Strategyproof Social Choice When Preferences and Outcomes May Contain Ties

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The Gibbard-Satterthwaite theorem implies that all anonymous, Pareto-optimal, and single-valued social choice functions (SCFs) can be strategically manipulated. In this paper, we investigate whether there exist *set-valued* SCFs that satisfy these conditions under various assumptions about how single alternatives are eventually selected from the choice set. These assumptions include even-chance lotteries as well as resolute choice functions and linear tie-breaking orderings unknown to the agents. For strict preferences, some SCFs are known to satisfy strategyproofness under these assumptions. Our findings for *weak preferences* are much more negative. We show that *(i)* all anonymous Pareto-optimal SCFs where ties are broken according to some linear tie-breaking ordering or by means of even-chance lotteries are manipulable, and that *(ii)* all pairwise Pareto-optimal SCFs are manipulable for any deterministic tie-breaking rule. These results are proved by reducing the statements to finite—yet very large—problems, which are encoded as formulas in propositional logic and then shown to be unsatisfiable by a computer.

1. Introduction

One of the most prominent results in social choice theory, the Gibbard-Satterthwaite theorem, shows that all anonymous, Pareto-optimal, and single-valued social choice functions (SCFs) are susceptible to strategic manipulation (Gibbard, 1973; Satterthwaite, 1975).[1] The restriction to single-valued SCFs has been identified as a major shortcoming of the theorem. For instance, Gärdenfors (1976) claims that "[resoluteness] is a rather restrictive

[1]The actual statement by Gibbard and Satterthwaite is somewhat stronger because it uses non-dictatorship instead of anonymity and non-imposition instead of Pareto-optimality.

and unnatural assumption." In a similar vein, Kelly (1977) writes that "the Gibbard-Satterthwaite theorem [. . .] uses an assumption of singlevaluedness which is unreasonable" and Taylor (2005) that "If there is a weakness to the Gibbard-Satterthwaite theorem, it is the assumption that winners are unique." This sentiment is echoed by various other authors (see, e.g., Barberà, 1977b; Duggan and Schwartz, 2000; Nehring, 2000; Barberà et al., 2001; Ching and Zhou, 2002). The problem with single-valuedness is that the SCF has to return a single alternative *based on the preferences only*. For example, if there are two alternatives, a and b, and two agents such that one prefers a and the other one b, there is no deterministic way of selecting a single alternative without violating basic fairness conditions such as anonymity and neutrality.

We investigate whether there exist anonymous, Pareto-optimal, and strategyproof *set-valued* SCFs under various assumptions on how ties are broken to eventually select a single alternative. Our main results are as follows.

1. There are no such SCFs when ties are broken according to some linear tie-breaking ordering or by means of an even-chance lottery (Theorem 1).

2. There are no such pairwise SCFs for any deterministic tie-breaking rule (Theorem 2).

In contrast to the Gibbard-Satterthwaite theorem, our results crucially rely on the possibility of ties in the preferences. In many settings, especially when there are large numbers of alternatives, it is natural or even inevitable to allow agents to express indifferences between alternatives. Moreover, in many subdomains of social choice such as coalition formation, matching, or allocation, agents are typically indifferent between outcomes in which their own coalition, matching partner, or assignment is identical. In comparison to other results in this stream of research, the assumption of weak preferences allows us to prove impossibility theorems that only require surprisingly weak assumptions with respect to the tie-breaking mechanism (see Section 2). In fact, it has turned out that for strict preferences, positive results can be obtained under the same assumptions (see, e.g., Feldman, 1979; Nehring, 2000; Brandt, 2015).

We have obtained our impossibilities using computer-aided theorem proving techniques that were pioneered by Tang and Lin (2009) and have been successfully used to tackle other problems in social choice (see, e.g., Tang and Lin, 2009; Geist and Endriss, 2011; Brandt and Geist, 2016; Brandl et al., 2015a; Brandt et al., 2017; Brandl et al., 2018).[2] The basic idea is to reduce the statement in question to a finite—yet very large—problem, which is encoded as a formula in propositional logic and then shown to be unsatisfiable by a so-called SAT solver. We then extract a minimal unsatisfiable set of constraints from the formula and translate this back into a human-readable proof of the result. Despite great efforts to simplify the proof of our main result as much as possible, it remains rather complex as it argues about 21 different preference profiles. We therefore verified the proof using the interactive theorem prover Isabelle, which releases any need to verify our program for

[2]An excellent introduction to these techniques has been provided by Geist and Peters (2017).

generating the proof. In contrast to previous papers, we are even able to give a *lower bound* on the proof complexity: no such proof is possible using less than 19 preference profiles. This can be considered as evidence that it is unlikely that the statement would have been proved without the help of computers, underlining the potential of computer-aided theorem proving in social choice theory.

2. Related Work

There already is a large body of literature dealing with impossibility theorems for strategyproof set-valued SCFs. In comparison to our theorems, these results require rather restrictive additional assumptions and/or utilize stronger notions of strategyproofness. As explained in Section 4, our tie-breaking assumptions result in strategyproofness notions based on two preference extensions: Kelly-strategyproofness and Fishburn-strategyproofness.

Early results by Barberà (1977a) and Kelly (1977) using Kelly-strategyproofness (or even weaker notions) required SCFs to be *quasi-transitively rationalizable*, a condition which is almost prohibitive on its own (see, e.g., Mas-Colell and Sonnenschein, 1972).[3] MacIntyre and Pattanaik (1981) and Bandyopadhyay (1983a) use similar—albeit slightly weaker—rationalizability conditions such as *minimal binariness* or *quasi-binariness* while Brandt (2015), improving on a result by Gärdenfors (1976), uses Condorcet-consistency. Barberà (1977b) restricts attention to *positively responsive* SCFs, which are almost always single-valued. Only very few commonly considered SCFs satisfy positive responsiveness, most notably Borda's rule and some of its variations.

Results with less restrictive assumptions typically require significantly stronger notions of strategyproofness. For example, while Ching and Zhou also use Fishburn's preference extension like we do in Theorem 1, they require that the original outcome is Fishburn-comparable to the choice set for *any* misrepresentation of preferences. This results in a very demanding notion of strategyproofness that is much stronger than Fishburn-strategyproofness and any other form of strategyproofness mentioned in this section. Duggan and Schwartz (2000) have shown that any non-imposing and non-dictatorial SCF that satisfies a condition they refer to as "residual resoluteness" can be manipulated by an optimist or by a pessimist. The corresponding notion of strategyproofness is stronger than Fishburn-strategyproofness (when preferences are strict) and stronger than Kelly-strategyproofness (without imposing any restrictions on preferences). Their remaining conditions—except residual resoluteness—are weaker than ours: non-imposition is weaker than Pareto-optimality and non-dictatorship is weaker than anonymity. Residual resoluteness is a rather technical condition that is difficult to justify (see, e.g., Rodríguez-Álvarez, 2007; Sato, 2008), and which is, for example, violated by the Pareto rule. To the best of

[3] This is acknowledged by Kelly (1977) who writes that "one plausible interpretation of such a theorem is that, rather than demonstrating the impossibility of reasonable strategy-proof social choice functions, it is part of a critique of the regularity [rationalizability] conditions."

our knowledge, the only impossibility using Fishburn-strategyproofness is a theorem by Brandt and Geist (2016), which only holds for the restricted class of majoritarian SCFs (aka tournament solutions).

The models considered by Barberà et al. (2001), Benoît (2002), and Sato (2014) differ from the ones considered so far in that they assume that agents submit *complete* preference relations over sets, subject to certain rationality constraints. Hence, their results can be interpreted as results about single-valued SCFs where the set of alternatives is defined as the set of all non-empty subsets of some set of candidates and the domain of admissible preferences is restricted. Barberà et al. (2001) prove a remarkable analogue of the Gibbard-Satterthwaite theorem for the domain of all preference relations over sets that concur with Fishburn's extension. However, Barberà et al.'s notion of strategyproofness is stronger than our Fishburn-strategyproofness because it rests on the assumption that all sets are comparable. The strategyproofness notions used by Benoît and Sato are weaker than those of Duggan and Schwartz (2000) and Barberà et al. (2001), but incomparable to both Fishburn- and Kelly-strategyproofness. However, their results rely on the comparatively strong assumption that an agent who prefers a to b to c may prefer the set $\{b\}$ to the set $\{a, c\}$ whereas we always deem these sets to be incomparable.[4] Benoît and Sato also employ additional technical conditions called *near unanimity* and *non-decisiveness*, respectively, which are reminiscent of Duggan and Schwartz's residual resoluteness and which are controversial when there are only few agents. For example, near unanimity is violated by Borda's rule for three agents and three alternatives (Sato, 2014).

No such conditions are required in our results. This is possible because we allow for weak preferences. In contrast to our theorems, the results by Duggan and Schwartz (2000), Barberà et al. (2001), Ching and Zhou (2002), Benoît (2002), Rodríguez-Álvarez (2007), Sato (2008, 2014), and Brandt and Geist (2016) all hold even when preferences are strict. While the assumption of strict preferences makes the statements stronger in the sense that they hold within a smaller domain of preferences, stronger conditions are required to derive an impossibility.

3. Preliminaries

Let $A = \{a, b, \dots\}$ be a finite set of m alternatives and $N = \{1, \dots, n\}$ a finite set of agents. A *(weak) preference relation* is a complete, reflexive, and transitive binary relation on A. The preference relation of agent i is denoted by $\succcurlyeq_i$, the set of all preference relations by $\mathcal{R}$. We write $\succ_i$ for the strict part of $\succcurlyeq_i$, i.e., $x \succ_i y$ if $x \succcurlyeq_i y$ but not $y \succcurlyeq_i x$, and $\sim_i$ for the indifference part of $\succcurlyeq_i$, i.e., $x \sim_i y$ if $x \succcurlyeq_i y$ and $y \succcurlyeq_i x$. A preference relation $\succcurlyeq_i$ is called *strict* if it additionally is anti-symmetric, i.e., $x \succ_i y$ or $y \succ_i x$ for all distinct alternatives x, y. The set of all strict preference relations is denoted by $\mathcal{L}$. We will compactly represent

[4] Within the domain of strict preferences, our proof of Theorem 1 does not make any assumptions on the comparability of sets that is not also made by Benoît (2002) and Sato (2014).

a preference relation as a comma-separated list where all alternatives among which an agent is indifferent are written as a set. For example $x \succ_i y \sim_i z$ is represented by $\succcurlyeq_i \colon x, \{y, z\}$. A *preference profile* R is a function from a set of agents N to the set of preference relations $\mathcal{R}$. The set of all preference profiles is denoted by $\mathcal{R}^N$. Our central objects of study are *social choice functions (SCFs)*, i.e., SCFs that map a preference profile to a set of alternatives called the *choice set*. Formally, an SCF is a function

$$f \colon \mathcal{R}^N \to \mathcal{C}$$

where $\mathcal{C} = 2^A \setminus \emptyset$ denotes the non-empty subsets of A. Such functions are sometimes also called *social choice correspondences*.

Given a preference profile R, an alternative x *Pareto-dominates* another alternative y if $x \succcurlyeq_i y$ for all $i \in N$ and $x \succ_j y$ for some $j \in N$. An alternative is *Pareto-optimal* if it is not Pareto-dominated by some other alternative. The notion of Pareto-optimality can be used to define a simple SCF that returns the set of all Pareto-optimal alternatives. Formally,

$$PO(R) = \{x \in A \colon x \text{ is not Pareto-dominated in } R\}.$$

An SCF f is said to be *Pareto-optimal* if $f(R) \subseteq PO(R)$ for all $R \in \mathcal{R}^N$.

Another simple SCF, with appealing strategic properties, is *serial dictatorship*, which is based on a fixed, but arbitrary, ordering of the agents. First, the set of alternatives is restricted to the ones top-ranked by the first agent. Then, the next agent successively refines the set of alternatives to the set of most preferred alternatives from the remaining set. Formally, serial dictatorship returns $\max_{\succcurlyeq_n} \circ \ \ldots \ \circ \max_{\succcurlyeq_1}(A)$, where $\max_{\succcurlyeq_i}(X)$ denotes the maximal elements of X according to the preference relation $\succcurlyeq_i$. Serial dictatorship satisfies Pareto-optimality and any reasonable form of strategyproofness, because choosing one's maximal elements is strategyproof for each agent, ruling out any possibility to manipulate. However, serial dictatorship is weakly dictatorial in the sense that it only returns alternatives top-ranked by a pre-determined agent.

Two common symmetry conditions for SCFs are anonymity and neutrality. An SCF is *anonymous* if the choice set does not depend on the identities of the agents and *neutral* if it is symmetric with respect to alternatives. Formally, an SCF is anonymous if $f(R) = f(R')$ for all $R, R' \in \mathcal{R}^N$ and all bijections $\pi \colon N \to N$ such that $\succcurlyeq_i = \succcurlyeq'_{\pi(i)}$ for all $i \in N$. For a permutation π on A and a preference relation $\succcurlyeq_i$, $\pi(x) \succcurlyeq_i^\pi \pi(y)$ if and only if $x \succcurlyeq_i y$. An SCF f is *neutral* if $f(R^\pi) = \pi(f(R))$ for all $R \in \mathcal{R}^N$ and all permutations $\pi \colon A \to A$ on the set of alternatives. PO is anonymous and neutral while serial dictatorship clearly violates anonymity.

For a preference profile $R \in \mathcal{R}^N$, let

$$n_R(x, y) = |\{i \in N \colon x \succcurlyeq_i y\}|$$

be the number of agents who prefer x to y. The majority margin of x over y in R is denoted by $g_R(x, y)$ where

$$g_R(x, y) = n_R(x, y) - n_R(y, x).$$

An SCF f is *pairwise* if for all $R, R' \in \mathcal{R}^N$, $f(R) = f(R')$ whenever $g_R(x,y) = g_{R'}(x,y)$ for all alternatives $x, y \in A$. In other words, the choice set of a pairwise SCF only depends on the anonymized comparisons between pairs of alternatives. Since majority margins are invariant under permutations of agents, pairwise SCFs are anonymous.[5] When ties are allowed, pairwiseness is slightly stronger than Fishburn's C2, which requires that the SCF only depends on n_R (Fishburn, 1977). This is due to the fact that a pair of opposed preferences affects the majority margin exactly like indifferences do. Hence, PO satisfies C2, but violates pairwiseness because you cannot tell whether x Pareto-dominates y by only looking at $g_R(x,y)$.

Nevertheless, there is a large number of attractive pairwise and Pareto-optimal SCFs (see, e.g., Fishburn, 1977; Fischer et al., 2016). Typical examples are Borda's rule, Kemeny's rule, the Simpson-Kramer rule (aka maximin), Nanson's rule, Schulze's rule, ranked pairs, and the essential set.

A very influential concept in social choice theory is that of a Condorcet winner, i.e., an alternative that is preferred to every other alternative by some majority of agents. Formally, an alternative x is a *Condorcet winner* in R if $g_R(x,y) > 0$ for all $y \in A \setminus \{x\}$. A *Condorcet extension* is an SCF that uniquely returns a Condorcet winner whenever one exists. All of the pairwise SCFs mentioned above, except Borda's rule, are Condorcet extensions.

4. Strategyproofness and Tie-Breaking

An important phenomenon in social choice is that agents misrepresent their preferences in order to obtain a more preferred outcome. Often, this is referred to as strategic manipulation.

When defining strategic manipulability for set-valued SCFs, one needs to specify how ties are broken and what the agents know about the tie-breaking mechanism. We will distinguish between two different types of breaking ties: deterministic rules (such as a fixed linear tie-breaking ordering) and randomized rules (such as returning an even-chance lottery over several alternatives).

4.1. Deterministic Tie-Breaking

Let us first consider deterministic tie-breaking functions $g : \mathcal{C} \to A$. If g is known by the agents, this model is equivalent to that of single-valued SCFs and the Gibbard-Satterthwaite theorem applies. We therefore make the weaker—yet reasonable—assumption that agents are unaware of the concrete tie-breaking function g, but only know that g belongs to a certain *class* of functions G. Based on this uncertainty, we define the following strong notion of strategy manipulability (which in turn results in a weak notion of strategyproofness).

[5] Note that, in contrast to other papers, we do not require pairwise SCFs to be neutral (cf. Aziz et al., 2014; Brandl et al., 2015b).

An SCF f is *G-manipulable* if there exist preference profiles $R, R' \in \mathcal{R}^N$, and an agent $i \in N$ with $\succcurlyeq_j = \succcurlyeq'_j$ for all $j \neq i$ such that

$$g(f(R')) \succcurlyeq_i g(f(R)) \text{ for all } g \in G \text{ and}$$
$$g(f(R')) \succ_i g(f(R)) \text{ for some } g \in G.$$

f is *G-strategyproof* if it is not G-manipulable.

A very weak notion of strategyproofness is obtained when quantifying over *all* possible tie-breaking functions G_{all} where

$$G_{\text{all}} = \left\{ g \in A^{\mathcal{C}} \colon g(X) \in X \text{ for all } X \in \mathcal{C} \right\}.$$

A natural subset of G_{all} is given by all tie-breaking functions that break ties according to some linear tie-breaking ordering $\geq$ on A (such as the alphabetical ordering). G_{lin} consists of all tie-breaking functions that consistently return the alternative in the choice set that is ranked highest according to some fixed tie-breaking ordering. Formally,

$$G_{\text{lin}} = \{ g \in G_{\text{all}} \colon \text{there is } \geq \, \in \mathcal{L} \text{ such that for all } X \in \mathcal{C} \text{ and } y \in X,\ g(X) \geq y \}.$$

If $\geq$ is known by the agents, this setting is again equivalent to that of single-valued SCFs. By quantifying over all possible linear tie-breaking orderings, we obtain G_{lin}-strategyproofness, which is stronger than G_{all}-strategyproofness. This notion can, for example, be motivated by the existence of a chairman, whose preferences are unknown and who eventually picks a single alternative from the choice set.

4.2. Randomized Tie-Breaking

Another common way to break ties is to make use of randomization. A randomized tie-breaking function is any function $g : \mathcal{C} \to \Delta(A)$ where $\Delta(A)$ is the set of all probability distributions (or lotteries) over alternatives in A. Lotteries are typically compared by assuming the existence of a utility function $u : A \to \mathbb{R}$ that assigns numeric values to alternatives and that is consistent with the agent's ordinal preferences. A utility function u is consistent with a preference relation $\succsim$ if, for all $x, y \in A$, $x \succsim y$ if and only if $u(x) \geq u(y)$. The set of all utility functions consistent with $\succsim$ will be denoted by $U(\succsim)$. With slight abuse of notation, we extend any utility function u to lotteries via *expected* utility. For some class of randomized tie-breaking functions G^{Δ}, an SCF f is *G^{Δ}-manipulable* if there exist preference profiles $R, R' \in \mathcal{R}^N$, and an agent $i \in N$ with $\succcurlyeq_j = \succcurlyeq'_j$ for all $j \neq i$ such that

$$u(g(f(R'))) > u(g(f(R))) \text{ for all } g \in G^{\Delta} \text{ and all } u \in U(\succsim_i).$$[6]

[6] For the classes of randomized tie-breaking functions we consider, we could equivalently demand that $u(g(f(R'))) \geq u(g(f(R)))$ for all $g \in G^{\Delta}$ and all $u \in U(\succsim_i)$ with at least one strict inequality.

G^{Δ}_{all}-strategyproofness is obtained when quantifying over *all* lotteries that randomize among alternatives in the choice set, i.e.,

$$G^{\Delta}_{\text{all}} = \left\{ g \in \Delta(A)^{\mathcal{C}} \colon \operatorname{supp}(g(X)) = X \text{ for all } X \in \mathcal{C} \right\}.$$

A very natural special case of randomized tie-breaking is to randomize uniformly among the alternatives, i.e. G consists of the unique function *uni*, which maps every choice set to an even-chance lottery over its alternatives. A more general set of tie-breaking functions is obtained by fixing an *a priori* weight function $w : A \to \mathbb{R}_{\geq 0}$ and then assign probabilities to alternatives in the choice set that are proportional to their weights.

$$G^{\Delta}_{\text{pro}} = \left\{ g \in G^{\Delta}_{\text{all}} \colon \text{there is } w \in (\mathbb{R}_{\geq 0})^A \text{ such that for all } X \in \mathcal{C}, g(X) \cong w|_X \right\}.$$

If the weight function assigns the same weight to all alternatives, this corresponds to an even-chance lottery.

In Section 8, we discuss generalizations of this model where SCFs directly map to the set of all lotteries $\Delta(A)$.

5. Preference Extensions

As it turns out, each variant of G-manipulation introduced in the previous section can be modeled using so-called *preference extensions* which extend the agents' preferences over alternatives to preferences over sets of alternatives. The three preference extensions considered in this paper are *Kelly's extension*, *Fishburn's extension*, and the *even-chance extension* (Kelly, 1977; Fishburn, 1972; Gärdenfors, 1979). For all $X, Y \in \mathcal{C}$ and $\succcurlyeq_i \in \mathcal{R}$,

$$X \succcurlyeq_i^K Y \text{ iff } x \succcurlyeq_i y \text{ for all } x \in X,\ y \in Y \tag{Kelly}$$
$$X \succcurlyeq_i^F Y \text{ iff } X \setminus Y \succcurlyeq_i^K Y \text{ and } X \succcurlyeq_i^K Y \setminus X, \text{ and} \tag{Fishburn}$$
$$X \succcurlyeq_i^E Y \text{ iff } |\{x \in X \colon x \succcurlyeq_i z\}|/|X| \geq |\{y \in Y \colon y \succcurlyeq_i z\}|/|Y| \text{ for all } z \in A. \tag{Even-chance}$$

The strict part of these relations will be denoted by $\succ_i^K$, $\succ_i^F$, and $\succ_i^E$ respectively. We thus have that

$$\begin{aligned}
X \succ_i^K Y &\text{ iff } X \succcurlyeq_i^K Y \text{ and there is } x \in X, y \in Y \text{ such that } x \succ_i y,\\
X \succ_i^F Y &\text{ iff } X \succcurlyeq_i^F Y \text{ and there is } x \in X \setminus Y, y \in Y \text{ or } x \in X, y \in Y \setminus X \text{ such that}\\
&\qquad x \succ_i y, \text{ and}\\
X \succ_i^E Y &\text{ iff } X \succcurlyeq_i^E Y \text{ and there is } z \in A \text{ such that}\\
&\qquad |\{x \in X \colon x \succcurlyeq_i z\}|/|X| > |\{y \in Y \colon y \succcurlyeq_i z\}|/|Y|.
\end{aligned}$$

It follows from the definitions that the even-chance extension is a refinement of Fishburn's extension, which in turn is a refinement of Kelly's extension. This inclusion relationship also holds for the strict parts of the three relations.

$$\succcurlyeq_i^K \subseteq \succcurlyeq_i^F \subseteq \succcurlyeq_i^E \text{ and } \succ_i^K \subseteq \succ_i^F \subseteq \succ_i^E \text{ for every } \succcurlyeq_i \in \mathcal{R}.$$

Even when $m = 3$ (which is sufficient for our impossibility theorems), the even-chance extension is incomplete because the sets $\{b\}$, $\{a, c\}$, and $\{a, b, c\}$ are incomparable when $\succcurlyeq_i\colon a, b, c$. Furthermore, the only difference between the even-chance extension and Fishburn's extension is that $\{a, b\} \succ_i^E \{a, c\}$ and $\{a, c\} \succ_i^E \{b, c\}$ (while these pairs of sets are incomparable according to Fishburn's extension).

With these extensions at hand, we can formally define strategyproofness of set-valued SCFs without making reference to tie-breaking functions. An SCF f is *Kelly-manipulable* if there exist preference profiles $R, R' \in \mathcal{R}^N$, and an agent $i \in N$ such that $\succcurlyeq_j = \succcurlyeq'_j$ for all $j \neq i$ and $f(R') \succ_i^K f(R)$. f is said to satisfy *Kelly-strategyproofness* if it is not Kelly-manipulable. Fishburn-strategyproofness and even-chance-strategyproofness are defined analogously. The relationship between the preference extensions implies that even-chance-strategyproofness is stronger than Fishburn-strategyproofness and that Fishburn-strategyproofness is stronger than Kelly-strategyproofness.

5.1. Tie-Breaking Rules and Preference Extensions

The connection between both strategyproofness notions and the tie-breaking assumptions given in Section 4 is as follows.

Proposition 1. *Let f be an SCF. Then,*

(i) f is Kelly-strategyproof iff it is G_{all}-strategyproof iff it is G^{Δ}_{all}-strategyproof,

(ii) f is Fishburn-strategyproof iff it is G_{lin}-strategyproof iff it is G^{Δ}_{pro}-strategyproof, and

(iii) f is even-chance-strategyproof iff it is $\{uni\}$-strategyproof.

The equivalences between Kelly-strategyproofness and G_{all}-strategyproofness and $G^{\Delta}_{\mathrm{all}}$-strategyproofness are relatively straightforward (see, e.g., Erdamar and Sanver (2009, Theorem 3.1) for the former and Gärdenfors (1979, Prop. 5) for the latter). The equivalence of Fishburn-strategyproofness and G_{lin}-strategyproofness is essentially due to the fact that for all $g \in G_{\mathrm{lin}}$ and $X, Y \in \mathcal{C}$, $g(X), g(Y) \in X \cap Y$ implies $g(X) = g(Y)$ (see, e.g., Erdamar and Sanver, 2009, Theorem 3.4). The equivalence between Fishburn-strategyproofness and $G^{\Delta}_{\mathrm{pro}}$-strategyproofness was shown by Ching and Zhou (2002, Lemma 1). Even-chance-strategyproofness and $\{uni\}$-strategyproof are equivalent because of standard stochastic dominance arguments (see, e.g., Gärdenfors, 1979).

5.2. Manipulation of the Pareto Rule

In order to illustrate the definitions of Kelly-strategyproofness and Fishburn-strategyproofness, consider the Pareto rule PO. Let us first have a look at the following preference profile R (due to Feldman, 1979).

$$\succcurlyeq_1 : a, \{b, c\} \qquad\qquad \succcurlyeq_2 : \{b, c\}, a$$

Clearly, $PO(R) = \{a, b, c\}$. Now assume that Agent 1 changes his preferences to $\succcurlyeq'_1$ resulting in preference profile R'.

$$\succcurlyeq'_1 : a, b, c \qquad\qquad \succcurlyeq'_2 : \{b, c\}, a$$

Alternative c is Pareto-dominated by alternative b in R' and $PO(R') = \{a, b\}$. This does not constitute a Kelly-manipulation because

$$\{a, b\} \not\succ_1^K \{a, b, c\}$$

In fact, these sets are incomparable according to $\succ_1^K$. This is in line with Proposition 1 because there could be a deterministic tie-breaking function that selects b from $\{a, b\}$ and a from $\{a, b, c\}$. The picture looks different for Fishburn's extension, however, as

$$\{a, b\} \succ_1^F \{a, b, c\}.$$

To see that this concurs with Proposition 1, consider the linear tie-breaking ordering $\geq$ with $c \geq a \geq b$. According to this tie-breaking ordering, a will be selected from $\{a, b\}$ and c from $\{a, b, c\}$. Since $a \succ_1 c$ and for all other tie-breaking orderings, Agent 1 is indifferent between the eventually chosen alternatives, we have a Fishburn-manipulation. $\{a, b\}$ is also preferred to $\{a, b, c\}$ when ties are broken by even-chance lotteries: for all utility functions consistent with $\succcurlyeq_1$, the expected utility for an even-chance lottery between a and b exceeds that of an even-chance lottery between all three alternatives.

The example shows that PO is Fishburn-manipulable (and consequently also even-chance-manipulable). By contrast, as first shown by Feldman (1979), PO does satisfy Kelly-strategyproofness. Since Feldman proves this statement by making reference to stronger strategyproofness notions, we give a self-contained proof below.

Proposition 2. *PO is Kelly-strategyproof.*

Proof. Assume for contradiction that there are two preference profiles R and R', and an agent $i \in N$ such that $\succcurlyeq_j = \succcurlyeq'_j$ for all $j \neq i$ and $PO(R') \succ_i^K PO(R)$. It is well-known that the Pareto dominance relation is transitive and that every Pareto-dominated alternative is Pareto-dominated by some alternative in PO. This also implies that PO contains at least one top-ranked alternative from every agent because top-ranked alternatives can only be Pareto-dominated by other top-ranked alternatives. Hence, $PO(R')$ contains only top-ranked alternatives of agent i while $PO(R)$ contains at least one alternative that is not top-ranked by agent i. This means that there is some $x \in PO(R) \setminus PO(R')$ and there is no $x' \in PO(R) \setminus \{x\}$ with $x' \sim_i x$. Since $x \notin PO(R')$, there has to be some $y \in PO(R')$ such that y Pareto-dominates x in R'. Moreover, y does not Pareto-dominate x in R. This implies that $x \succ_i y$. Since $x \in PO(R)$ and $y \in PO(R')$, it is impossible that $PO(R') \succ_i^K PO(R)$. □

Remark 1. *PO* is not the most discriminating Kelly-strategyproof SCF. For example, using a proof similar to that of Proposition 2, it can be shown that the SCF that returns all Pareto-optimal alternatives that are top-ranked by at least one agent is Kelly-strategyproof as well.

Remark 2. Proposition 2 and Remark 1 also hold when replacing strategyproofness with *group-strategyproofness* where a group of agents can manipulate such that all of them are strictly better off (see, also, Bandyopadhyay, 1983b; Umezawa, 2009).

6. Computer-aided Theorem Proving

Our results are obtained using the computer-aided proving methodology described by Brandt and Geist (2016). Understanding the proofs does not require any knowledge about how they were obtained. Readers who are only interested in the results may therefore skip to Section 7.

Our methodology is based on first providing a reduction argument (Lemma 1), which allows us to prove a statement for general domain sizes by restricting ourselves to a finite number of agents and alternatives. However, in these restricted domains, the number of anonymous SCFs is enormous (see Table 1), which renders any type of exhaustive search infeasible. For our main results, we require three agents and three alternatives, which already admits about $3.3 \cdot 10^{384}$ possible anonymous SCFs.[7] Thus, heuristic search algorithms as provided by state-of-the-art SAT solvers are required. Apart from allowing us to deal with enormous search spaces, the computer-aided approach has the major advantage that related conjectures and hypotheses, e.g., statements including additional axioms, can be checked quickly using the same framework.

m	n	Preference profiles	Pareto-optimal SCFs	SCFs
3	3	455	$\sim 5.0 \cdot 10^{179}$	$\sim 3.3 \cdot 10^{384}$
3	4	1,820	$\sim 2.8 \cdot 10^{933}$	$\sim 1.2 \cdot 10^{1,538}$
4	3	73,150	$\sim 2.2 \cdot 10^{42,914}$	$\sim 1.2 \cdot 10^{86,031}$

Table 1: Number of different profiles and Pareto-optimal SCFs when assuming anonymity.

[7] For comparison, this search space exceeds that of Theorem 3 by Brandt and Geist (2016) and lies in between that of Theorems 1 and 2 by Brandl et al. (2015a).

6.1. SAT-Solving and Proof Extraction

Basically, the core of the computer-aided approach is the aforementioned encoding of the problems to be solved as a SAT instances in *conjunctive normal form (CNF)*. For this, all axioms involved need to be stated in propositional logic. All variables are of the form $c_{R,X}$ with a preference profile R and a set of alternatives $X \subseteq A$. The semantics of these variables are that $c_{R,X}$ is true if and only if $f(R) = X$, i.e., the SCF f selects the set of alternatives X as the choice set for the preference profile R.

Although an encoding with variables $c_{R,x}$ for single alternatives x rather than choice sets would require less variable symbols, it would significantly increase the complexity of the clauses for some axioms, especially for strategyproofness. Due to the fact that strategyproofness clauses outnumber all other clauses combined, we chose the former encoding with more variables but much easier clauses. First, we ensure that the variables $c_{R,X}$ indeed model a function rather than an arbitrary relation, i.e., for each preference profile R, there is exactly one choice set X such that the variable $c_{R,X}$ is set to true. We split this into *choice set existence*,

$$\left(\forall R \in \mathcal{R}^N\right) (\exists X \subseteq A)\ c_{R,X} \equiv \bigwedge_{R \in \mathcal{R}^N} \bigvee_{X \subseteq A} c_{R,X},$$

and *uniqueness*,

$$\begin{aligned} &\left(\forall R \in \mathcal{R}^N\right) ((\forall Y, Z \subseteq A)\ Y \neq Z \to \neg(c_{R,Y} \wedge c_{R,Z})) \\ \equiv &\bigwedge_{R \in \mathcal{R}^N} \bigwedge_{Y \neq Z} (\neg c_{R,Y} \vee \neg c_{R,Z}). \end{aligned}$$

By contrast to these rather elaborate axioms, the formalization of Pareto-optimality can be easily written without logical disjunctions as

$$\begin{aligned} &\left(\forall R \in \mathcal{R}^N\right) (\forall x \notin PO(R))\ \ x \notin f(R) \\ \equiv &\bigwedge_{R \in \mathcal{R}^N} \ \bigwedge_{x \notin PO(R)} \ \bigwedge_{X \ni x} \neg c_{R,X}. \end{aligned}$$

Similar to the choice set uniqueness axiom, strategyproofness from some preference extension $\mathcal{E}$ can be encoded by

$$\begin{aligned} &\left(\forall R \in \mathcal{R}^N\right) (\forall \succcurlyeq_i \in R) \left(\forall \succcurlyeq_i' \in \mathcal{R}\right) \neg \left(f\left(R_{i \mapsto \succcurlyeq_i'}\right) \succ_i^{\mathcal{E}} f(R)\right) \\ \equiv &\bigwedge_{R \in \mathcal{R}^N} \ \bigwedge_{\succcurlyeq_i \in R} \ \bigwedge_{\succcurlyeq_i' \in \mathcal{R}} \ \bigwedge_{Y \succ_i^{\mathcal{E}} X} \left(\neg c_{R_{i \mapsto \succcurlyeq_i'},Y} \vee \neg c_{R,X}\right), \end{aligned}$$

with $R_{i \mapsto \succcurlyeq_i'}$ denoting the preference profile R where agent i's preference relation is replaced with $\succcurlyeq_i'$.

After encoding the axioms using a Java program, satisfiability of the SAT instance is checked with the Lingeling solver family by Biere (2013). If an instance turns out to be unsatisfiable, we extract a *minimal unsatisfiable core* (also called a *minimal unsatisfiable set (MUS)*), a feature which is offered by a range of SAT solvers. A subset of clauses is an *unsatisfiable core* if it is already unsatisfiable by itself. If removing any clause from the unsatisfiable core renders it satisfiable, it is called minimal. However, although an MUS is inclusion-minimal, it is not necessarily a *smallest* unsatisfiable core, i.e., a core with a minimal number of clauses or variables. In particular, an MUS does not have to be unique.

Especially with regard to proof extraction later on, we aimed at finding a smallest minimal unsatisfiable set (SMUS), for which we used the software tool Marco by Liffiton et al. (2016). Rather than merely minimizing the number of clauses of the CNF formula, we are interested in proofs that minimize the number of required preference profiles. One of the reasons behind this is that strategyproofness is responsible for most of the clauses in our SAT instances, resulting in Marco spending most of the runtime on optimizing the size of the MUS concerning the number of applications of strategyproofness only, instead of rather concentrating on the number of different preference profiles involved in it. We realized this optimization objective by using *group-oriented* CNF formulas and declaring clauses of the choice set existence axioms as *interesting* groups and all the remaining clauses as a single *don't care* group. This technique significantly increases the performance of our search for a (group-oriented) SMUS (see Liffiton and Sakallah (2008) for more details on group-oriented SAT solving).

Moreover, with the group-oriented approach, we can now also give lower bounds for the number of profiles needed in such impossibility proofs. The number of profiles seems to be a reasonable measure of proof complexity, even though it is, of course, very well possible that proofs using more profiles turn out to be "easier," e.g., by requiring fewer case distinctions. We achieve the lower bound with the tool Forqes by Ignatiev et al. (2015), as it supports a restricted version of group-oriented SAT solving, namely the specification of *don't care* clauses. In contrast to Marco, it does not compute or return approximations of an SMUS during its runtime, but rather iteratively rules out the existence of an MUS of a given size starting with the trivial size of just one clause (and finally returns an SMUS if not aborted before).

After finding a sufficiently small MUS, a proof trace can be extracted from the MUS with the help of certain SAT solvers like PicoSAT by Biere (2008). If this yields a reasonably sized proof trace, we can directly create a pen-and-paper proof by going through its main steps and translating the clauses back to the preference profile level. For this, we use a dictionary containing the correspondences between SAT variables and preference profile/choice set combinations. We used this method to find the proofs for Theorem 1 and Theorem 2.

6.2. Formal Verification

If computer-generated proofs exceed a certain size, it becomes a tedious and error-prone task for humans to translate the output of the SAT solver to a human-readable proof and thereby checking correctness. Simply accepting the black-box-like output of the SAT solver as a proof is not sufficient, as one has *(i)* to trust the correctness of the SAT solver and *(ii)* to rely on the correctness of the Java program that generates the CNF formula in the first place. The first concern is less problematic and is addressed by using a verified SAT solver (Marić, 2010). However, more importantly, there is no guarantee that the Java program meets its specification. Even a verified SAT solver may produce an overall unsound proof due to a bug in the program for encoding the axioms. To tackle this issue, we make use of the interactive theorem prover ISABELLE (see, e.g., Nipkow et al., 2002) to produce a machine-verified proof. The main application of the generic proof assistant ISABELLE is the formalization of mathematical proofs and formal verification. Building on the framework introduced by Brandl et al. (2018), the set of preference profiles and conditions obtained from the MUS is translated to higher-order logic and the user interactively develops the proof. This approach entirely removes the dependence on the unverified Java program and we obtain an independent ISABELLE proof that can even be checked manually step by step. Trustworthiness of ISABELLE is considerably high as it is widely used for verification tasks.[8]

7. Results

We start by showing that for impossibility results using Pareto-optimality and strategyproofness, it suffices to prove that the axioms are incompatible for some fixed number of alternatives and agents.

Lemma 1. *Let f be an anonymous SCF that satisfies Pareto-optimality and strategyproofness for A and N. Then there is an anonymous SCF f' that satisfies these axioms for any $A' \subseteq A$ and $N' \subseteq N$.*

Proof. We define an embedding φ of preference profiles $R' = (\succcurlyeq'_1, \dots, \succcurlyeq'_{n'})$ over N' and A' into preference profiles R over N and A by means of extending the existing preferences with $D = A \setminus A'$ as new bottom-ranked, hence Pareto-dominated, alternatives and adding indifferent agents:

$$\varphi\left(R'\right) = R$$
$$\text{with } \succcurlyeq_i = \begin{cases} \succcurlyeq'_i \,\cup\, (A \times D) & \text{if } i \in N', \\ A \times A & \text{otherwise.} \end{cases}$$

[8]Using higher-order proof assistants such as ISABELLE to prove these theorems in the first place is currently completely out of reach due to performance limitations.

Now let $f'(R') = f(\varphi(R'))$. f' is anonymous since f is anonymous and agents in N only differ by their preferences over A'. Pareto-optimality of f' holds because f is Pareto-optimal and $PO(R) = PO(R')$. Finally, f' is strategyproof because f is strategyproof and the choice sets of f' under the two profiles R' and $(R')_{i \mapsto \succcurlyeq'_i}$ are equal to the choice sets of f under the two (extended) profiles R and $R_{i \mapsto \succcurlyeq_i}$, respectively. □

It is easily seen that Lemma 1 also holds for neutral SCFs.

7.1. Fishburn-strategyproofness

Recall from Section 5 that PO is Fishburn-manipulable. Our main theorem is a much more general statement showing that *every* anonymous and Pareto-optimal SCF is Fishburn-manipulable.

Theorem 1. *There is no anonymous SCF that satisfies Pareto-optimality and Fishburn-strategyproofness for $m \geq 3$ and $n \geq 3$.*

The full proof of Theorem 1 is given in Appendix A. Here, we provide some information on its size and structure and prove a weaker version of Theorem 1 for neutral SCFs (Corollary 1). Starting from the initial, unsatisfiable SAT instance, we used Marco to find a small (group-oriented) MUS, which utilizes the 21 profiles listed in Table 3. Although the MUS does not guarantee that this is the minimal number of profiles needed, no significantly easier proof of this form exists, because we were able to compute a lower bound of 19 profiles with Forqes. With the help of PicoSAT, we extracted a proof out of the MUS which is divided in 22 main proof steps.[9] This proof has been verified by Isabelle and proof replication data is publicly available (Brandt et al., 2018).

Remark 3. The main result by Brandt and Geist (2016) shows that all Pareto-optimal *majoritarian* SCFS are Fishburn-manipulable when $m \geq 5$ and $n \geq 7$. An SCF is majoritarian if it is neutral and its outcome only depends on the (unweighted) pairwise majority relation. Examples of majoritarian SCFs are Copeland's rule, the top cycle, and the uncovered set. This result is weaker than Theorem 1, except that it even holds for strict preferences.

Remark 4. The axioms of Theorem 1 are independent of each other. PO satisfies all axioms except Fishburn-strategyproofness, serial dictatorship satisfies all axioms except anonymity, and the trivial SCF which always returns all alternatives satisfies all axioms except Pareto-optimality. Also, the bounds used in the theorem ($m \geq 3$ and $n \geq 3$) are tight, as confirmed by the SAT solver.

[9] We applied Pareto-optimality constraints manually before these proof steps to make the proof as compact as possible.

Remark 5. When assuming strict preferences, there are various Fishburn-strategyproof SCFs satisfying Pareto-optimality, e.g., PO or the SCF that returns all top-ranked alternatives (Feldman, 1979; Brandt and Brill, 2011). Theorem 1 shows that these SCFs cannot be extended to weak preferences without giving up one of these desirable properties.

Remark 6. *Weak Pareto-optimality* requires that an alternative y should not be selected whenever there is another alternative x such that $x \succ_i y$ for all $i \in N$. Theorem 1 does not hold when replacing Pareto-optimality with weak Pareto-optimality because the SCF that returns all first-ranked alternatives satisfies weak Pareto-optimality and Fishburn-strategyproofness. Note that the SCF that returns all weakly Pareto-optimal alternatives violates Fishburn-strategyproofness. This can be seen by replacing the second agent's preferences in the example given in Section 5 with $\succcurlyeq_2\colon b, \{a, c\}$.

For the reader's benefit, we now give a full, human-readable proof of a significantly weaker version of Theorem 1 which additionally assumes neutrality.[10] This proof is based on only three preference profiles (rather than 21) and requires only five strategyproofness applications (rather than 89).

Corollary 1. *There is no neutral and anonymous SCF that satisfies Pareto-optimality and Fishburn-strategyproofness for $m \geq 3$ and $n \geq 2$.*

Proof. Let $N = \{1, 2\}$ and $A = \{a, b, c\}$ and assume for contradiction that f is a neutral and anonymous SCF that satisfies Pareto-optimality and Fishburn-strategyproofness. First, consider preference profile R^1.

$$\succcurlyeq_1^1\colon a, b, c \qquad\qquad \succcurlyeq_2^1\colon b, a, c$$

By anonymity and neutrality, $a \in f(R^1)$ if and only if $b \in f(R^1)$. Together with c being Pareto-dominated (by both a and b), this implies $f(R^1) = \{a, b\}$. This already determines the choice set for the following preference profile R^2.

$$\succcurlyeq_1^2\colon a, b, c \qquad\qquad \succcurlyeq_2^2\colon \{b, c\}, a$$

Both $f(R^2) = \{a\}$ and $f(R^2) = \{b\}$ would allow for manipulations since the second agent prefers $\{a, b\}$ to $\{a\}$ in R^2 and $\{b\}$ to $\{a, b\}$ in R^1. Furthermore, $c \notin f(R^2)$ since alternative c is Pareto-dominated by b, hence $f(R^2) = \{a, b\}$. Lastly, we consider preference profile R^3.

$$\succcurlyeq_1^3\colon a, \{b, c\} \qquad\qquad \succcurlyeq_2^3\colon \{b, c\}, a$$

By anonymity and neutrality, $b \in f(R^3)$ if and only if $c \in f(R^3)$. However, if $\{b, c\} \subseteq f(R^3)$, then the first agent can deviate from $\succcurlyeq_1^3$ to $\succcurlyeq_1^2$. This only leaves $f(R^3) = \{a\}$, which allows the first agent to deviate from $\succcurlyeq_1^2$ to $\succcurlyeq_1^3$, a contradiction. □

[10] Neutrality may seem like an innocuous fairness criterion, but it can be overly restrictive in some settings (see, e.g., Sen, 1970, Section 6.1 (Critique of Anonymity and Neutrality)). In fact, many SCFs used in the real world such as supermajority rules are not neutral.

Remark 7. Using a more complicated proof, it can be shown that Corollary 1 even holds for a weakening of Fishburn-strategyproofness where choice sets can only be compared when they are disjoint or contained in each other. We are unable to prove the same for Theorem 1, even when $m = 4$ and $n = 3$ or when $m = 3$ and $n = 4$.

Since even-chance-strategyproofness is stronger than Fishburn-strategyproofness, we have the following immediate corollary.

Corollary 2. *There is no anonymous SCF that satisfies Pareto-optimality and even-chance-strategyproofness for $m \geq 3$ and $n \geq 3$.*

Remark 8. Proving Corollary 2 is not significantly easier than proving Theorem 1. Forqes provides a lower bound of 15 preference profiles.

7.2. Kelly-strategyproofness

It is not possible to replace Fishburn-strategyproofness with Kelly-strategyproofness in Theorem 1 because PO is Kelly-strategyproof (Proposition 2). We therefore focus on pairwise SCFs when dealing with Kelly-strategyproofness and directly present a human-readable proof of the following impossibility.

Theorem 2. *There is no pairwise SCF that satisfies Pareto-optimality and Kelly-strategyproofness for $m \geq 3$ and $n \geq 3$.*

Proof. Let $N = \{1, 2, 3\}$ and $A = \{a, b, c\}$ and assume for contradiction that there is a pairwise SCF f that satisfies Pareto-optimality and Kelly-strategyproofness. If not stated otherwise, the absolute values of the majority margins in the following applications of pairwiseness are always one. First, consider the classic Condorcet profile R^1.

$$\succsim_1^1 : a, b, c \qquad \succsim_2^1 : c, a, b \qquad \succsim_3^1 : b, c, a.$$

Due to the symmetry of the profile, we may assume without loss of generality that $b \in f(R^1)$. Now consider R^2.

$$\succsim_1^2 : a, b, c \qquad \succsim_2^2 : c, a, b \qquad \succsim_3^2 : b, \{a, c\},$$

R^2 and R^1 only differ in the third agent's preferences. By Fishburn-strategyproofness, $b \in f(R^2)$, as otherwise Agent 3 could obtain a preferred choice set by changing his preferences from $\succsim_3^2$ to $\succsim_3^1$. Now consider R^3, which has the same majority margins as R^2.

$$\succsim_1^3 : a, b, c \qquad \succsim_2^3 : \{a, c\}, b \qquad \succsim_3^3 : b, c, a,$$

Since $g_{R^2} = g_{R^3}$, $b \in f(R^3)$. Now consider R^4.

$$\succsim_1^4 : a, b, c \qquad \succsim_2^4 : a, c, b \qquad \succsim_3^4 : b, c, a,$$

R^4 differs from R^3 by the second agent's preferences $\succcurlyeq_2^4$. It follows that $b \in f\big(R^4\big)$. Otherwise, the second agent could misrepresent his preferences $\succcurlyeq_2^3$ by $\succcurlyeq_2^4$ and obtain the Kelly-preferred choice set $f\big(R^4\big)$ without b. Finally, consider R^5.

$$\succcurlyeq_1^5 \colon a, b, c \qquad\qquad \succcurlyeq_2^5 \colon \{a, b, c\} \qquad\qquad \succcurlyeq_3^5 \colon \{a, b, c\}.$$

Since $g_{R^5} = g_{R^4}$, $b \in f\big(R^5\big)$ holds as well. However, b is Pareto-dominated by a in R^5, a contradiction. □

The original proof of Theorem 2 found by the SAT solver consisted of nine preference profiles, and we used Forqes to verify that no proof with less than nine profiles exists. The given proof only argues about five profiles because the first step ("without loss of generality") implicitly makes reference to profiles that are not spelled out explicitly.

Remark 9. The conjunction of pairwiseness and Pareto-optimality implies that Condorcet winners should be chosen whenever the pairwise majority relation is transitive and its margins have absolute value 1. We have shown that Theorem 2 also holds when pairwiseness is replaced with this weaker, but technical, condition and $n \geq 4$. Note that this technical assumption is weaker than requiring the SCF to be a Condorcet extension.[11] Interestingly, the SMUS we found for this statement also consists of nine profiles.

Remark 10. The axioms of Theorem 2 are independent of each other. Borda's rule satisfies all axioms except Kelly-strategyproofness, *PO* satisfies all axioms except pairwiseness, and the trivial SCF which always returns all alternatives satisfies all axioms except Pareto-optimality. Also, the bounds used in the theorem ($m \geq 3$ and $n \geq 3$) are tight, as confirmed by the SAT solver.

Remark 11. Theorem 2 implies Theorem 4 by Aziz et al. (2014), who use a stronger notion of strategyproofness and furthermore require $m, n \geq 4$. Brandl et al. (2015a, Table 2) mention a consequence of this theorem for Fishburn-strategyproof SCFs. Interestingly, this consequence follows from both Theorem 1 and Theorem 2.

Remark 12. When assuming strict preferences, there are attractive pairwise Kelly-strategyproof SCFs satisfying Pareto-optimality, e.g., the uncovered set, the minimal covering set, and the essential set (Brandt, 2015). Theorem 2 shows that these SCFs cannot be extended to weak preferences without giving up one of these desirable properties. The same is true if we instead define Kelly's extension by requiring that X is preferred to Y if and only if every alternative in X is *strictly* preferred to every alternatives in Y (Brandt, 2015, Remark 6). This preference extension can be obtained via the framework introduced in Section 4 either by defining G-manipulability using $g(f(R')) \succ_i g(f(R))$ for all $g \in G$ or by changing the definition of G^{Δ}_{all} to $G^{\Delta}_{\text{all}} = \big\{g \in \Delta(A)^{\mathcal{C}} \colon \text{supp}(g(X)) \subseteq X \text{ for all } X \in \mathcal{C}\big\}$.

[11] Brandt (2015, Theorem 2) has also shown that every Condorcet extension is Kelly-manipulable. While his proof needs $3m$ agents, we require Pareto-optimality for our reduction argument.

Model	G-Manipulation	Pref. extension	Result
$\mathcal{R}^N \xrightarrow{f} \mathcal{C} \xrightarrow{g} A$	$\lvert G\rvert = 1$	none	Gibbard (1973) and Satterthwaite (1975)
	$G \subseteq G_{\text{lin}}$	Fishburn	Theorem 1
	$G \subseteq G_{\text{all}}$	Kelly	Theorem 2 (pairwise SCFs)
$\mathcal{R}^N \xrightarrow{f} \mathcal{C} \xrightarrow{g} \Delta(A)$	$G = \{uni\}$	even-chance	Corollary 2
	$G \subseteq G^{\Delta}_{\text{pro}}$	Fishburn	Theorem 1
	$G \subseteq G^{\Delta}_{\text{all}}$	Kelly	Theorem 2 (pairwise SCFs)
$\mathcal{R}^N \xrightarrow{f} \Delta(A)$	n/a	stochastic dom.	Brandl et al. (2018) (neutral, *SD*-efficient SCFs)
$U^N \xrightarrow{f} \Delta(A)$	n/a	expected utility	Hylland (1980) (*ex ante* efficient SCFs)

Table 2: Summary of main results plus two results from probabilistic social choice for comparison. Each line corresponds to an impossibility theorem using anonymity, Pareto-optimality, and strategyproofness.

Remark 13. Theorem 2 does not hold when replacing Pareto-optimality with weak Pareto-optimality (see Remark 6). The SCF that returns all weakly Pareto-optimal alternatives satisfies pairwiseness, weak Pareto-optimality, and Kelly-strategyproofness.

Remark 14. When preferences are dichotomous (i.e., each preference relation admits at most two indifference classes), both impossibilities do not hold because approval voting satisfies all desired conditions. PO, however, still violates Fishburn-strategyproofness, which can be seen by modifying the example given in Section 5.

8. Conclusion and Discussion

We investigated the existence of anonymous, Pareto-optimal, and strategyproof SCFs when there may be ties in the preferences as well as in the outcomes. Our main results are negative and, together with existing positive results, sharpen the boundary of strategyproof social choice. The computer-aided proof of Theorem 1 is rather complex and we have shown that no significantly easier proof exists.

Our results and their interpretation based on the tie-breaking rules introduced in Section 4 are summarized in Table 2. For comparison, the table also contains the Gibbard-Satterthwaite theorem and two related results from probabilistic social choice where SCFs may return *arbitrary* lotteries. When agents submit expected utility functions (or, equivalently, complete preferences over lotteries that adhere to the von Neumann-Morgenstern axioms), Hylland (1980) has proved that no SCF satisfies non-dictatorship, strategyproofness, and a strengthening of Pareto-optimality called *SD*-efficiency. Brandl et al. (2018) showed that this impossibility still holds when agents only submit preferences over alternatives, strategyproofness is weakened to *SD*-strategyproofness, and non-dictatorship is strengthened to anonymity and neutrality. When comparing this result to Corollary 2, it turns out that Corollary 2 is weaker in that it only allows for even-chance lotteries, but it is stronger in that it only requires Pareto-optimality (rather than *SD*-efficiency) and dispenses with neutrality. Even-chance lotteries are the most natural—and sometimes the only acceptable—form of randomized tie-breaking (see, e.g., Fishburn, 1972). Moreover, it may be difficult to implement non-uniform lotteries in practice.

Our results are tight in the sense that omitting any of the axioms, weakening Pareto-optimality to weak Pareto-optimality, reducing the number of agents or alternatives, or requiring strict preferences immediately allows for positive results. This underlines the adequacy of impossibility results to improve our understanding of what *can* be achieved and to guide practitioners looking for attractive SCFs. For example, the *essential set* (Dutta and Laslier, 1999; Laslier, 2000) and its probabilistic counterpart *maximal lotteries* (Fishburn, 1984; Brandl et al., 2016) satisfy Kelly-strategyproofness when preferences are strict and a weakening of Kelly-strategyproofness when preferences are weak (see Remark 12 and Aziz et al. (2018)). Our theorems show that, when insisting on Pareto-optimality, one cannot hope for reasonable SCFs that satisfy higher degrees of strategyproofness.

There are only few opportunities to strengthen our results even further. One is to try to weaken anonymity to a non-dictatorship condition for weak preferences. This would, however, require more than three agents or more than three alternatives (as confirmed by the SAT solver). Furthermore, it is unclear whether one can find a reduction argument that only uses non-dictatorship.

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APPENDIX

A. Proof of Theorem 1

Proof. Let f be a Fishburn-strategyproof and Pareto-optimal SCF. A recurrent step in the proof is to argue that, for two preference profiles $R, R' \in \mathcal{R}^N$ and some agent $i \in N$ such that $\succcurlyeq_j = \succcurlyeq'_j$ for all $j \neq i$, $f(R) = \{x\}$ implies that $f(R') = \{y\}$ for some $y \in A$. In particular, it can be shown that the following four implications hold when $f(R) = \{x\}$ and $m = 3$.

(i) If agent i top-ranks x, possibly with other Pareto-dominated alternatives, in R', then $f(R') = \{x\}$.

(ii) If agent i does not top-rank x in R and some alternative is Pareto-dominated both in R and R', then $f(R') = \{x\}$.

(iii) If agent i's unique least-preferred alternative in R is x, then $f(R') = \{x\}$.

(iv) If y Pareto-dominates x in R' and agent i top-ranks exactly x and y in R', then $f(R') = \{y\}$.

All four implications already hold for Kelly's extension and are easily proved. Implications *(i)* and *(iv)* make use of strategyproofness from R' to R: If agent i ranks alternative x top in R' and for $f(R) = \{x\}$ holds, then strategyproofness implies that $f(R')$ must consist only of top ranked alternatives of agent i in R': Otherwise, agent i can get a strictly preferred outcome by changing his preferences from R' to R.

For implication *(ii)*, observe that there is some alternative y in the top indifference class of agent i in R, while, by assumption, alternative x is not top ranked. Since $f(R) = \{x\}$, strategyproofness with respect to agent i from R to R' implies that neither $f(R') = \{x, y\}$ nor $f(R') = \{y\}$. By Pareto-optimality, we have that the remaining alternative $z \notin f(R')$, therefore we have that $f(R') = \{x\}$.

Lastly, implication *(iii)* is an immediate consequence of strategyproofness from R to R': Any outcome other than the singleton consisting of the last ranked alternative would be strictly preferred by agent i.

Now, for the proof of the theorem, let $N = \{1, 2, 3\}$ and $A = \{a, b, c\}$ and consider the preference profiles given in Table 3. We determine restrictions on the outcome of f imposed by Pareto-optimality and Fishburn-strategyproofness until we arrive at a contradiction. We will use the four implications from above by referring to their numbers and instantiating with the appropriate preference profiles. To improve readability, it will not be explicitly mentioned that Pareto-dominated alternatives, which are are marked in gray, must not be contained in any choice set. Anonymity is used implicitly by identifying preference profiles with *sets* of individual preference relations.

Profile	Preference Relations		
R^1	$\{a, c\}, b$	a, b, c	b, c, a
R^2	$c, \{a, b\}$	a, b, c	b, c, a
R^3	$c, \{a, b\}$	a, b, c	c, b, a
R^4	$c, \{a, b\}$	b, c, a	c, b, a
R^5	$\{a, c\}, b$	$\{a, c\}, b$	b, c, a
R^6	$b, \{a, c\}$	$c, \{a, b\}$	b, c, a
R^7	$\{a, c\}, b$	$b, \{a, c\}$	b, c, a
R^8	$\{b, c\}, a$	$a, \{b, c\}$	a, c, b
R^9	$\{b, c\}, a$	$b, \{a, c\}$	a, b, c
R^{10}	$\{a, c\}, b$	b, c, a	c, b, a
R^{11}	$\{a, c\}, b$	a, b, c	c, b, a
R^{12}	$c, \{a, b\}$	b, a, c	c, b, a
R^{13}	$\{a, c\}, b$	b, a, c	c, b, a
R^{14}	$a, \{b, c\}$	$c, \{a, b\}$	a, c, b
R^{15}	$\{b, c\}, a$	$a, \{b, c\}$	a, b, c
R^{16}	$\{a, b\}, c$	$c, \{a, b\}$	a, b, c
R^{17}	$a, \{b, c\}$	a, b, c	b, c, a
R^{18}	$\{a, c\}, b$	$b, \{a, c\}$	b, a, c
R^{19}	$a, \{b, c\}$	$c, \{a, b\}$	a, b, c
R^{20}	$b, \{a, c\}$	a, b, c	b, a, c
R^{21}	$\{b, c\}, a$	a, b, c	b, c, a

Table 3: Anonymous preference profiles for three agents used in the proof of Theorem 1. Pareto-dominated alternatives are marked in gray.

Interestingly, Kelly-strategyproofness suffices for the entire proof except for two implications involving the possibility that $R^2 = \{a, b, c\}$ in the proof of Claim (2). We will explicitly mention that Fishburn's extension is used in these cases. It turns out that only profiles R^1 to R^4 are directly affected in the results of the main proof steps, which can be broken down into four claims. The rest of the profiles are needed in intermediate steps.

(1) $\underline{f(R^1) \neq \{a\}}$

Assume for contradiction that $f(R^1) = \{a\}$. The following chain of implications show that this entails $f(R^{12}) = \{c\}$. For convenience, we restate the preference profiles involved in these implications on the right-hand side. Preference relations that changed from one profile to another are highlighted in gray.

$$
\begin{array}{rlccc}
 & f(R^1) = \{a\} & a, b, c & \{a, c\}, b & b, c, a \\
\overset{(iv)}{\Longrightarrow} & f(R^5) = \{c\} & \{a, c\}, b & \{a, c\}, b & b, c, a \\
\overset{(i)}{\Longrightarrow} & f(R^{10}) = \{c\} & c, b, a & \{a, c\}, b & b, c, a \\
\overset{(i)}{\Longrightarrow} & f(R^4) = \{c\} & c, b, a & c, \{a, b\} & b, c, a \\
\overset{(ii)}{\Longrightarrow} & f(R^{12}) = \{c\} & c, b, a & c, \{a, b\} & b, a, c
\end{array}
$$

From $f(R^{12}) = \{c\}$ we can infer that $f(R^{13}) \subseteq \{a, c\}$ because otherwise there is a manipulation from R^{13} to R^{12}.

$$
\begin{array}{rlccc}
 & f(R^{12}) = \{c\} & c, \{a, b\} & b, a, c & c, b, a \\
\Rightarrow & f(R^{13}) \subseteq \{a, c\} & \{a, c\}, b & b, a, c & c, b, a
\end{array}
$$

Further, $f(R^1) = \{a\}$ implies that $f(R^{11}) = \{a\}$.

$$
\begin{array}{rlccc}
 & f(R^1) = \{a\} & b, c, a & \{a, c\}, b & a, b, c \\
\overset{(iii)}{\Longrightarrow} & f(R^{11}) = \{a\} & c, b, a & \{a, c\}, b & a, b, c
\end{array}
$$

Now $f(R^{13}) \subseteq \{a, c\}$ and $f(R^{11}) = \{a\}$ imply $f(R^{13}) = \{a\}$, since both $f(R^{13}) = \{c\}$ and $f(R^{13}) = \{a, c\}$ would violate strategyproofness from R^{13} to R^{11}.

$$
\begin{array}{rlccc}
 & f(R^{11}) = \{a\} & a, b, c & \{a, c\}, b & c, b, a \\
\Rightarrow & f(R^{13}) = \{a\} & b, a, c & \{a, c\}, b & c, b, a
\end{array}
$$

From $f(R^{13}) = \{a\}$ we deduce that $f(R^2) = \{a\}$ using the following chain of implications.

$$
\begin{array}{rlccc}
 & f(R^{13}) = \{a\} & c, b, a & \{a, c\}, b & b, a, c \\
\overset{(iii)}{\Longrightarrow} & f(R^{18}) = \{a\} & b, \{a, c\} & \{a, c\}, b & b, a, c \\
\overset{(i)}{\Longrightarrow} & f(R^{20}) = \{a\} & b, \{a, c\} & a, b, c & b, a, c \\
\overset{(i)}{\Longrightarrow} & f(R^9) = \{a\} & b, \{a, c\} & a, b, c & \{b, c\}, a \\
\overset{(ii)}{\Longrightarrow} & f(R^{21}) = \{a\} & b, c, a & a, b, c & \{b, c\}, a \\
\overset{(iii)}{\Longrightarrow} & f(R^2) = \{a\} & b, c, a & a, b, c & c, \{a, b\}
\end{array}
$$

Also, $f(R^{12}) = \{c\}$ implies $f(R^3) = \{c\}$.

$$
\begin{array}{rlccc}
 & f(R^{12}) = \{c\} & b, a, c & c, \{a, b\} & c, b, a \\
\overset{(iii)}{\Longrightarrow} & f(R^3) = \{c\} & a, b, c & c, \{a, b\} & c, b, a
\end{array}
$$

However, $f(R^3) = \{c\}$ and $f(R^2) = \{a\}$ violate strategyproofness from R^2 to R^3.

$$\begin{array}{llll} f(R^2) = \{a\} & b, c, a & c, \{a, b\} & a, b, c \\ f(R^3) = \{c\} & c, b, a & c, \{a, b\} & a, b, c \end{array}$$

Hence, the assumption that $f(R^2) = \{a\}$ was incorrect.

(2) $\underline{f(R^2) \notin \{\{b\}, \{a, b\}, \{a, b, c\}\}}$

Assume for contradiction that (2) is false. This implies $f(R^{16}) = \{a\}$, since $f(R^{16}) = \{a, c\}$ and $f(R^{16}) = \{c\}$ would violate Fishburn-strategyproofness from R^{16} to R^2.

$$\begin{array}{lllll} & f(R^2) \in \{\{b\}, \{a, b\}, \{a, b, c\}\} & b, c, a & c, \{a, b\} & a, b, c \\ \Rightarrow & f(R^{16}) = \{a\} & \{a, b\}, c & c, \{a, b\} & a, b, c \end{array}$$

Now $f(R^{16}) = \{a\}$ implies $f(R^{17}) = \{a\}$.

$$\begin{array}{lllll} & f(R^{16}) = \{a\} & \{a, b\}, c & a, b, c & c, \{a, b\} \\ \xRightarrow{(i)} & f(R^{19}) = \{a\} & a, \{b, c\} & a, b, c & c, \{a, b\} \\ \xRightarrow{(i)} & f(R^{14}) = \{a\} & a, \{b, c\} & a, c, b & c, \{a, b\} \\ \xRightarrow{(ii)} & f(R^{8}) = \{a\} & a, \{b, c\} & a, c, b & \{b, c\}, a \\ \xRightarrow{(i)} & f(R^{15}) = \{a\} & a, \{b, c\} & a, b, c & \{b, c\}, a \\ \xRightarrow{(iii)} & f(R^{17}) = \{a\} & a, \{b, c\} & a, b, c & b, c, a \end{array}$$

From (1) we know that $f(R^1) \neq \{a\}$, therefore $f(R^{17}) = \{a\}$ yields either $f(R^1) = \{c\}$ or $f(R^1) = \{a, c\}$ by strategyproofness from R^1 to R^{17}.

$$\begin{array}{lllll} & f(R^{17}) = \{a\} & a, \{b, c\} & a, b, c & b, c, a \\ \Rightarrow & f(R^1) \in \{\{c\}, \{a, c\}\} & \{a, c\}, b & a, b, c & b, c, a \end{array}$$

However, $f(R^1) \in \{\{c\}, \{a, c\}\}$ contradicts the assumption that $f(R^2)$ is either $\{b\}$, $\{a, b\}$, or $\{a, b, c\}$. In each of the cases, Fishburn-strategyproofness from R^2 to R^1 is violated.

$$\begin{array}{llll} f(R^2) \in \{\{b\}, \{a, b\}, \{a, b, c\}\} & c, \{a, b\} & a, b, c & b, c, a \\ f(R^1) \in \{\{c\}, \{a, c\}\} & \{a, c\}, b & a, b, c & b, c, a \end{array}$$

(3) $\underline{f(R^1) \neq \{c\}}$

Assume for contradiction that $f(R^1) = \{c\}$. The following chain of implications shows that this implies $f(R^7) = \{b\}$.

$$\begin{array}{rlccc}
 & f(R^1) = \{c\} & \{a,c\},b & b,c,a & a,b,c \\
\xRightarrow{(iv)} & f(R^{21}) = \{b\} & \{b,c\},a & b,c,a & a,b,c \\
\xRightarrow{(i)} & f(R^9) = \{b\} & \{b,c\},a & b,\{a,c\} & a,b,c \\
\xRightarrow{(i)} & f(R^{20}) = \{b\} & b,a,c & b,\{a,c\} & a,b,c \\
\xRightarrow{(ii)} & f(R^{18}) = \{b\} & b,a,c & b,\{a,c\} & \{a,c\},b \\
\xRightarrow{(i)} & f(R^7) = \{b\} & b,c,a & b,\{a,c\} & \{a,c\},b
\end{array}$$

However, $f(R^7) = \{b\}$ and $f(R^1) = \{c\}$ contradicts strategyproofness from R^1 to R^7.

$$\begin{array}{lccc}
f(R^1) = \{c\} & a,b,c & \{a,c\},b & b,c,a \\
f(R^7) = \{b\} & b,\{a,c\} & \{a,c\},b & b,c,a
\end{array}$$

(4) $\underline{f(R^2) \notin \{\{c\},\{b,c\}\}}$

Assume for contradiction that (4) does not hold. This implies $f(R^{21}) = \{b\}$, since $a \in f(R^{21})$ would violate strategyproofness from R^{21} to R^2.

$$\begin{array}{rlccc}
 & f(R^2) \in \{\{c\},\{b,c\}\} & c,\{a,b\} & a,b,c & b,c,a \\
\Rightarrow & f(R^{21}) = \{b\} & \{b,c\},a & a,b,c & b,c,a
\end{array}$$

Now $f(R^{21}) = \{b\}$ implies $f(R^6) = \{b\}$:

$$\begin{array}{rlccc}
 & f(R^{21}) = \{b\} & b,c,a & \{b,c\},a & a,b,c \\
\xRightarrow{(i)} & f(R^9) = \{b\} & b,\{a,c\} & \{b,c\},a & a,b,c \\
\xRightarrow{(i)} & f(R^{20}) = \{b\} & b,\{a,c\} & b,a,c & a,b,c \\
\xRightarrow{(ii)} & f(R^{18}) = \{b\} & b,\{a,c\} & b,a,c & \{a,c\},b \\
\xRightarrow{(i)} & f(R^7) = \{b\} & b,\{a,c\} & b,c,a & \{a,c\},b \\
\xRightarrow{(iii)} & f(R^6) = \{b\} & b,\{a,c\} & b,c,a & c,\{a,b\}
\end{array}$$

However, $f(R^6) = \{b\}$ and the assumption that $f(R^2)$ is either $\{c\}$ or $\{b,c\}$ contradicts strategyproofness from R^2 to R^6 in either case.

$$\begin{array}{rlccc}
 & f(R^2) \in \{\{c\},\{b,c\}\} & a,b,c & c,\{a,b\} & b,c,a \\
\Rightarrow & f(R^6) = \{b\} & b,\{a,c\} & c,\{a,b\} & b,c,a
\end{array}$$

(5) $\underline{b \notin f(R^1)}$

Assume for contradiction that (5) does not hold. From (2) and (4) we deduce that either $f(R^2) = \{a\}$ or $f(R^2) = \{a, c\}$. This contradicts strategyproofness from R^1 to R^2.

$$\begin{array}{rlccc} & f(R^1) \ni b & \{a,c\}, b & a, b, c & b, c, a \\ \Rightarrow & f(R^2) \in \{\{a\}, \{a, c\}\} & c, \{a, b\} & a, b, c & b, c, a \end{array}$$

(6) $\underline{f(R^3) \neq \{c\}}$

Assume for contradiction that $f(R^3) = \{c\}$. From (2) and (4) we have again that either $f(R^2) = \{a\}$ or $f(R^2) = \{a, c\}$. In both cases, strategyproofness from R^2 to R^3 is violated.

$$\begin{array}{lccc} f(R^2) \in \{\{a\}, \{a, c\}\} & b, c, a & c, \{a, b\} & a, b, c \\ f(R^3) = \{c\} & c, b, a & c, \{a, b\} & a, b, c \end{array}$$

(7) $\underline{f(R^4) \neq \{c\}}$

Assume for contradiction that $f(R^4) = \{c\}$. This implies $f(R^3) = \{c\}$, a contradiction to (6).

$$\begin{array}{rlccc} & f(R^4) = \{c\} & b, c, a & c, \{a, b\} & c, b, a \\ \xRightarrow{(ii)} & f(R^{12}) = \{c\} & b, a, c & c, \{a, b\} & c, b, a \\ \xRightarrow{(iii)} & f(R^3) = \{c\} & a, b, c & c, \{a, b\} & c, b, a \end{array}$$

(8) $\underline{f(R^4) = \{c\}}$

From (1), (3), and (5) we know that $f(R^1) = \{a, c\}$. This implies $f(R^5) = \{c\}$, as otherwise strategyproofness from R^5 to R^1 would be violated.

$$\begin{array}{rlccc} & f(R^1) = \{a, c\} & a, b, c & \{a, c\}, b & b, c, a \\ \Rightarrow & f(R^5) = \{c\} & \{a, c\}, b & \{a, c\}, b & b, c, a \end{array}$$

Now $f(R^5) = \{c\}$ implies $f(R^4) = \{c\}$.

$$\begin{array}{rlccc} & f(R^5) = \{c\} & \{a, c\}, b & \{a, c\}, b & b, c, a \\ \xRightarrow{(i)} & f(R^{10}) = \{c\} & c, b, a & \{a, c\}, b & b, c, a \\ \xRightarrow{(i)} & f(R^4) = \{c\} & c, b, a & c, \{a, b\} & b, c, a \end{array}$$

Since $f(R^4) = \{c\}$ and $f(R^4) \neq \{c\}$ is a contradiction, (8) and (7) conclude the proof, showing that no Fishburn-strategyproof and Pareto-optimal SCF exists. $\square$