On the Tradeoff Between Efficiency and Strategyproofness

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(joint work with Haris Aziz, Florian Brandl, and Markus Brill)

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Agents have complete and transitive preference relations $\succeq_i$ over a finite set of alternatives $A$.

A social decision scheme $f$ maps a preference profile $(\succeq_1, \ldots, \succeq_n)$ to a lottery $\Delta(A)$.

Special case: Random assignment (aka house allocation). $A$ is the set of deterministic assignments.

- Agents are indifferent between all assignments in which they are assigned the same object.
efficiency

No agent can be made better off without making another one worse off

strategyproofness

No agent can obtain a more preferred outcome by misreporting his preferences.
Only Dictatorship
strict preferences; Gibbard (1973), Satterthwaite (1975)
Efficiency:

There is no $p \in \Delta(A)$ such that $p \succeq_i f(\cdot)$ for all $i \in N$ and $p \succ_i f(\cdot)$ for some $i \in N$.

Strategyproofness:

There is no $\succ_i'$ such that $f(\succ_i', \cdot) \succ_i f(\succ_i, \cdot)$.

Extend preferences over alternatives to (incomplete) preferences over lotteries!
there is no \( p \in \Delta(A) \) such that
\[ p \succeq_i f(\cdot) \]
for all \( i \in N \) and
\[ p \succ_i f(\cdot) \]
for some \( i \in N \)

weak: there is no \( \succ_i' \) such that
\[ f(\succ_i', \cdot) \succ_i f(\succ_i, \cdot) \]

strong: for all \( \succ_i' \) it holds that
\[ f(\succ_i, \cdot) \succ_i f(\succ_i', \cdot) \]

Extend preferences over alternatives to (incomplete) preferences over lotteries!
Sure Thing (ST)

\[
\begin{array}{c}
a > b > c \\
p = (\frac{2}{3}, \frac{1}{3}, 0) \\
q = (0, \frac{1}{3}, \frac{2}{3})
\end{array}
\]

\[p \succeq_{ST} q \iff \forall x \in \text{supp}(p) \setminus \text{supp}(q), y \in \text{supp}(q): x > y\]
\[\land \forall x \in \text{supp}(p), y \in \text{supp}(q) \setminus \text{supp}(p): x > y\]
\[\land \forall x \in \text{supp}(p) \cap \text{supp}(q): p(x) = q(x)\]

- loosely based on Savage’s sure-thing principle
- inspired by non-probabilistic preference extensions due to Fishburn (1972) and Gärdenfors (1979)
Bilinear Dominance (BD)

\[
\begin{align*}
\begin{array}{ccc}
a & b & c \\
p & \frac{1}{2} & \frac{1}{2} & 0 \\
q & \frac{1}{3} & \frac{1}{3} & \frac{1}{3}
\end{array}
\end{align*}
\]

\[\forall x, y \in A : x > y \Rightarrow p(x) \cdot q(y) \geq p(y) \cdot q(x)\]

- for every pair of alternatives, it’s more likely that \(p\) yields the better alternative and \(q\) the worse alternative
- \(p\) is preferred to \(q\) for every consistent SSB utility function
- Fishburn (1984), Aziz et al. (2015)

\[
\forall \succeq : \succeq^{ST} \subseteq \succeq^{BD}
\]
Stochastic Dominance (SD)

\[
\begin{align*}
 a > b > c \\
p &= \left( \frac{1}{2}, 0, \frac{1}{2} \right) \\
q &= \left( 0, \frac{1}{2}, \frac{1}{2} \right)
\end{align*}
\]

\[ p \succeq^{SD} q \iff \forall x \in A: \sum_{y \succeq x} p(y) \geq \sum_{y \succeq x} q(y) \]

\[ \forall \succ: \succeq^{ST} \subset \succeq^{BD} \subset \succeq^{SD} \]

- for every alternative, it's more likely that \( p \) yields something better
- \( p \) yields more expected utility for every consistent vNM function
- Bogomolnaia & Moulin (2001) and many others
Pairwise Comparison (PC)

\[
an > b > c
\]
\[
p = \begin{pmatrix} \frac{2}{3} & 0 & \frac{1}{3} \end{pmatrix}
\]
\[
q = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}
\]

\[
p \succeq_{PC} q \iff \forall x \in A: \sum_{x \succeq y} p(x) q(y) \geq \sum_{x \succeq y} q(x) p(y)
\]

- it’s more likely that \( p \) yields a better alternative
- minimizes ex ante regret
- \( \succeq_{PC} \) is a complete relation for all \( \succeq \)

\[
\forall \succeq: \succeq^{ST} \subset \succeq^{BD} \subset \succeq^{SD} \subset \succeq^{PC}
\]
Only Random Dictatorship
strict preferences; Gibbard (1977)
No assignment rule
strict preferences; Bogomolnaia & Moulin (2001)
Probabilistic Serial (PS) assignment rule
strict preferences, Bogomolnaia & Moulin (2001)
No anonymous and neutral social decision scheme
Aziz, Brandl, & B. (2014)
No anonymous and neutral social decision scheme

Brandl, B., & Geist (2016)
SD Impossibility

- requires at least 4 agents and at least 4 alternatives
  - more than 31 million possible preferences profiles
- was shown with the help of a computer (SMT solver)
- proof has been extracted from the solver’s output and brought into human-readable form
- operates on 47 canonical preference profiles and is very tedious to check
- has been verified by a computer (Isabelle/HOL)
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Proven Complexity of Related Results

• Bogomolnaia and Moulin [2001]
• Zhou [1990]
• Katta and Sethuraman [2006]
• Aziz et al. [2013]
• Aziz et al. [2014, Theorems 2 & 4]
• Aziz et al. [2014, Theorem 3]
• this paper

• Social Choice
• House Allocation

The Incompatibility of Efficiency and Strategyproofness

Florian Brandl
No pairwise social decision scheme
Aziz, Brandl, & B. (2014)
Random Serial Dictatorship

- Extension of random dictatorship to weak preferences
  - pick an ordering of agents uniformly at random
  - sequentially narrow down the set of alternatives by letting each agent restrict it to his most preferred ones.
- Widespread assignment rule (aka random priority)

\[
\begin{array}{ccc}
1 & 1 & 1 \\
\frac{1}{2} a + \frac{1}{6} b + \frac{1}{3} c
\end{array}
\]

1,2,3: c
1,3,2: a
2,1,3: c
2,3,1: b
3,1,2: a
3,2,1: a
Random Serial Dictatorship
Aziz, B., & Brill (2013)
Maximal Lotteries

- First studied by Kremeras (1965) and Fishburn (1984)
- preference profiles induce **symmetric zero-sum games**
- maximal lotteries correspond to **mixed maximin strategies** in these games

\[
\begin{array}{ccc}
2 & 2 & 1 \\
\hline
a & b & c \\
b & c & a \\
c & a & b \\
\end{array}
\]

\[
\begin{array}{ccc|ccc}
& a & b & c \\
\hline
a & 0 & 1 & -1 \\
b & -1 & 0 & 3 \\
c & 1 & -3 & 0 \\
\end{array}
\]

\[
\frac{3}{5} a + \frac{2}{5} b + \frac{1}{5} c
\]
Maximal Lotteries
Aziz, B., & Brill (2013)
Conclusion

- No social decision scheme satisfies moderate degrees of efficiency and strategyproofness.
- \textit{RSD} is very strategyproof, but only a little efficient.
- \textit{ML} is very efficient, but only a little strategyproof.
- Further results
  - \textit{RSD} and \textit{ML} are \textit{ST}-group-strategyproof, but not \textit{SD}-group-strategyproof.
  - No anonymous and neutral social decision scheme is \textit{ex post} efficient and \textit{BD}-group-strategyproof, even when preferences are dichotomous.