Bounds on the Disparity and Separation of Tournament Solutions

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Abstract

A tournament solution is a function that maps a tournament, i.e., a directed graph representing an asymmetric and connex relation on a finite set of alternatives, to a non-empty subset of the alternatives. Tournament solutions play an important role in social choice theory, where the binary relation is typically defined via pairwise majority voting. If the number of alternatives is sufficiently small, different tournament solutions may return overlapping or even identical choice sets. For two given tournament solutions, we define the *disparity index* as the order of the smallest tournament for which the solutions differ and the *separation index* as the order of the smallest tournament for which the corresponding choice sets are disjoint. Isolated bounds on both values for selected tournament solutions are known from the literature. In this paper, we address these questions systematically using an exhaustive computer analysis. Among other results, we provide the first tournament in which the bipartisan set and the Banks set are not contained in each other.

Keywords: Tournament solutions, disjointness, disparity, minimal examples

1. Introduction

An important area in the mathematical social sciences concerns solution concepts that identify desirable sets of alternatives based on the preferences of multiple agents. Many of these concepts are defined in terms of a so-called dominance

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relation, where one alternative dominates another if a strict majority of the agents prefer the former to the latter. This relation can be nicely represented as an oriented graph whose vertices are the alternatives and there is a directed edge from a to b if and only if a dominates b. Whenever there is an odd number of agents with linear preferences, the dominance relation is asymmetric and connex, i.e., there is exactly one directed edge between any pair of distinct vertices, and the graph thus constitutes a tournament. A tournament solution is a function that maps a tournament to a non-empty subset of its vertices or alternatives. Application areas of tournament solutions include voting [39, 35], multi-criteria decision analysis [2, 4], zero-sum games [27, 34, 23], and coalitional games [8].

A wide variety of tournament solutions have been proposed in the literature. Even though many of them are based on vastly different ideas, they share some similarities. For instance, all tournament solutions considered in this paper uniquely select the Condorcet winner, i.e., an alternative that dominates every other alternative, whenever such an alternative exists. Moreover, some tournament solutions return completely identical or at least overlapping choice sets if the number of alternatives is sufficiently small. In this paper, we aim at formalizing and systematically investigating the similarity of any given pair of tournament solutions by studying the minimal number of alternatives that are required for the disparity and the separation of the corresponding choice sets. To this end, we define the *disparity index* as the order of the smallest tournament for which the solutions differ and the *separation index* as the order of the smallest tournament for which the corresponding choice sets are disjoint.

Isolated bounds on both values for selected tournament solutions have been provided in previous work. In particular, the construction of tournaments for which certain tournament solutions return disjoint choice sets has been addressed by several researchers. For example, the first tournament proposed in the literature for which the Banks set and the Slater set are disjoint consists of 75 alternatives [33].[1] Later, this bound on the separation index was improved to 16 alternatives by Charon et al. [19] and, more recently, to 14 alternatives by Östergård and Vaskelainen [40]. Östergård and Vaskelainen have also provided a lower bound of 11 by means of an exhaustive computer analysis. In other work, Hudry [31] has shown that the separation index for the Banks set and the Copeland set is 13. Dutta [25] provided a tournament of order 8 in which the Banks set and the tournament equi-

[1]Laffond and Laslier [33] presented a similar tournament on 139 alternatives in which the Banks set, the Slater, and the Copeland set are all disjoint from each other.

librium set are both strictly contained in the minimal covering set. Among other facts, our study has shown that Dutta's example is minimal.

Perhaps the most interesting open problem regarding the relationships between tournament solutions concerns the bipartisan set and the Banks set. In all examples studied so far, either the Banks set is contained in the bipartisan set or the Banks set is contained in the bipartisan set (see, e.g., [35]). In particular, it is unknown whether these tournament solutions always intersect. In this paper, we provide the first tournament in which the bipartisan set and the Banks set are *not* contained in each other. This tournament is of order 8. The minimal covering set (a superset of the bipartisan set) has been shown to always intersect with the Banks set. We show that the smallest tournament in which neither choice set is contained in the other is of order 10. Our findings are summarized in Sections 4 and 5.

2. Preliminaries

A (finite) *tournament* T is a pair $(A, \succ)$, where A is a set of alternatives and $\succ$ is an asymmetric and connex (but not necessarily transitive) binary relation on A, usually referred to as the *dominance relation*. Intuitively, $a \succ b$ signifies that alternative a is preferable to alternative b. The dominance relation can be extended to sets of alternatives by writing $A \succ B$ when $a \succ b$ for all $a \in A$ and $b \in B$. Moreover, for a subset of alternatives $B \subseteq A$, we will sometimes consider the restriction of the dominance relation $\succ_B = \succ \cap (B \times B)$.

For a tournament $(A, \succ)$ and an alternative $a \in A$, we denote by $D(a)$ the *dominion* (or out-neighborhood) of a, i.e.,

$$D(a) = \{\, b \in A \mid a \succ b \,\},$$

and by $\overline{D}(a)$ the set of *dominators* (or in-neighborhood) of a, i.e.,

$$\overline{D}(a) = \{\, b \in A \mid b \succ a \,\}.$$

The *order* $|T|$ of a tournament $T = (A, \succ)$ refers to the cardinality of A, and $\mathcal{T}_n$ denotes the set of all tournaments of order n or less. The set of all linear orders on some set A is denoted by $\mathcal{L}(A)$ and the maximal element of A according to a linear order $L \in \mathcal{L}(A)$ is denoted by $\max(L)$.

The elements of the *adjacency matrix* $M(T) = (m_{ab})_{a,b \in A}$ of a tournament T are 1 whenever $a \succ b$ and 0 otherwise. The *skew-adjacency matrix* $G(T)$ of the corresponding tournament graph is skew-symmetric and defined as the difference of the adjacency matrix and its transpose, i.e., $G(T) = M(T) - M(T)^t$.

A *tournament solution* is a function that maps a tournament to a nonempty subset of its alternatives. For two tournament solutions S_1 and S_2, we define the *disparity index* $d(S_1, S_2)$ as the order of the smallest tournament T for which S_1 and S_2 differ, i.e.,

$$d(S_1, S_2) = \min\{n \in \mathbb{N} \mid \exists\, T \in \mathcal{T}_n \text{ such that } S_1(T) \neq S_2(T)\}.$$

Similarly, we define the *separation index* $s(S_1, S_2)$ as the order of the smallest tournament T for which the two respective choice sets are disjoint. Formally,

$$s(S_1, S_2) = \min\{n \in \mathbb{N} \mid \exists\, T \in \mathcal{T}_n \text{ such that } S_1(T) \cap S_2(T) = \emptyset\}.$$

Obviously, $d(S_1, S_2) \leq s(S_1, S_2)$ for all tournament solutions S_1 and S_2.

We now define the tournament solutions considered in this paper and address the question of how to compute them. For an overview and more details on most concepts, we refer to Laslier [35] and Brandt et al. [13]. Computational issues are discussed by Brandt et al. [13], Hudry [32], and Brandt [5].

Copeland set. The *Copeland set* $CO(T)$ [21] of a tournament T consists of all alternatives whose dominion is of maximum size, i.e.,

$$CO(T) = \arg\max_{a \in A} |D(a)|.$$

$|D(a)|$ is also called the *Copeland score* of a. This set can be easily computed in time $O(|T|^2)$ by determining all out-degrees and choosing the alternatives with maximum out-degree.

Top cycle. A nonempty subset of alternatives $B \subseteq A$ is called *dominant* if $B \succ A \setminus B$. The *top cycle* $TC(T)$ [29, 43] of a tournament T is defined as the smallest dominant set, i.e.,

$$TC(T) = \{B \text{ is dominant} \mid \forall C \subsetneq B : C \text{ is not dominant}\}.$$

Uniqueness of the minimal dominant set is straightforward and was first shown by Good [29]. The top cycle can be computed in linear time by identifying the strongly connected components of T [44].

Uncovered set. The *uncovered set* $UC(T)$ [26] of a tournament T consists of all alternatives that reach all other alternatives in at most two steps,[2] i.e.,

$$UC(T) = \{a \in A \mid \{a\} \cup D(a) \cup \bigcup_{b \in D(a)} D(b) = A\}.$$

It is easily seen that $a \in UC(T)$ if and only if $(M^2 + M)_{ab} \neq 0$ for all $b \in A \setminus \{a\}$. Consequently, the running time for computing UC is governed by matrix multiplication, i.e., it is in $O(|T|^{2.38})$ [22].

Iterated uncovered set. UC is not idempotent and one can therefore define a sequence of tournament solutions by letting $UC^1(T) = UC(T)$ and $UC^k = UC(UC^{k-1}(T))$.[3] The *iterated uncovered set* $UC^\infty(T)$ (see [35]) of a tournament T is then defined as

$$UC^\infty(T) = \bigcap_{k \in \mathbb{N}} UC^k(T).$$

Due to the finiteness of T, $UC^\infty(T) = UC^{|T|}$, i.e., computing UC^∞ requires at most $|T|$ successive UC-computations. Therefore, UC^∞ can be computed in time $O(|T|^{1+2.38})$.

Bipartisan set. Let $\Delta(A)$ be the set of all probability distributions over A. Laffond et al. [34] and Fisher and Ryan [27] have shown independently that every tournament T admits a unique probability distribution $p_T \in \Delta(A)$ such that

$$\sum_{a,b \in A} p_T(a)q(b)G(T)_{a,b} \geq 0 \text{ for all } q \in \Delta(A).$$

p_T then corresponds to the unique mixed Nash equilibrium of the zero-sum game $G(T)$. The *bipartisan set* $BP(T)$ of a tournament T is defined as the support of this equilibrium, i.e.,

$$BP(T) = \{a \in A \mid p_T(a) > 0\}.$$

BP can be computed in polynomial time using a linear feasibility program [7].

[2] In the original definition by Fishburn [26], $UC(T)$ consists of the alternatives which are not *covered* by any other alternative. An alternative a covers an alternative b if $\{b\} \cup D(b) \subseteq D(a)$.

[3] It is understood that $S'(S(T))$ denotes $S'(T|_{S(T)})$.

Markov set. The tournament matrix can be used to define the transition matrix of a Markov chain as $N = \frac{1}{|T|-1} \cdot (M + I_{CO})$ where I_{CO} is the diagonal matrix of the Copeland scores. The *Markov set* $MA(T)$ [35] of a tournament T consists of the alternatives that have maximum probability in the chain's unique stationary distribution, i.e.,

$$MA(T) = \arg\max_{a \in A} \{p(a) \mid p \in \Delta(A) \text{ and } N \cdot p = p\}.$$

Computing p as the eigenvector of N associated with the eigenvalue 1 is again governed by matrix multiplication [32] and therefore in $O(|T|^{2.38})$.

Kendall-Wei set. Based on ideas by Kendall and Wei, the alternatives with maximum entries in the eigenvector of the (unique) largest positive eigenvalue λ of the adjacency matrix of T comprise the *Kendall-Wei set* $KW(T)$. Formally,

$$KW(T) = \arg\max_{a \in A} \{p(a) \mid p \in \Delta(A) \text{ and } (M - \lambda I)\, p = 0\}$$

where I is the identity matrix. Alternatively, the Kendall-Wei scores reflect the outcome of the following process: for each alternative a, count all paths of length k starting from a, then normalize these numbers and consider $k \to \infty$. For this reason, Laslier [35] has called it the *long-path* method. Computing the eigenvector in sufficient precision can again be done in polynomial time.

Banks set. Define $\mathcal{B}_T(a) = \{B \subseteq A \mid \succ_B \in \mathcal{L}(B) \text{ and } \max(\succ_B) = a\}$ as the set of all transitive subsets with maximal element a. The *Banks set* $BA(T)$ [3] of a tournament is then defined as the set of all alternatives, that are maximal in some maximal transitive subset, i.e.,

$$BA(T) = \{a \in A \mid \exists B \in \mathcal{B}_T(a) \text{ such that } \nexists b\colon b \succ B\}.$$

Computing BA is known to be NP-hard [45]. Our implementation is based on a recent algorithm by Gaspers and Mnich [28] that enumerates all *feedback vertex sets*, each of which is the complement of a maximal transitive subset.

Slater set. The *Slater set* $SL(T)$ [42] of a tournament T consists of the maximal elements of those linear orders that have as many directed edges as possible in common with T, i.e.,

$$SL(T) = \{\max(L) \mid L \in \arg\max_{L' \in \mathcal{L}(A)} |L' \cap \succ|\}.$$

Finding these linear orders is equivalent to solving an instance of the NP-complete problem *feedback arc set* [1, 14, 20], which implies that checking membership in the Slater set is NP-hard [16]. Yet, there are implementations that are sufficiently fast on small instances (e.g., [17]).

Minimal stable sets. A subset of alternatives $B \subseteq A$ is called S-*stable* for tournament solution S if $a \notin S(B \cup \{a\})$ for all $a \in A \setminus B$. Stable sets can be used to define a new tournament solution $\widehat{S}$ that returns the union of all minimal S-stable sets, i.e.,

$$\widehat{S}(T) = \bigcup \{B \text{ is } S\text{-stable} \mid \forall C \subsetneq B : C \text{ is not } S\text{-stable}\}.$$

This enables the definition of the *minimal covering set* $MC(T)$ [24] and the *minimal extending set* $ME(T)$ [6] of a tournament T by letting

$$MC(T) = \widehat{UC}(T) \quad \text{and} \quad ME(T) = \widehat{BA}(T).$$

A polynomial-time algorithm for computing MC using the BP algorithm as a subroutine was proposed by Brandt and Fischer [7]. Computing the minimal extending set is a tedious task. It was recently shown to be an NP-hard problem while the best known upper bound is Σ_3^p [12]. We compute minimal extending sets using a naive implementation.

Minimal retentive sets. A nonempty subset of alternatives $B \subseteq A$ is called S-*retentive* for tournament solution S if $S(\overline{D}(b)) \subseteq B$ for all $b \in B$ such that $\overline{D}(b) \neq \emptyset$. Just like stable sets, retentive sets can be used to define a new tournament solution $\mathring{S}$ that returns the union of all minimal S-retentive sets, i.e.,

$$\mathring{S}(T) = \bigcup \{B \text{ is } S\text{-retentive} \mid \forall C \subsetneq B : C \text{ is not } S\text{-retentive}\}.$$

This enables the definition of $\mathring{TC}$ [11] and the *tournament equilibrium set* $TEQ = \mathring{TEQ}$ [41]. Note that the latter is a well-defined recursion as the order of the subtournament on $\overline{D}(b)$ in a tournament T is always strictly smaller than the order of T.

A general method for computing $\mathring{S}$, given an implementation for S, is to compute the corresponding relation $\xrightarrow{S}$ and return the maximal elements of that relation's transitive closure, as suggested by Brandt et al. [9]. In order to compute $\mathring{TC}$, we consider $\xrightarrow{TC}$ where for any $a, b \in A$, $a \xrightarrow{TC} b$ if and only if $a \in TC(\overline{D}(b))$.

This takes polynomial time. Due to its recursive nature, computing TEQ is much harder than computing $\mathring{TC}$. The problem is known to be NP-hard while the best known upper bound is PSPACE [9]. For general tournaments with more than 100 alternatives, computing TEQ is currently out of reach. For structured tournaments this changes drastically [10].

All of the aforementioned tournament solutions return subsets of TC and all except TC and $\mathring{TC}$ return subsets of UC.[4] On top of that, the following inclusion relationships are known:

$$BP \subseteq MC \subseteq UC^{\infty} \quad \text{and} \quad TEQ \subseteq BA.$$

Furthermore, it has been shown that

$$BA(MC) \subseteq BA \quad \text{and} \quad TEQ(UC^{\infty}) = TEQ,$$

which implies that

$$BA \cap MC \neq \emptyset \text{ , and } TEQ \subseteq BA \cap UC^{\infty} \neq \emptyset$$

(see [35]).

3. Methodology

For some pairs of tournament solutions, we can easily show that they always intersect. As a consequence, their separation index is ∞.

Proposition 1. *The following statements hold:*

1. $s(MC, ME) = \infty$
2. $s(UC^{\infty}, ME) = \infty$
3. $s(\mathring{TC}, TEQ) = \infty$
4. $s(BA, ME) = \infty$

Proof 1. *We prove each statement separately.*

1. *Since $BA \subseteq UC$, every UC-stable set is also BA-stable.*

[4]The fact that $KW \subseteq UC$ is not yet mentioned in the literature. It follows directly from the long-path interpretation of the Kendall-Wei scores.

2. *Since $MC \subseteq UC^\infty$, this follows from Statement 1.*
3. *Since $TEQ \subseteq TC$, every TC-retentive set is also TEQ-retentive.*
4. *For all tournaments T, $BA(T)$ is BA-stable [6].*

Apart from these theoretical results, we exhaustively searched for minimal examples with disparate or disjoint choice sets. To this end, we implemented algorithms for computing all the considered tournament solutions. Some of them were implemented directly (CO and TC) or with the help of a fast matrix multiplication library (UC, UC^∞, and MA). For BP and MC our implementation constructs linear programs that are passed to an LP solver (Gurobi [30]). For BA, we implemented the elaborate algorithm of Gaspers and Mnich, which was also used for the ME implementation. Computing the complete Slater set is achieved with the help of a tailored branch-and-bound algorithm by Charon and Hudry [15]. Finally, we implemented the $\circ$-operator in its general form, which allows us to compute $\mathring{TC}$ and TEQ.

Obviously, the number of non-isomorphic tournaments of order n grows exponentially ([38], p. 87). We generated all non-isomorphic tournaments of order ten or less using McKay's *nauty* toolkit [36]. In total, we analyzed about 10^7 tournaments. For each pair of tournament solutions and all tournaments in increasing order, we examined the choice sets for disparity and disjointness. Some of the most interesting tournaments we encountered were rearranged using a graphical tournament tool until the respective statements seemed most intuitive. Figures of these tournaments are included in Sections 4 and 5. We believe that these might also be of didactic value when teaching the basics of tournament solutions.

4. Results

Our results are summarized in Table 1. When the exact value of an index is unknown, we provide lower and upper bounds.

TC*, *CO*, *SL*, *MA*, *KW* *vs. the rest. CO, SL, MA, and KW tend to select significantly smaller choice sets than the other tournament solutions whereas TC is not very discriminative. This is witnessed by the tournament of order 4 depicted in Figure 1 where CO, SL, MA, and KW are smaller and TC is larger than all the remaining tournament solutions. This tournament accounts for all '4' entries in Table 1.

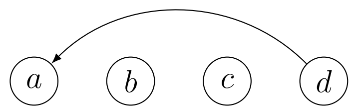

Figure 1: In this tournament, $SL(T) = MA(T) = KW(T) = \{a\} \subsetneq CO(T) = \{a, b\} \subsetneq UC(T) = \{a, b, d\} \subsetneq TC(T) = \{a, b, c, d\}$. All other tournament solutions considered in this paper coincide with UC. Omitted directed edges point rightwards.

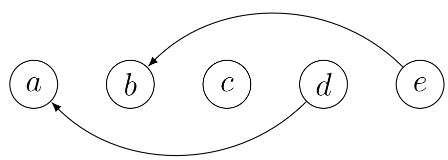

Figure 2: In this tournament, $UC^\infty(T) = MC(T) = BP(T) = \mathring{TC}(T) = ME(T) = TEQ(T) = \{a, b, d\}$ whereas $UC(T) = BA(T) = \{a, b, c, d\}$. Omitted directed edges point rightwards.

***UC*, *BA* vs. *UC*$^\infty$, *MC*, *BP*, $\mathring{TC}$, *ME*, *TEQ*.** A smallest tournament for which BA (and UC) differs from MC (and UC^∞, BP, TEQ, ME, $\mathring{TC}$) is shown in Figure 2. It is easy to verify that $\{a, b, d\}$ is UC-stable. Alternative c, however, is in $BA(T)$ because $B = \{c, d, e\} \in \mathcal{B}_T(c)$ and neither a nor b dominates B.

***UC*, *MA*, *KW* vs. *BA*.** There is an interesting family of tournaments that serve as minimal examples for a number of set-theoretic relationships between different tournament solutions. The first is the disparity of UC and BA—two solutions that return identical choice sets for all tournaments of order up to 6.

The basic variant of this tournament family is shown in Figure 3 and constitutes a minimal tournament for which $BA \subsetneq UC$.[5] The difference is that $d \notin BA(T)$ as for all $B \in \mathcal{B}_T(d)$ there is some $x \in \overline{D}(d)$ with $x \succ B$. Note that in this tournament $|D(x)| = 4$ for all $x \in BA(T) = \{a, b, c\}$ and $|D(x)| \leq 3$ for all $x \notin BA(T)$, i.e., $CO(T) = BA(T)$.

When each gray alternative is replaced by the unique tournament of order 2, the resulting tournament of order 10 is a minimal example for $BA \subsetneq CO$, as $CO(T) = \{a, b, c, d\}$. (This result is not part of Table 1.)

If we go one step further and replace each gray alternative with any tournament of order 3, the resulting tournament has order 13 and is a known minimal example

[5] Other examples of the same order can be found in Moulin [39] and Miller et al. [37].

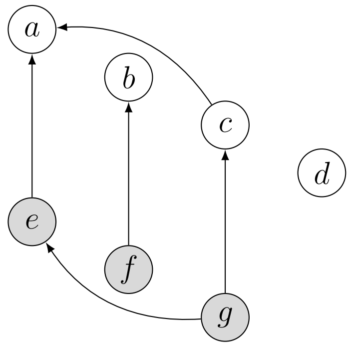

Figure 3: Minimal example for $BA(T) = \{a, b, c\} \subsetneq UC(T) = \{a, b, c, d\}$. If e, f, and g each get replaced by any tournament of order 3, the resulting tournament of order 13 is the minimal example for $BA \cap CO = \emptyset$ by Hudry [31]. If e, f, and g are instead replaced with a tournament of order 4, we get a tournament of order 16 in which $MA(T) = KW(T) = \{d\}$ is disjoint from $BA(T) = \{a, b, c\}$. Omitted directed edges point downwards.

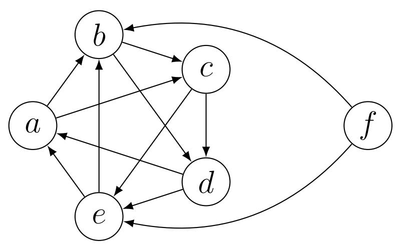

Figure 4: In this tournament, $BP(T) = \{a, b, c, d, e\} \subsetneq A = MC(T) = ME(T) = TEQ(T)$. Omitted directed edges point rightwards. Note that the subtournament on $BP(T)$ constitutes the only regular tournament of order 5.

for the separation of BA and CO proposed by Moulin [39] and Hudry [31].

Finally, if we put any tournament of order 4 in place of the gray alternatives, we get a tournament of order 16 where still $BA(T) = \{a, b, c\}$ but $MA(T) = KW(T) = \{d\}$. Since any one of the new alternatives can be removed without changing $BA(T)$ or $MA(T)$, this gives an upper bound of 15 for the separation of MA and BA and of 16 for the separation of KW and BA.

***BP* vs. *MC*, *ME*, *TEQ*.** Consider the tournament in Figure 4. The unique equilibrium strategy of the tournament game $G(T)$ is $(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, 0)$ and therefore $BP(T) = \{a, b, c, d, e\}$. However, this set is not UC-stable as f can reach every other alternative in $BP(T) \cup \{f\} = A$ in at most two steps. This is a minimal tournament for which MC differs from BP. The same holds for ME and TEQ as they coincide with MC for tournaments up to order 7.

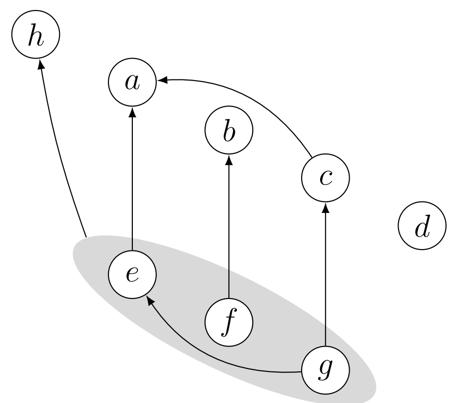

Figure 5: Minimal tournament for which $ME(T) = TEQ(T) \neq MC(T)$. Here, $ME(T) = TEQ(T) = A \setminus \{d\}$ whereas $MC(T) = A$. The ellipse indicates $\{e, f, g\} \succ h$ and omitted directed edges point downwards. If we change the dominance relation slightly to $e \succ g$, we get a minimal tournament T' for which ME and TEQ do not coincide as $TEQ(T') = A \setminus \{d\} \neq ME(T') = A$.

***MC* vs. *ME*, *TEQ*.** The minimal tournament for which TEQ and ME differ from MC is of order 8 and depicted in Figure 5. This tournament is again a variant of the tournament from Figure 3, this time expanded with an additional vertex h.[6] In this tournament $B = A \setminus \{d\}$ is the only BA-stable set. It is easy to check that B is not UC-stable as d does reach every other vertex in A in at most two steps. In fact, only A is UC-stable and therefore $MC(T) = A$. This implies that $d(MC, ME) = 8$. The reader can also verify that d does not dominate any vertex according to the TEQ-relation $\xrightarrow{TEQ}$ and therefore $d \notin TEQ(T)$, implying $d(MC, TEQ) = 8$. While TEQ and ME actually coincide for this tournament, a small modification gives a minimal tournament T' for which this is not the case, similar to the one reported by Brandt [5]. The only necessary change in the dominance relation is $e \succ g$, then $TEQ(T') = A \setminus \{d\} \subsetneq ME(T') = A$, accounting for $d(ME, TEQ) = 8$.

***CO*, *MA*, *KW* vs. UC^∞, *MC*, *BP*, $\mathring{TC}$, *ME*, *TEQ*.** For the separation of these tournament solutions, we found the tournaments depicted in Figure 6 and Figure 7. For the tournament T shown in Figure 6, it is easy to verify that alternative b has the largest dominion but is not contained in the UC-stable set $\{a, c, d, e, f\}$. Therefore, $CO(T) \cap MC(T) = \emptyset$ which gives $s(CO, MC) = 8$.

[6]The edge between the vertices a and c can be inverted without changing the result.

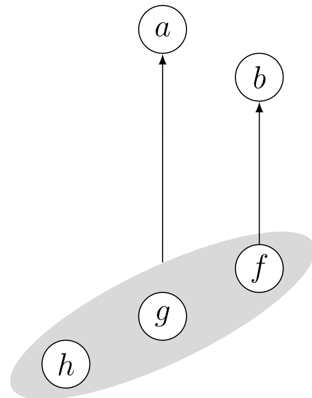

Figure 6: In this tournament, $CO(T) = MA(T) = \{b\}$ whereas $UC^\infty(T) = MC(T) = BP(T) = \mathring{TC}(T) = ME(T) = TEQ(T) = \{a, c, d, e, f\}$. This is a smallest tournament for which the respective choice sets are disjoint. The ellipse indicates $\{f, g, h\} \succ a$ and omitted directed edges point downwards.

As for this tournament $CO(T) = MA(T)$ and $UC^\infty(T) = MC(T) = BP(T) = \mathring{TC}(T) = ME(T) = TEQ(T)$, this also induces a few other separation indices in Table 1.

Figure 7 contains a similar example in which MA and KW are disjoint from UC^∞, MC, BP, $\mathring{TC}$, ME, and TEQ.

5. Further Findings

***BP* and *BA*.** Apart from values and bounds for the disparity and separation indices, our exhaustive search also revealed a number of other tournaments with interesting properties.

For instance, we have found the first tournament where BP and BA have a proper intersection, i.e., they are not contained in each other. The tournament is depicted in Figure 8, has 8 alternatives, and is minimal. The equilibrium strategy is $(\frac{7}{23}, \frac{3}{23}, \frac{1}{23}, \frac{7}{23}, 0, \frac{1}{23}, \frac{1}{23}, \frac{3}{23})$, i.e., $BP(T) = A \setminus \{e\}$. It is, however, easy to verify that $e \in BA$ as no other alternative dominates $\{e, f, g, h\} \in \mathcal{B}_T(e)$. At the same time, every set in $\mathcal{B}_T(f)$ is dominated by some alternative in $\{b, c, e\} \subseteq \overline{D}(f)$ and therefore $f \notin BA$. In fact, $BA = A \setminus \{f\}$.

***BA* and *MC*.** It was known already that BA and MC always intersect but none of them always chooses a subset of the other [35]. Our experiments showed that a proper intersection can only be observed for tournaments of order at least 10. A tournament of this kind is depicted in Figure 9. The reader can easily check that

$s \setminus d$	TC	UC	UC^∞	MC	BP	$\mathring{TC}$	BA	ME	TEQ	CO	SL	MA	KW
TC	—	**4** (Fig. 1)	**4** (Fig. 1)	**4** (Fig. 1)	**4** (Fig. 1)	**4** (Fig. 1)	**4** (Fig. 1)	**4** (Fig. 1)	**4** (Fig. 1)	**4** (Fig. 1)	**4** (Fig. 1)	**4** (Fig. 1)	**4** (Fig. 1)
UC	∞	—	**5** (Fig. 2)	**5** (Fig. 2)	**5** (Fig. 2)	**5** (Fig. 2)	**7**[a] (Fig. 3)	**5** (Fig. 2)	**5** (Fig. 2)	**4** (Fig. 1)	**4** (Fig. 1)	**4** (Fig. 1)	**4** (Fig. 1)
UC^∞	∞	∞	—	**6**	**6**	**6**	**5** (Fig. 2)	**6**	**6**	**4** (Fig. 1)	**4** (Fig. 1)	**4** (Fig. 1)	**4** (Fig. 1)
MC	∞	∞	∞	—	**6**[b]	**6**	**5** (Fig. 2)	**8** (Fig. 5)	**8**[c] (Fig. 5)	**4** (Fig. 1)	**4** (Fig. 1)	**4** (Fig. 1)	**4** (Fig. 1)
BP	∞	∞	∞	∞	—	**6**	**5** (Fig. 2)	**6**	**6**[b]	**4** (Fig. 1)	**4** (Fig. 1)	**4** (Fig. 1)	**4** (Fig. 1)
$\mathring{TC}$	∞	$[\mathbf{11}, \infty]$	$[\mathbf{11}, \infty]$	$[\mathbf{11}, \infty]$	$[\mathbf{11}, \infty]$	—	**5** (Fig. 2)	**6**	**6**	**4** (Fig. 1)	**4** (Fig. 1)	**4** (Fig. 1)	**4** (Fig. 1)
BA	∞	∞	∞	∞	$[\mathbf{11}, \infty]$	$[\mathbf{11}, \infty]$	—	**5** (Fig. 2)	**5** (Fig. 2)	**4** (Fig. 1)	**4** (Fig. 1)	**4** (Fig. 1)	**4** (Fig. 1)
ME	∞	∞	∞ (Prop. 1)	∞ (Prop. 1)	$[\mathbf{11}, \infty]$	$[\mathbf{11}, \infty]$	∞ (Prop. 1)	—	**8**[d] (Fig. 5)	**4** (Fig. 1)	**4** (Fig. 1)	**4** (Fig. 1)	**4** (Fig. 1)
TEQ	∞	∞	∞	$[\mathbf{11}, \infty]$	$[\mathbf{11}, \infty]$	∞ (Prop. 1)	∞	$[\mathbf{11}, \infty]$	—	**4** (Fig. 1)	**4** (Fig. 1)	**4** (Fig. 1)	**4** (Fig. 1)
CO	∞	∞	**8** (Fig. 6)	**8** (Fig. 6)	**8** (Fig. 6)	**8** (Fig. 6)	**13**[e] (Fig. 3)	**8** (Fig. 6)	**8** (Fig. 6)	—	**4** (Fig. 1)	**4** (Fig. 1)	**4** (Fig. 1)
SL	∞	∞	**8**[b]	**8**[b]	**8**[b]	**8**	$[\mathbf{11}, \mathbf{14}]$[f]	**8**	**8**[b]	**6**[g]	—	**5**	**5**
MA	∞	∞	**8** (Fig. 6)	**8** (Fig. 6)	**8** (Fig. 6)	**8** (Fig. 6)	$[\mathbf{11}, \mathbf{15}]$ (Fig. 3)	**8** (Fig. 6)	**8** (Fig. 6)	**8**	**6**	—	**6**
KW	∞	∞	**8**	**8**	**8**	**8**	$[\mathbf{11}, \mathbf{16}]$ (Fig. 3)	**8**	**8**	**7**	**6**	**7**	—

[a] Shown by Miller et al. [37]
[b] Shown (without minimality) by Laslier [35]
[c] Shown (without minimality) by Dutta [25]
[d] Shown by Brandt [5]
[e] Shown by Hudry [31]
[f] Shown by Östergård and Vaskelainen [40]
[g] Shown by Charon et al. [18]

Table 1: Overview of all disparity indices (d) and separation indices (s) currently known for the tournament solutions considered.

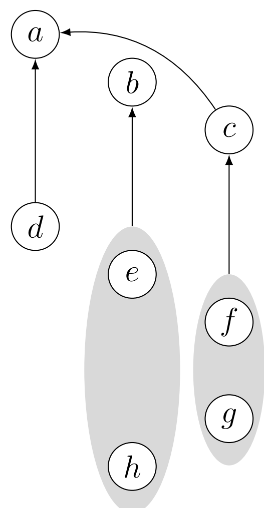

Figure 7: In this tournament, $MA(T) = KW(T) = \{d\}$ whereas $UC^\infty(T) = MC(T) = BP(T) = \mathring{TC}(T) = ME(T) = TEQ(T) = \{a, b, c\}$. This is a smallest tournament for which the respective choice sets are disjoint. The ellipses indicate $\{e, h\} \succ b$ and $\{f, g\} \succ c$ and omitted directed edges point downwards.

$A \setminus \{c, i\}$ is UC-stable. On the other hand, i obviously is in $BA(T)$, witnessed by the maximal transitive subset $\{i, c, j\}$. Alternative f, however, is not in $BA(T)$ as for each $B \in \mathcal{B}_T(f)$, there is an alternative from $\{b, d, e\} \subseteq \overline{D}(f)$ that dominates B. In fact, $MC(T) = A \setminus \{c, i\}$ and $BA(T) = A \setminus \{c, f\}$. The choice sets overlap.

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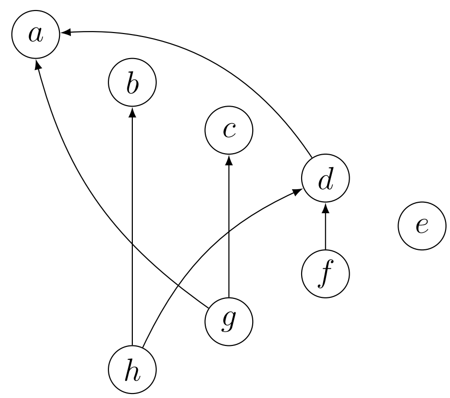

Figure 8: The first reported tournament where BA and BP are not contained in each other. In this tournament, $BP(T) = A \setminus \{e\}$ whereas $BA(T) = A \setminus \{f\}$. Omitted directed edges point downwards.

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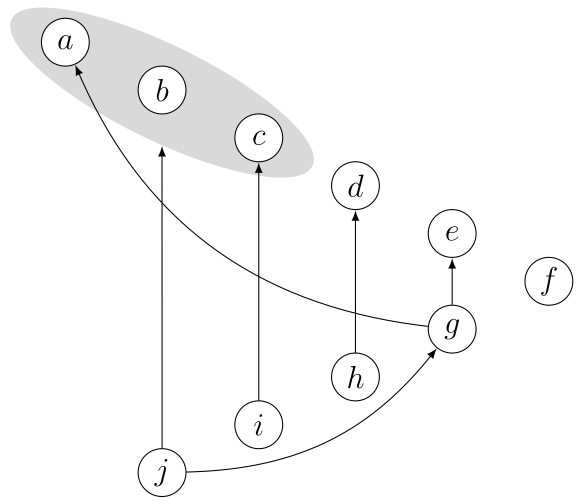

Figure 9: A minimal tournament for which BA and MC properly intersect. $BA(T) = A \setminus \{c, f\}$ whereas $MC(T) = A \setminus \{c, i\}$. Omitted directed edges point downwards.

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