an analogous weaker result for the special case of tournaments (i.e., antisymmetric majority relations R_{Maj}), which was used as a Lemma by Brandt and Geist (2014).

⁵ The whole example was obtained from and proved minimal by an automated computer search based on the method developed by Brandt et al. (2014).

⁶ In fact, any tournament of size 7 can be realized by 3 agents.

(a) majority relation R_{Maj}

1	1	1
e	c	b
f	a	d
c	d	g
g	f	a
a	g	e
b	b	f
d	e	c

(b) a consistent preference profile R

Fig. 2: A minimal example of a majority relation R_{Maj} for which the transition from $b \notin UC(R)$ to $b \notin PO(R)$ increases the minimal number of agents required to realize this relation.

If we, however, require b to be Pareto dominated, i.e., $x P_{\text{Par}} b$ for *at least one* of the alternatives $x P_{\text{Maj}} b$, then the minimal number of agents required to realize R_{Maj} (together with the additional requirement of $x P_{\text{Par}} b$) rises to 5, which has been verified on a computer.⁷

This example also shows that Corollary 1 does not in general hold for all constant electorates. To see this, let the number of agents be fixed at three and define an SCF f such that for all preference profiles R' ,

$$f(R') = \begin{cases} A & \text{if } R' \text{ is consistent with } R_{\text{Maj}}, \\ UC(R') & \text{otherwise,} \end{cases}$$

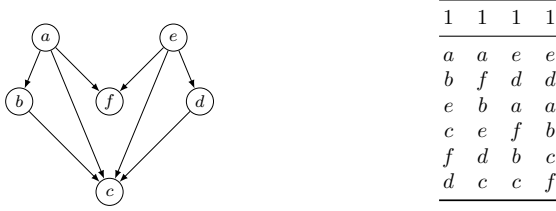
where R_{Maj} is the majority relation in Figure 2. Thus, f is a coarsening of UC but, due to the electorate being fixed at three agents, f is still Pareto optimal.

4.2 Equivalence of the covering relation and the Pareto relation

With Theorem 1 in mind, it is a natural question whether, given a preference profile R , one can even obtain a consistent preference profile R' (i.e., $R_{\text{Maj}} = R'_{\text{Maj}}$) in which the Pareto relation coincides with the covering relation (i.e., $P'_{\text{Par}} = C_{R_{\text{Maj}}}$). Unfortunately, the answer to this question is negative as the following counterexample shows.

Consider the preference profile R and the corresponding majority relation R_{Maj} in Figure 3 and note that all strict majority edges are also covering edges, i.e., $P_{\text{Maj}} = C_{R_{\text{Maj}}}$. Therefore, constructing a consistent preference profile R' with $P'_{\text{Par}} = C_{R_{\text{Maj}}}$, in this particular example, means to construct

⁷ This even holds when individual preferences are allowed to be *weak* orders.



(a) majority relation R_{Maj} , whose strict part P_{Maj} coincides with the covering relation $C_{R_{\text{Maj}}}$

(b) a consistent preference profile R

Fig. 3: An example of a majority relation R_{Maj} whose covering relation $C_{R_{\text{Maj}}}$ cannot be obtained as the Pareto relation of a consistent preference profile.

a consistent preference profile R' such that $P'_{\text{Par}} = C_{R_{\text{Maj}}} = P_{\text{Maj}}$. We now show—using computer-aided solving techniques—that such a profile does not exist.

For $P'_{\text{Par}} = P_{\text{Maj}}$ to hold, R' may only contain agents with individual preferences R'_i that respect all given strict majority edges as Pareto edges, i.e., for which $P'_i \supseteq P_{\text{Maj}}$. This will stand in conflict with being able to maintain the majority ties in $R'_{\text{Maj}} = R_{\text{Maj}}$. We used an ASP program (Answer Set Programming, a declarative problem-solving paradigm; see, e.g., Gebser et al. (2011, 2012); in our case the packaged grounder/solver CLASP) to compute all 22 candidates R''_i for complete, antisymmetric, and transitive preference relations such that $P''_i \supseteq P_{\text{Maj}}$. (One such ordering is e, d, a, f, b, c .) We would now have to construct a preference profile using only these 22 candidates for individual preference relations as building blocks. A small IP (Integer Program, see, e.g., Schrijver (1986); the performance of the open-source solver LPSOLVE satisfied our requirements) suffices to show that this is not possible: it contains 22 integer-valued variables $x_{R''_i}$, which denote, for each of the 22 preference relations, how many agents with this particular preference relation are contained in R' . Furthermore, it contains one constraint per indifference edge in the majority relation (of which there are 7, namely (a, d) , (a, e) , (b, d) , (b, e) , (b, f) , (c, f) , and (d, f)). Each of these constraints postulates for one indifference edge (y, z) that $0 = \sum_{R''_i} s((y, z), R''_i) \cdot x_{R''_i}$, where $s((y, z), R''_i) = 1$ if $(y, z) \in R''_i$, and $s((y, z), R''_i) = -1$ otherwise. With the additional constraints that $x_{R''_i} \geq 0$ for all R''_i and $\sum_{R''_i} x_{R''_i} \geq 1$ the IP solver returns that there are no feasible solutions to this problem, which completes the proof.⁸

⁸ The same counterexample also applies to the case of weak individual orders (i.e., without the assumption of antisymmetry of R_i). In this case there are 256 instead of 22 candidates for individual preferences relations, and additional constraints for the Pareto edges are

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required, which makes the resulting IP significantly larger, but still solvable within less than one second.