On the Tradeoff between Efficiency and Strategyproofness*

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We study *social decision schemes (SDSs)*, i.e., functions that map a collection of individual preferences over alternatives to a lottery over the alternatives. Depending on how preferences over alternatives are extended to preferences over lotteries, there are varying degrees of efficiency and strategyproofness. In this paper, we consider four such preference extensions: stochastic dominance (SD), a strengthening of SD based on pairwise comparisons (PC), a weakening of SD called bilinear dominance (BD), and an even weaker extension based on Savage's sure-thing principle (ST). While *random serial dictatorships* are PC-strategyproof, they only satisfy *ex post* efficiency. On the other hand, we show that *strict maximal lotteries* satisfy PC-efficiency and ST-strategyproofness. We also prove the incompatibility of *(i)* PC-efficiency and PC-strategyproofness for neutral SDSs, *(ii)* *ex post* efficiency and BD-strategyproofness for pairwise SDSs, and *(iii)* *ex post* efficiency and BD-group-strategyproofness for neutral SDSs.

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1. Introduction

Two fundamental notions in microeconomic theory are *efficiency*—no agent can be made better off without making another one worse off—and *strategyproofness*—no agent can obtain a more preferred outcome by misrepresenting his preferences. The conflict between these two notions is already apparent in Gibbard and Satterthwaite's seminal theorem, which states that the only single-valued social choice functions (SCFs) that satisfy non-imposition—a weakening of efficiency—and strategyproofness are dictatorships (Gibbard, 1973; Satterthwaite, 1975). In this paper, we study efficiency and strategyproofness in the context of *social decision schemes (SDSs)*, i.e., functions that map a preference profile to a probability distribution (or lottery) over a fixed set of alternatives (e.g., Gibbard, 1977; Barberà, 1979a). Randomized voting methods have a surprisingly long tradition going back to ancient Greece and have recently gained increased attention in social choice (see, e.g., Bogomolnaia et al., 2005; Chatterji et al., 2014; Brandl et al., 2016b) and political science (see, e.g., Goodwin, 2005; Stone, 2011; Guerrero, 2014). Randomization is particularly natural in subdomains of social choice that are concerned with the assignment of objects to agents such as house allocation (see, e.g., Abdulkadiroğlu and Sönmez, 1998; Bogomolnaia and Moulin, 2004; Che and Kojima, 2010; Budish et al., 2013). Positive results, such as our results on maximal lotteries, are inherited from the general social choice domain to these subdomains.

In order to identify efficient lotteries and argue about incentives with respect to lotteries, one needs to know the agents' preferences over lotteries. There are various problems associated with asking the agents to submit complete preference relations over all lotteries. For example, the preferences may not allow for a concise representation and agents may not even be aware of these preferences in the first place.[1] We therefore follow the common approach which only assumes ordinal preferences over alternatives, which are then systematically extended to possibly incomplete preferences over lotteries. We will refer to these extensions as *lottery extensions* (see also Cho, 2016). One of the most studied lottery extensions is *stochastic dominance (SD)*, which states that one lottery is preferred to another iff the former first-order stochastically dominates the latter. This extension is of particular importance because it coincides with the extension in which one lottery is preferred to another iff, for any von Neumann-Morgenstern (vNM) utility function consistent with the ordinal preferences, the former yields at least as much expected utility as the latter. Settings in which the existence of an underlying vNM utility function cannot be assumed may call for other lottery extensions. A natural candidate is the *pairwise comparison (PC)* extension, which arises as a special case of skew-symmetric bilinear (SSB) utility functions, a generalization of vNM utility functions proposed by Fishburn (1982). According to this extension lottery p is preferred to lottery q iff it is more likely that p yields a better alternative than q. Clearly, each of these lottery extensions gives to rise to

[1] Even if agents *think* they can competently assign von Neumann-Morgenstern utilities to alternatives, these assignments are prone to be based on arbitrary choices.

different variants or degrees of efficiency and strategyproofness.

Since many lottery extensions are incomplete, i.e., some pairs of lotteries are incomparable, there are two fundamentally different ways how to define strategyproofness. The strong notion, first advocated by Gibbard (1977), requires that every misreported preference relation of an agent will result in a lottery that is comparable and weakly less preferred by that agent to the original lottery. According to the weaker notion, first used by Postlewaite and Schmeidler (1986) and then popularized by Bogomolnaia and Moulin (2001), no agent can misreport his preferences to obtain another lottery that is strictly preferred to the original one. In other words, the strong version always interprets incomparabilities in the worst possible manner (such that they violate strategyproofness) while the weak version interprets them as actual incomparabilities that cannot be resolved.[2] Usually, the strong notion is much more demanding than the weak one. Whenever a lottery extension is *complete*, however, both notions coincide.

Perhaps the most well-known SDS, which is only defined for linear preferences, is *random dictatorship (RD)*. In *RD*, one of the agents is chosen uniformly at random and this agent's most preferred alternative is selected. Gibbard (1977) has shown that *RD* is the only strongly *SD*-strategyproof SDS that never puts positive probability on Pareto-dominated alternatives. In other words, it is the only strongly *SD*-strategyproof SDS that is *ex post efficient*.[3] It is easily verified that *RD* even satisfies the stronger condition of *SD*-efficiency.

Gibbard's proof requires the universal domain of linear preferences and cannot be extended to arbitrary subdomains (see, e.g., Chatterji et al., 2014). In many important subdomains of social choice such as house allocation, matching, and coalition formation, ties are unavoidable because agents are indifferent among all outcomes in which their allocation, match, or coalition is the same (see Section 2). In the presence of ties, *RD* is typically extended to *random serial dictatorship (RSD)*, where dictators are invoked sequentially and ties between most-preferred alternatives are broken by subsequent dictators. While *RSD* still satisfies strong *SD*-strategyproofness, it violates *SD*-efficiency. This was first observed by Bogomolnaia and Moulin (2001) in the restricted domain of house allocation. The example by Bogomolnaia and Moulin (2001) can be translated to a preference profile with 24 alternatives in the general social choice domain. We give a minimal example for four agents and four alternatives and completely characterize under which configurations *RSD* satisfies *SD*-efficiency. Recently Brandl et al. (2016a) have shown a sweeping impossibility: no anonymous and neutral SDS simultaneously satisfies *SD*-efficiency and *SD*-strategyproofness whenever there are at least four alternatives and four agents. This result was obtained with the help of computers and it is extremely tedious to verify the computer-generated proof. We give manual and simpler proofs for two related statements: there is no *PC*-efficient and *PC*-strategyproof neutral SDS and there is no *ex post* efficient

[2] The weak notion of strategyproofness has often been considered in the context of set-valued social choice where preferences over alternatives are extended to incomplete preference relations over sets of alternatives (see, e.g., Gärdenfors, 1976; Barberà, 1977a,b; Kelly, 1977; Feldman, 1979a,b). For relatively weak preference extensions, it allows for rather positive results (Nehring, 2000; Brandt, 2015).

[3] Alternative proofs of this theorem were given by Duggan (1996), Nandeibam (1997), and Tanaka (2003).

and *BD*-strategyproof pairwise SDS, where *BD*-strategyproofness is a weakening of *SD*-strategyproofness. While the first result uses stronger conditions than that by Brandl et al. (2016a), it requires less agents and alternatives. The second result uses weaker conditions, but only holds for the restricted class of pairwise SDSs.

In order to obtain positive results we then introduce a new lottery extension that is weaker than the stochastic dominance extension and is based on Savage's sure-thing principle (ST). All three lottery extensions are then used to demonstrate an interesting tradeoff (see Figure 1): Random serial dictatorship is strongly *SD*-strategyproof, but only satisfies *ex post* efficiency. On the other hand, *strict maximal lotteries (SML)* as defined by Kreweras (1965) and Fishburn (1984a), satisfy PC-efficiency and ST-strategyproofness. Strict maximal lotteries correspond to the quasi-strict mixed equilibria of the symmetric zero-sum game induced by the pairwise majority margins. While ST-strategyproofness is quite weak, it is important to note that most common *ex post* efficient SDSs (except *RSD*) violate much weaker notions of strategyproofness. Moreover, *SML* satisfies a number of other desirable properties violated by *RSD* such as Condorcet-consistency and composition-consistency (Laslier, 2000; Brandl et al., 2016b). Figure 1 summarizes our findings.

We also consider manipulation by groups of agents. We prove that both *RSD* and *SML* satisfy ST-group-strategyproofness and that no *ex post* efficient SDS satisfies the slightly stronger notion of *BD*-group-strategyproofness. These results are visualized in Figure 2.

All of our impossibility results assume anonymity. *Serial dictatorship*, an extreme example of a non-anonymous SDS, is defined for a fixed sequence of the agents and lets each agent narrow down the set of alternatives by picking his most preferred of the alternatives selected by the previous agents. Serial dictatorship trivially satisfies all reasonable notions of efficiency and strategyproofness. Since lotteries can guarantee *ex ante* fairness via randomization, anonymity and neutrality are typically two minimal conditions that fair SDSs are expected to satisfy.

2. Related Work

Starting with the Gibbard-Satterthwaite impossibility (Gibbard, 1973; Satterthwaite, 1975), there is remarkable number of results that reveal a tradeoff between efficiency and strategyproofness.

As mentioned above, *RD*, as proposed by Gibbard (1977), satisfies *SD*-efficiency and (strong) *SD*-strategyproofness when preferences are linear. Brandl et al. (2016c) have shown that *RD* cannot be extended to the full domain of weak preferences without violating at least one of these properties.[4] This theorem has been strengthened by Brandl et al. (2016a) who showed that no anonymous and neutral SDS satisfies *SD*-efficiency and *SD*-

[4] A natural candidate is *RSD*, which as we discuss in Section 4, violates *SD*-efficiency. Aziz (2013) proposes another SDS that satisfies a stronger notion of efficiency and a weaker notion of strategyproofness than *RSD*. However, it also violates *SD*-efficiency.

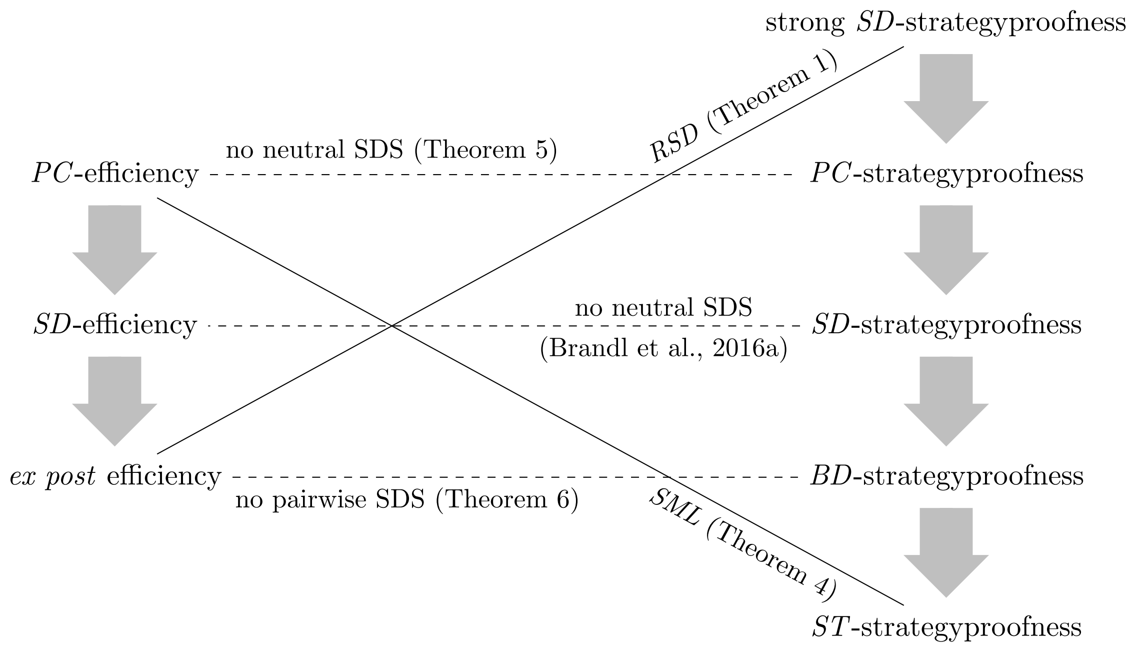


Figure 1: Relationships between varying degrees of efficiency and strategyproofness. An arrow from one notion of efficiency or strategyproofness to another denotes that the former implies the latter. Solid lines represent positive results whereas dashed lines represent impossibility results. The result by Brandl et al. (2016a) was shown using computer-aided solving techniques.

strategyproofness by leveraging computer-aided solving techniques.[5]

When preferences are *dichotomous*, efficiency and strategyproofness are compatible. The utilitarian mechanism described by Bogomolnaia et al. (2005) satisfies the strongest degrees of efficiency and strategyproofness considered in this paper (PC-efficiency and strong SD-strategyproofness). Interestingly, this mechanism always return a maximal lottery (which are of a particularly simple form for dichotomous preferences). When replacing strategyproofness with group-strategyproofness and weakening SD-efficiency to *ex post* efficiency, this possibility turns into an impossibility (Bogomolnaia et al., 2005). We strengthen this result in Section 7.

Another subdomain of social choice that has been thoroughly studied in the literature is the *assignment* (aka house allocation or two-sided matching with one-sided preferences) domain. An assignment profile can be associated with a social choice profile by letting

[5]The theorem by Brandl et al. (2016a) also implies analogous impossibilities for the upward lexicographic UL and downward lexicographic DL extensions introduced by Cho (2016). The impossibility for UL even holds for linear preferences while this is not the case for DL since RD satisfies both DL-efficiency and DL-strategyproofness (Brandl, 2013).

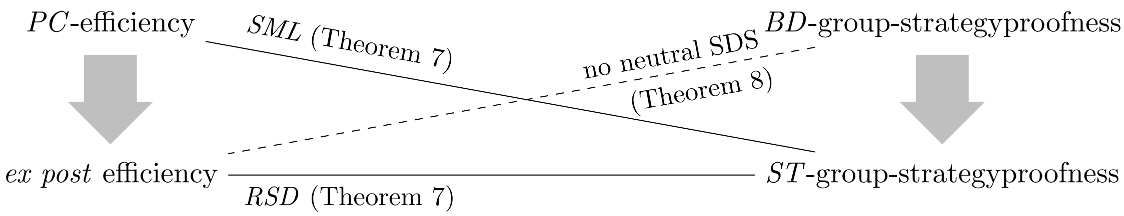


Figure 2: Relationships between efficiency and group-strategyproofness concepts. Solid lines represent positive results whereas dashed lines represent impossibility results.

the set of alternatives be the set of deterministic allocations and postulating that agents are indifferent among all allocations in which they receive the same object (see, e.g., Aziz et al., 2013).[6] Thus, impossibility results for the assignment setting imply impossibility results for the social choice setting.

Bogomolnaia and Moulin (2001) have shown that no random assignment rule satisfies both SD-efficiency and strong SD-strategyproofness while satisfying equal treatment of equals. The result by Bogomolnaia and Moulin even holds when preferences over objects are linear and single-peaked (Kasajima, 2013). (Nevertheless, when transferred to the social choice domain, the preferences over allocations will contain ties and not be single-peaked anymore.) In a related paper, Katta and Sethuraman (2006) proved that no assignment rule satisfies SD-efficiency, SD-strategyproofness, and strong SD-envy-freeness for the full domain of weak preferences over objects.[7] Nesterov (2016) showed similar impossibilities: *ex post* efficiency, strong SD-strategyproofness, and strong SD-envy-freeness as well as SD-efficiency, strong SD-strategyproofness, and weak SD-envy-freeness are incompatible with each other. Bogomolnaia and Moulin (2001) have introduced the probabilistic serial (PS) assignment rule which satisfies SD-efficiency and SD-strategyproofness when preferences over objects are linear. Natural extensions of the probabilistic serial rule to the full assignment domain and to the general social choice domain fail to satisfy SD-strategyproofness (Katta and Sethuraman, 2006; Aziz and Stursberg, 2014). Thus, it is an interesting open question whether there is any SD-efficient and SD-strategyproof assignment rule on the full assignment domain that satisfies equal treatment of equals. We conjecture that no such rule exists.

There is an extensive literature showing impossibility results for *set-valued* SCFs. Just like in probabilistic social choice, strategyproofness is defined by lifting the preference relation, in this case to sets of alternatives. It turns out that some of the resulting notions

[6]Note that this transformation turns assignment profiles with linear preferences over k objects into social choice profiles with non-linear preferences over $k!$ allocations.

[7]Strong envy-freeness is a fairness property that is stronger than equal treatment of equals as used by Bogomolnaia and Moulin (2001). Weak envy-freeness and equal treatment of equals are incomparable.

of strategyproofness are logically related to notions considered in this paper. For example, every strongly SD-strategyproof SDS induces a set-valued SCF (by just taking the support of the resulting lottery) that is strategyproof with respect to the optimist and pessimist extensions as used by Duggan and Schwartz (2000), Rodríguez-Álvarez (2007), Rodríguez-Álvarez (2009), Sato (2008), and others. Similarly, every ST-strategyproof SDS induces a set-valued SCF that is strategyproof with respect to the simple extension where one set is preferred to another iff all alternatives in the former are strictly preferred to all alternatives in the latter (see, e.g., Nehring, 2000; Brandt, 2015, Remark 6). The relationship between BD-strategyproofness and strategyproofness with respect to Fishburn's set extension as used by Feldman (1979a), Ching and Zhou (2002), Sanver and Zwicker (2012), Brandt and Geist (2016), and others works the other way round: Every Fishburn-strategyproof SCF induces a BD-strategyproof SDS by taking the uniform lottery over the resulting choice set.[8] As a consequence, Theorem 6 implies Theorem 3 by Brandt and Geist (2016). However, the proof of Theorem 6 uses weak preferences while the result by Brandt and Geist (2016) even holds for linear preferences.

Mennle and Seuken (2015) proposed a different approach to trade off efficiency and strategyproofness by quantifying manipulation losses and considering convex combinations of random assignment rules. There also has been some recent work on the tradeoff between participation (resistance against strategic abstention) and efficiency (Brandl et al., 2015a,b).

3. Preliminaries

Let $N = \{1, \dots, n\}$ be a set of agents who entertain ordinal preferences over a finite set A of m alternatives. Every agent $i \in N$ is equipped with a complete and transitive *preference relation* $R_i \subseteq A \times A$. The set of all preference relations will be denoted by $\mathcal{R}$. In accordance with conventional notation, we write P_i for the strict part of R_i, i.e., $a \; P_i \; b$ if $a \; R_i \; b$ but not $b \; R_i \; a$ and I_i for the indifference part of R_i, i.e., $a \; I_i \; b$ if $a \; R_i \; b$ and $b \; R_i \; a$. Preference relations are straightforwardly extended to sets of alternatives X, Y where $X \; R_i \; Y$ denotes that $x \; R_i \; y$ holds for all $x \in X$ and $y \in Y$. Similarly, $X \; P_i \; Y$ iff $x \; P_i \; y$ for all $x \in X$ and $y \in Y$. We will compactly represent a preference relation as a comma-separated list where all alternatives among which an agent is indifferent are represented by a set. For example $a \; P_i \; b \; I_i \; c$ is written as $R_i \colon a, \{b, c\}$. A preference relation R_i is *linear* if $x \; P_i \; y$ or $y \; P_i \; x$ for all distinct alternatives $x, y \in A$. A preference relation R_i is *dichotomous* if $x \; R_i \; y \; R_i \; z$ implies $x \; I_i \; y$ or $y \; I_i \; z$. A *preference profile* $R = (R_1, \dots, R_n)$ is an n-tuple containing a preference relation R_i for every agent $i \in N$. The set of all preference profiles is thus given by $\mathcal{R}^n$. By R_{-i} we denote the preference profile obtained from R by removing the preference relation of agent i, i.e., $R_{-i} = R \setminus \{(i, R_i)\}$.

The set of all *lotteries* (or *probability distributions*) over A is denoted by $\Delta(A)$. We will write lotteries as convex combinations of alternatives, e.g., $1/2\, a + 1/2\, b$ denotes the

[8] Following Gärdenfors (1979), we refer to this frequently reinvented extension as Fishburn's extension.

uniform distribution over $\{a, b\}$. For a given lottery p and alternative x, $p(x)$ denotes the probability that p assigns to x. The support of a lottery $p \in \Delta(A)$, denoted by $\widehat{p}$, is the set of all alternatives to which p assigns positive probability, i.e., $\widehat{p} = \{x \in A \colon p(x) > 0\}$. A lottery p is *degenerate* if $|\widehat{p}| = 1$.

Our central object of study are social decision schemes, i.e., functions that map a preference profile to a lottery. Thus, a *social decision scheme (SDS)* is a function

$$f \colon \mathcal{R}^n \to \Delta(A).$$

A minimal fairness condition for SDSs is *anonymity*, which requires that $f(R) = f(R \circ \pi)$ for all $R \in \mathcal{R}^n$ and all permutations $\pi \colon N \to N$. Another fairness requirement is *neutrality*. For a permutation π of A and a preference relation R_i, we define R_i^π as the preference relation where alternatives are renamed according to π, i.e., $\pi(x)\ R_i^\pi\ \pi(y)$ iff $x\ R_i\ y$. An SDS f is *neutral* if, for all $R \in \mathcal{R}^n$, $x \in A$, and all permutations $\pi \colon A \to A$, $f(R)(x) = f(R^\pi)(\pi(x))$.

3.1. Lottery Extensions

In order to reason about the outcomes of SDSs, we need to make assumptions on how agents compare lotteries given their preferences over alternatives. A *lottery extension* maps a preference relation to a (possibly incomplete) preference relation over lotteries. We will now define the lottery extensions considered in this paper. Throughout this section, let $R_i \in \mathcal{R}$ and $p, q \in \Delta(A)$. For the examples we assume that the underlying preference relation is $R_i \colon a, b, c$.

A very simple and crude lottery extension prescribes that p is preferred to q iff every alternative in the support of p is preferred to every alternative in the support of q, i.e., $\widehat{p}\ P_i\ \widehat{q}$. This extension only allows the comparison of lotteries with disjoint supports. We slightly generalize this definition by requiring that p and q assign the same probability to all alternatives that are contained in both supports and $(\widehat{p} \setminus \widehat{q})\ P_i\ (\widehat{p} \cap \widehat{q})\ P_i\ (\widehat{q} \setminus \widehat{p})$. Following Savage's sure-thing principle, the resulting lottery extension will be referred to as the *sure-thing (ST)* extension. Formally,

$$p\ R_i^{ST}\ q \quad \text{iff} \quad (\widehat{p} \setminus \widehat{q})\ P_i\ (\widehat{p} \cap \widehat{q})\ P_i\ (\widehat{q} \setminus \widehat{p}) \ \wedge\ \forall x \in \widehat{p} \cap \widehat{q} \colon p(x) = q(x). \qquad (ST)$$

The idea underlying ST is that the comparison of two lotteries should be independent of the part in which they coincide. This is strongly related to von Neumann and Morgenstern's independence axiom (von Neumann and Morgenstern, 1947) and has also been used for defining preference extensions from alternatives to sets of alternatives (Fishburn, 1972; Gärdenfors, 1979). For example, $1/2\, a + 1/2\, b\ P_i^{ST}\ 1/2\, b + 1/2\, c$.[9]

The second extension we consider, called *bilinear dominance (BD)*, requires that for every pair of alternatives the probability that p yields the more preferred alternative and q

[9] A more complete version of the ST extension can be defined by demanding that $(\widehat{p} \setminus \widehat{q})\ R_i\ (\widehat{p} \cap \widehat{q})\ R_i\ (\widehat{q} \setminus \widehat{p})$ instead of $(\widehat{p} \setminus \widehat{q})\ P_i\ (\widehat{p} \cap \widehat{q})\ P_i\ (\widehat{q} \setminus \widehat{p})$. However, Theorems 4 and 7 do not hold under this extension.

the less preferred alternative is at least as large as the other way round. Formally,

$$p \; R_i^{BD} \; q \quad \text{iff} \quad \forall x, y \in A\colon \; (x \; P_i \; y \Rightarrow p(x)q(y) \geq p(y)q(x)) \, . \tag{BD}$$

Apart from its intuitive appeal, the main motivation for BD is that p bilinearly dominates q iff p is preferable to q for every SSB utility function consistent with R_i (cf. Fishburn, 1984b; Aziz et al., 2015). For example, $1/2\, a + 1/2\, b \; P_i^{BD} \; 1/3\, a + 1/3\, b + 1/3\, c$.

Perhaps the best-known lottery extension is *stochastic dominance (SD)*, which prescribes that for each alternative $x \in A$, the probability that p selects an alternative that is at least as good as x is greater or equal to the probability that q selects such an alternative (Hadar and Russell, 1969). Formally,

$$p \; R_i^{SD} \; q \quad \text{iff} \quad \forall x \in A\colon \sum_{y R_i x} p(y) \geq \sum_{y R_i x} q(y). \tag{SD}$$

It is well-known that $p \; R_i^{SD} \; q$ iff the expected utility for p is at least as large as that for q for every von-Neumann-Morgenstern utility function compatible with R_i. For example, $1/2\, a + 1/2\, c \; P_i^{SD} \; 1/2\, b + 1/2\, c$.

A novel strengthening of SD is the *pairwise comparison (PC)* extension (Aziz et al., 2015). The reasoning behind PC is that p should be preferred to q if the probability that p yields an alternative preferred to the alternative returned by q is at least as large as the other way round. In other words, p is preferred to q if choosing p results in lower *ex ante* regret. Formally,

$$p \; R_i^{PC} \; q \quad \text{iff} \quad \sum_{x R_i y} p(x)q(y) \geq \sum_{x R_i y} q(x)p(y). \tag{PC}$$

For example, $2/3\, a + 1/3\, c \; P_i^{PC} \; b$.

The PC extension can alternatively be defined using skew-symmetric bilinear (SSB) utility functions as defined by Fishburn (1982). Blavatskyy (2006) gave a characterization of the PC extension which relies on the axioms that characterize SSB utility functions (cf. Fishburn, 1982, 1988) plus an additional axiom that singles out PC. In contrast to the previous three extensions, PC yields *complete* preference relations over lotteries.

The four lottery extensions introduced here form a hierarchy.[10] For all $R_i \in \mathcal{R}$,

$$R_i^{ST} \;\subseteq\; R_i^{BD} \;\subseteq\; R_i^{SD} \;\subseteq\; R_i^{PC}.$$

3.2. Efficiency and Strategyproofness

Arguably one of the most fundamental axioms in microeconomic theory is Pareto-efficiency. An alternative *Pareto-dominates* another alternative if every agent weakly prefers the for-

[10] To see that $R_i^{ST} \subseteq R_i^{BD}$, let $p, q \in \Delta(A)$ such that $p \; R_i^{ST} \; q$ and $x, y \in A$ with $x \; P_i \; y$. We need to show that $p(x)q(y) \geq p(y)q(x)$. If $p(y) = 0$ or $q(x) = 0$ this is trivial, since the right hand side is zero. So we consider the case where $y \in \widehat{p}$ and $x \in \widehat{q}$. Because $p \; R_i^{ST} \; q$ and $x \; P_i \; y$ we have that $x \in \widehat{p}$ and $y \in \widehat{q}$. This implies that $p(x) = q(x)$ and $p(y) = q(y)$ and in particular $p(x)q(y) = p(y)q(x)$. We refer to Aziz et al. (2015) for the remaining inclusions.

mer to the latter and at least one agent strictly prefers the former to the latter. *Pareto-efficiency* prescribes that Pareto-dominated alternatives are not chosen. There are various reasonable ways to define Pareto-efficiency in probabilistic social choice. In particular, every lottery extension defines a corresponding notion of Pareto-efficiency.

Definition 1. Let $\mathcal{E} \in \{ST, BD, SD, PC\}$, $R \in \mathcal{R}^n$, and $p, q \in \Delta(A)$. Then, p $\mathcal{E}$-*dominates* q if $p\ R_i^{\mathcal{E}}\ q$ for all $i \in N$ and $p\ P_i^{\mathcal{E}}\ q$ for some $i \in N$. An SDS f is $\mathcal{E}$-*efficient* if, for every $R \in \mathcal{R}^n$, there does not exist a lottery that $\mathcal{E}$-dominates $f(R)$.

Since $R_i^{ST} \subseteq R_i^{BD} \subseteq R_i^{SD} \subseteq R_i^{PC}$, PC-efficiency implies SD-efficiency which in turn implies BD-efficiency which in turn implies ST-efficiency.

A standard efficiency notion that cannot be formalized using lottery extensions is *ex post* efficiency. An SDS is *ex post efficient* if, for every preference profile, it assigns probability zero to all Pareto-dominated alternatives. It can be shown that SD-efficiency implies *ex post* efficiency and *ex post* efficiency implies BD-efficiency (cf. Aziz et al., 2015).

Efficiency essentially requires that outcomes are *socially* optimal. This can be contrasted with strategyproofness, which is concerned with the *individual* behavior of agents. Strategyproofness prescribes that no agent can obtain a more preferred outcome by misrepresenting his preferences. Again, we obtain varying degrees of this property depending on the underlying lottery extension.

Definition 2. Let $\mathcal{E} \in \{ST, BD, SD, PC\}$. An SDS f is $\mathcal{E}$-*manipulable* if there are $R, R' \in \mathcal{R}^n$ and $i \in N$ with $R_j = R'_j$ for all $j \neq i$ such that $f(R')\ P_i^{\mathcal{E}}\ f(R)$. An SDS is $\mathcal{E}$-*strategyproof* if it is not $\mathcal{E}$-manipulable.

Since $R_i^{ST} \subseteq R_i^{BD} \subseteq R_i^{SD} \subseteq R_i^{PC}$, PC-strategyproofness implies SD-strategyproofness which in turn implies BD-strategyproofness which in turn implies ST-strategyproofness (see Figure 1). Note that our definition of strategyproofness does *not* require that $f(R)\ R_i^{\mathcal{E}}\ f(R')$ for all R' with $R'_j = R_j$ for all $j \neq i$. We refer to this stronger notion as *strong strategyproofness*, but only use it in the context of the SD extension. For coarser extensions, in which most lotteries are incomparable, it seems unduly restrictive. The weaker notion employed here is for example also used by Postlewaite and Schmeidler (1986) and Bogomolnaia and Moulin (2001) for the SD extension. Note that due to the completeness of the PC extension, strong PC-strategyproofness and PC-strategyproofness coincide. Moreover, strong SD-strategyproofness is stronger than PC-strategyproofness while (weak) SD-strategyproofness is weaker.

4. Random Serial Dictatorship

In this section, we examine random serial dictatorship (RSD)—an extension of random dictatorship to the case where agents may express indifference among alternatives. RSD is commonly used in house allocation, matching, and coalition formation domains where ties are ubiquitous (see, e.g., Abdulkadiroğlu and Sönmez, 1998; Bogomolnaia and Moulin, 2004; Che and Kojima, 2010; Budish et al., 2013). In these contexts, *RSD* is sometimes also referred to as the *random priority* mechanism. *RSD* is defined by picking a sequence of the agents uniformly at random and then invoking serial dictatorship (i.e., each agent narrows down the set of alternatives by picking his most preferred of the alternatives selected by the previous agents).

For a formal definition of *RSD*, let $\max_{R_i}(A)$ denote the set of maximal alternatives according to R_i, i.e., $\max_{R_i}(A) = \{x \in A \colon x \mathrel{R_i} y \text{ for all } y \in A\}$. For given $R \in \mathcal{R}$, $A' \subseteq A$, and $N' \subseteq N$, *RSD* is then recursively defined as follows.[11]

$$RSD(R, A', N') \quad = \quad \begin{cases} \Delta(A') & \text{if } N' = \emptyset, \\ \frac{1}{|N'|} \sum\limits_{i \in N'} RSD(R, \max_{R_i}(A'), N' \setminus \{i\}) & \text{otherwise.} \end{cases}$$

Then, $RSD(R) = RSD(R, A, N)$.

Clearly, the set $RSD(R)$ can only contain more than one lottery if there are two alternatives among which all agents are indifferent. Otherwise, $RSD(R)$ consists of a single lottery. An SDS is called an *RSD* scheme if it always selects a lottery from the set $RSD(R)$ that furthermore only depends on $RSD(R)$.

Definition 3. An SDS f is an *RSD scheme* if for every $R \in \mathcal{R}^n$, $f(R) \in RSD(R)$, and for all $R, R' \in \mathcal{R}^n$, $RSD(R) = RSD(R')$ implies $f(R) = f(R')$.

If Φ is a property such as efficiency or strategyproofness, we write "*RSD* satisfies Φ" if every *RSD* scheme satisfies Φ. Similarly, we say that "*RSD* violates Φ" if every *RSD* scheme violates Φ.

It is well-known that, if the outcome is determined by *RSD*, truth telling is a weakly dominant strategy for every agent when lotteries are compared according to the *SD* extension (see, e.g., Bogomolnaia et al., 2005). Moreover, *RSD* is *ex post* efficient.

Theorem 1. *RSD is* ex post *efficient and strongly SD-strategyproof.*

The proofs of all theorems are deferred to the appendix.

It has been shown by Bogomolnaia and Moulin (2001) that *RSD* violates *SD*-efficiency within the domain of house allocation. The example by Bogomolnaia and Moulin can be translated to a preference profile on 24 alternatives in the general social choice setting. We give an independent example with four alternatives and four agents and completely characterize under which configurations *RSD* satisfies *SD*-efficiency.[12] Consider the following

[11] Here, the sum of two sets A and B is defined as the Minkowski sum, i.e., $A + B = \{x + y \colon x \in A, y \in B\}$.

[12] Bogomolnaia et al. (2005) provide an example with six agents and five alternatives for the special case of dichotomous preferences.

preference profile.

$$\begin{aligned}
R_1 &: \{a, c\}, b, d \\
R_2 &: \{a, d\}, b, c \\
R_3 &: \{b, c\}, a, d \\
R_4 &: \{b, d\}, a, c
\end{aligned}$$

The unique *RSD* lottery is $p = {}^1\!/_3\, a + {}^1\!/_3\, b + {}^1\!/_6\, c + {}^1\!/_6\, d$, which is *SD*-dominated by ${}^1\!/_2\, a + {}^1\!/_2\, b$. In fact, it is even the case that *all* agents *strictly* prefer the latter lottery according to *SD*. In other words, there exists a lottery which gives *strictly more* expected utility than p to *each* agent and for *every* utility representation consistent with the ordinal preferences of the agents. Therefore *RSD* is not *SD*-efficient for $n = 4$ and $m = 4$.

Theorem 2. *RSD is SD-efficient iff $n \leq 2$, or $m \leq 3$, or $n = 3$ and $m \leq 5$.*

The stronger notion of *PC*-efficiency is violated by *RSD* even when preferences are linear. In other words, *RD* already fails to satisfy *PC*-efficiency. To see this, consider the following preference profile with three agents and three alternatives.

$$\begin{aligned}
R_1 &: a, b, c \\
R_2 &: b, c, a \\
R_3 &: c, b, a
\end{aligned}$$

Clearly, *RD* (and hence *RSD*) returns ${}^1\!/_3\, a + {}^1\!/_3\, b + {}^1\!/_3\, c$, which is *PC*-dominated by the degenerate lottery b.[13] *RD* does satisfy *SD*-efficiency, however.

While implementing *RSD* by uniformly selecting a sequence of agents and then running serial dictatorship is straightforward, it was recently shown that computing the *ex ante* *RSD* probabilities is #P-complete and therefore computationally intractable (Aziz et al., 2013). Even the question of whether the *RSD* probability of a given alternative exceeds some fixed value $\lambda \in (0, 1)$ is NP-complete and hence cannot be answered in polynomial time unless P equals NP.

5. Strict Maximal Lotteries

Maximal lotteries were first considered by Kreweras (1965) and independently proposed and studied in more detail by Fishburn (1984a). Interestingly, maximal lotteries or variants thereof have been rediscovered again by economists (Laffond et al., 1993), mathematicians (Fisher and Ryan, 1995), political scientists (Felsenthal and Machover, 1992), and computer scientists (Rivest and Shen, 2010). An axiomatic characterization of maximal lotteries was recently given by Brandl et al. (2016b).

[13] A similar example in which all agents are *strictly* better off can easily be constructed with four alternatives and four agents.

In order to define maximal lotteries, we need some notation. For a preference profile $R \in \mathcal{R}^n$ and two alternatives $x, y \in A$, the *majority margin* $g_R(x, y)$ is defined as the difference between the number of agents who prefer x to y and the number of agents who prefer y to x, i.e.,

$$g_R(x, y) = |\{i \in N \colon x \mathrel{R_i} y\}| - |\{i \in N \colon y \mathrel{R_i} x\}|.$$

Thus, $g_R(y, x) = -g_R(x, y)$ for all $x, y \in A$. A *maximal alternative*, aka *(weak) Condorcet winner*, is an alternative $x \in A$ with $g_R(x, y) \geq 0$ for all alternatives $y \in A$. It is well known that maximal alternatives may fail to exist. This drawback can however be remedied by considering lotteries over alternatives. The function g_R can be naturally extended to pairs of lotteries by considering its bilinear form, which corresponds to *expected* majority margins. For $p, q \in \Delta(A)$, let

$$g_R(p, q) = \sum_{x,y \in A} p(x)q(y)g_R(x, y).$$

The set of maximal lotteries is then defined as

$$ML(R) = \{p \in \Delta(A) \colon g_R(p, q) \geq 0 \text{ for all } q \in \Delta(A)\}.^{14}$$

The minimax theorem (von Neumann, 1928) implies that $ML(R)$ is non-empty for all $R \in \mathcal{R}^n$. In fact, g_R can be interpreted as the payoff matrix of a symmetric zero-sum game and maximal lotteries as the mixed maximin strategies (or Nash equilibria) of this game. As a consequence, elements of $ML(R)$ can be found in polynomial time using linear programming. Interestingly, $ML(R)$ is a singleton in most cases. This holds, in particular, if all agents have linear preferences and the number of agents is odd (Laffond et al., 1997; Le Breton, 2005).

A particularly interesting subclass of $ML(R)$ is given by the set of *strict* maximal lotteries $SML(R)$, which corresponds to the set of *quasi-strict* Nash equilibria of g_R, i.e., all equilibria p in which every action in the support of p yields strictly more payoff than every action outside of the support of p.[15] In zero-sum games, quasi-strict equilibria constitute a subset of equilibria with maximal support (see, e.g., Dutta and Laslier, 1999; Brandt and Fischer, 2008).

Definition 4. Let $R \in \mathcal{R}^n$ and $p \in \Delta(A)$. Then $p \in SML(R)$ if, for all $x \in A$,

$$\begin{aligned} p(x) > 0 \quad &\text{iff} \quad g_R(x, p) = 0, \text{ and} \\ p(x) = 0 \quad &\text{iff} \quad g_R(x, p) < 0. \end{aligned}$$

An SDS is called an *SML scheme* if it always selects a lottery from the set $SML(R)$ and furthermore only depends on $SML(R)$.[16]

[14] Since ML only depends on g_R, it does neither require transitivity nor completeness of individual preferences.

[15] Geometrically, $SML(R)$ is the relative interior of $ML(R)$.

[16] The second assumption is not critical because $SML(R)$ is almost always a singleton (see, e.g., Brandl et al., 2016b).

Definition 5. An SDS f is an *SML scheme* if for every $R \in \mathcal{R}^n$, $f(R) \in SML(R)$, and for all $R, R' \in \mathcal{R}^n$, $SML(R) = SML(R')$ implies $f(R) = f(R')$.

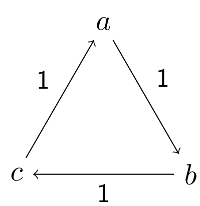
If Φ is a property such as efficiency or strategyproofness, we write "SML satisfies Φ" if every SML scheme satisfies Φ. Similarly, we say that "SML violates Φ" if every SML scheme violates Φ. Note that every SML scheme only depends on g_R, since $SML(R)$ only depends on g_R.

We prove that every SML scheme is PC-efficient. This result contrasts with our earlier observation that RSD fails to be SD-efficient and hence PC-efficient.

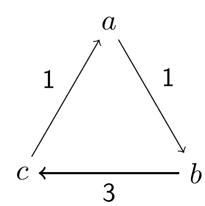
Theorem 3. *SML is PC-efficient.*

While SML satisfies a very high degree of efficiency, it does not do as well in terms of strategyproofness. In fact, SML already violates BD-strategyproofness. To see this, let $A = \{a, b, c\}$ and consider the preference profiles given below. The set of maximal lotteries only depends on g_R, which can be nicely represented as a weighted majority graph: For every pair of alternatives x and y with $g_R(x, y) > 0$, there is an edge from x to y labeled with $g_R(x, y)$.

$$\begin{aligned} R_1 &\colon a, c, b \\ R_2 &\colon a, b, c \\ R_3, R_4 &\colon b, c, a \\ R_5 &\colon c, a, b \end{aligned}$$



$$\begin{aligned} R'_1 &\colon a, \mathbf{b}, \mathbf{c} \\ R'_2 &\colon a, b, c \\ R'_3, R'_4 &\colon b, c, a \\ R'_5 &\colon c, a, b \end{aligned}$$



It can be verified that $SML(R) = \{\frac{1}{3}a + \frac{1}{3}b + \frac{1}{3}c\}$. However, if agent 1 misrepresents his preferences between b and c by reporting $R'_1\colon a, b, c$, the outcome for the new preference profile R' is $SML(R') = \{\frac{3}{5}a + \frac{1}{5}b + \frac{1}{5}c\}$. Thus, $f(R')\ P_1^{BD}\ f(R)$ for any SML scheme f.

On the other hand, we show that SML is ST-strategyproof.

Theorem 4. *SML is ST-strategyproof.*

The proof of Theorem 4 reveals some interesting properties of SML and we verified that SML does not satisfy minimal strengthenings of ST-strategyproofness. It seems as if this is the highest degree of strategyproofness one can hope for when also insisting on PC-efficiency.[17] While ST-strategyproofness does allow manipulators to skew the resulting distribution, crude manipulative attacks such as distorting the outcome from one degenerate lottery to another—an attack that many common SDSs suffer from (see Section 6)—or from one support to another disjoint one are futile. Moreover, SML is PC-strategyproof for all preference profiles that admit a Condorcet winner (Hoang, 2017).

[17]For example, SML does not satisfy strategyproofness with respect to the strengthening of ST given in Footnote 9. When preference are strict, SML satisfies strategyproofness with respect to the Kelly extension (p is preferred to q iff $\widehat{p}\ R_i\ \widehat{q}$) (Brandt, 2015). This extensions is incomparable to the ST extension, but only allows the comparison of lotteries whose supports overlap in at most one alternative.

6. Negative Results

Brandl et al. (2016a) have shown that there is no anonymous, neutral, *SD*-efficient, and *SD*-strategyproof SDS. As shown in Section 4, *RSD* violates *PC*-efficiency, even when preferences are assumed to be linear. In fact, no anonymous, *PC*-efficient, and *PC*-strategyproof SDS is known under this assumption. Randomizing over the winning sets of various commonly used SCFs such as *Borda's rule*, *Copeland's rule*, or *Hare's rule* (aka instant runoff) fails to be *PC*-strategyproof because all these rules can be manipulated with respect to *any* lottery extension (Taylor, 2005, pp. 44–51). Known *PC*-strategyproof SDSs that are assigning probabilities to alternatives in proportion to their Borda or Copeland scores (see, e.g., Barberà, 1979b), on the other hand, trivially fail to satisfy *ex post* efficiency (and therefore also *PC*-efficiency). Still, *PC*-efficiency is not unduly restrictive as it is satisfied by *ML*.

We now show that *PC*-efficiency and *PC*-strategyproofness are indeed incompatible with each other.

Theorem 5. *There is no anonymous, neutral, PC-efficient, and PC-strategyproof SDS for $n \geq 3$ and $m \geq 3$.*

We conjecture that this impossibility even holds for linear preferences and even when giving up neutrality. Brandl et al. (2016a) have leveraged computer-aided solving techniques to prove a similar statement for *SD*-efficiency and *SD*-strategyproofness.[18] While their theorem uses considerably weaker notions of efficiency and strategyproofness, their computer-generated proof is extremely tedious to check. We give a more accessible manual proof of Theorem 5 in the Appendix.

An important subclass of SDSs consists of pairwise SDSs. An SDS f is *pairwise* (or a neutral C2 function) if it is neutral and $f(R) = f(R')$ for all $R, R' \in \mathcal{R}^n$ such that for all $x, y \in A$,

$$|\{i \in N : x \ R_i \ y\}| - |\{i \in N : y \ R_i \ x\}| = |\{i \in N : x \ R_i' \ y\}| - |\{i \in N : y \ R_i' \ x\}|.$$

In other words, the outcome of a pairwise SDS only depends on the anonymized comparisons between pairs of alternatives (see, e.g., Young, 1974; Fishburn, 1977; Zwicker, 1991). Hence, pairwiseness is stronger than both anonymity and neutrality. Many SCFs such Borda's rule, Copeland's rule, and Kemeny's rule are pairwise. Moreover, *ML* is a pairwise SDS.

The following theorem shows that the conditions in Theorem 5 (and also those in the theorem by Brandl et al., 2016a) can be significantly strengthened when restricting attention to pairwise SDSs.

Theorem 6. *There is no pairwise,* ex post *efficient, and BD-strategyproof SDS for $n \geq 4$ and $m \geq 4$.*

[18] Strictly speaking, Theorem 5 is not implied by the theorem of Brandl et al. (2016a) because their theorem requires at least four agents and four alternatives.

Our results, both positive and negative, concerning the tradeoff between efficiency and strategyproofness are summarized in Figure 1.

7. Group-strategyproofness

A strengthening of strategyproofness that is often considered is *group-strategyproofness.* It prescribes that no group of agents should be able to jointly benefit from misrepresenting their preferences.

Definition 6. Let $\mathcal{E} \in \{ST, BD, SD, PC\}$. An SDS f is $\mathcal{E}$-*group-manipulable* if there are $R, R' \in \mathcal{R}^n$ and $S \subseteq N$ with $R_j = R'_j$ for all $j \notin S$ and $f(R')\ P_i^{\mathcal{E}}\ f(R)$ for all $i \in S$. An SDS is $\mathcal{E}$-*group-strategyproof* if it is not $\mathcal{E}$-group-manipulable.

Perhaps surprisingly, *RSD* and *SML* satisfy the same degree of group-strategyproofness (with respect to the preference extensions considered in this paper): both *RSD* and *SML* satisfy ST-group-strategyproofness and violate BD-group-strategyproofness.

It can easily be seen that *RSD* is ST-group-strategyproof because every *RSD* lottery assigns positive probability to at least one most preferred alternative of every agent. There can be no lottery that some agent prefers to this lottery according to the ST extension.

For the case of *SML*, it can be verified that the proof of Theorem 4 straightforwardly carries over to group-strategyproofness. Hence, we have the following theorem.

Theorem 7. *RSD and SML satisfy ST-group-strategyproofness*

For our final result we consider the same conditions as in Theorem 6, but replace BD-strategyproofness with BD-group-strategyproofness. It turns out that pairwiseness is no longer required for an impossibility.

Theorem 8. *For $n \geq 3$ and $m \geq 3$, there is no anonymous, neutral,* ex post *efficient, and BD-group-strategyproof SDS, even when preferences are dichotomous.*

An immediate consequence of this theorem is that *RSD* violates BD-group-strategyproofness.[19] Theorem 8 is a strengthening of a theorem by Bogomolnaia et al. (2005) who showed that same statement for *SD*-group-strategyproofness and at least four agents and six alternatives. For the stronger (but less reasonable) notion of group-strategyproofness in which only one of the deviating agents has to be strictly better off, we were able to show the previous impossibility even without requiring anonymity and neutrality. The proof can be found in the Appendix. Our results on group-strategyproofness are summarized in Figure 2.

[19]In general, *RSD* schemes need not be neutral, but the proof of Theorem 8 only uses profiles in which there is a unique *RSD* lottery.

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APPENDIX

A. Proofs

A.1. Random Serial Dictatorship

Theorem 1. *RSD is* ex post *efficient and strongly SD-strategyproof.*

Proof. *RSD* can be seen as a convex combination of serial dictatorships for all permutations of agents. Serial dictatorships are Pareto-efficient because whenever x Pareto-dominates y, y will never be removed before x and x can only be removed together with y. The convex combination of Pareto-efficient SCFs constitutes an *ex post* efficient SDS.

Serial dictatorships are strategyproof because at any stage of the mechanism, an agent can only get a less preferred alternatives selected by lying about his preferences. The convex combination of strategyproof SCFs constitutes a strongly *SD*-strategyproof SDSs by the linearity of expectation. □

Theorem 2. *RSD is SD-efficient iff $n \leq 2$ or $m \leq 3$ or $n = 3$ and $m \leq 5$. RSD is not SD-efficient for $n \geq 4$ and $m \geq 4$ and $n \geq 3$ and $m \geq 6$.*

Proof. For the cases where *RSD* is *SD*-efficient, we in fact show that *SD*-efficiency is equivalent to *ex post* efficiency. The statement then follows from the fact that *RSD* is *ex post* efficient. We assume throughout the proof that there are no two alternatives among which every agent is indifferent. This assumption is without loss of generality, any two such alternatives may be regarded as the same alternative for all conclusions drawn throughout this proof. Let $R \in \mathcal{R}^n$ be a preference profile and let $p \in \Delta(A)$ be *ex post* efficient in R. Assume for contradiction that p is not *SD*-efficient, i.e., there is $q \in \Delta(A)$ that *SD*-dominates p, i.e., $q\ R_i^{SD}\ p$ for all $i \in N$ and $q\ P_i^{SD}\ p$ for some $i \in N$. Let $S^- = \{x \in A\colon q(x) < p(x)\}$ and $S^+ = \{x \in A\colon q(x) > p(x)\}$. Since $q \neq p$, both S^- and S^+ are non-empty. All elements of S^- are Pareto-efficient because $S^- \subseteq \widehat{p}$. Moreover, observe that, for every agent $i \in N$, there are $x \in S^+$ and $y \in S^-$ such that x is ranked highest and y is ranked lowest among the elements of $S^+ \cup S^-$, since $q\ {R_i}^{SD}\ p$. We now consider the three cases from the statement of the theorem.

- $n \leq 2$: The argument for $n = 1$ is trivial. Consider $n = 2$. Since all $x, x' \in S^-$ are Pareto-efficient and $n = 2$, we have that $x\ R_1\ x'$ implies $x'\ P_2\ x$. Let $x \in S^-$ be the highest ranked alternative in R_1 among the alternatives in S^- and let $y \in S^+$ such that $y\ R_1\ x$. Such a y exists because $q\ {R_1}^{SD}\ p$. Since x is Pareto-efficient, we have that $x\ P_2\ y$. But this implies that $x'\ P_2\ y$ for all $x' \in S^-$, which contradicts $q\ {R_2}^{SD}\ p$.

- $m \leq 3$: Clearly, $S^- \cap S^+ = \emptyset$. Hence either $|S^-| = 1$ or $|S^+| = 1$. But then x Pareto-dominates y for all $x \in S^+$ and $y \in S^-$, which contradicts *ex post* efficiency of p.

- $n = 3$ and $m \leq 5$: Clearly, $S^- \cap S^+ = \emptyset$. If $|S^-| = 1$ or $|S^+| = 1$, then x Pareto-dominates y for all $x \in S^+$ and $y \in S^-$, which contradicts *ex post* efficiency of p. Consider the case $|S^-| = 2$ and $|S^+| = 2$. For every $x \in S^-$ and $y \in S^+$, there is $i \in N$ such that $x \; P_i \; y$, since x is Pareto optimal. From $n = 3$, it follows that there are $i \in N$ such that either $x \; P_i \; y$ for some $x \in S^-$ and all $y \in S^+$ or $x \; P_i \; y$ for all $x \in S^-$ and some $y \in S^+$. Either case contradicts $q \; {R_i}^{SD} \; p$. Now consider the case $|S^-| = 2$ and $|S^+| = 3$ and let $S^- = \{a, b\}$. Without loss of generality, $a \; P_1 \; b$ and $b \; R_i \; a$ for $i \in \{2, 3\}$. As stated above, $q \; {R_i}^{SD} \; p$ and $b \; R_i \; a$ imply that a is ranked last in R_i for $i \in \{2, 3\}$. Also, for every $y \in S^+$, there is $i \in N$ such that $a \; P_i \; y$. It follows that $a \; P_1 \; y$ for all $y \in S^+$. This contradicts the fact that $q \; {R_1}^{SD} \; p$. The case $|S^-| = 3$ and $|S^+| = 2$ is analogous.

To show that *RSD* does not satisfy *SD*-efficiency for $n = 3$ and $m = 6$, consider the following preference profile R.

$$
\begin{aligned}
R_1 &\colon \{a, d\}, c, e, b, f \\
R_2 &\colon \{b, e\}, a, f, c, d \\
R_3 &\colon \{c, f\}, b, d, a, e
\end{aligned}
$$

$RSD(R) = 1/6(a + b + c + d + e + f) = p$. However, for the lottery $q = 1/3(a + b + c)$ we have $q \; P_i^{SD} \; p$ for all $i \in \{1, 2, 3\}$, i.e., p is not *SD*-efficient.

An example that shows that *RSD* does not satisfy *SD*-efficiency for $n = 4$ and $m = 4$ has been given in the main part of the paper.

If *RSD* is *SD*-inefficient for some number of agents and alternatives, then it is also inefficient for any larger numbers. To see this for the number of agents, observe that adding agents that are indifferent between all alternatives does not change the set of *SD*-efficient lotteries. And to see this for the number of alternatives, observe that adding an alternative at the bottom of each agent's preference relation does not change the set of *SD*-efficient lotteries. Hence, the statements follow from induction on the number of agents and the number of alternatives with the above examples for inefficiency of *RSD* as the base cases. ☐

A.2. Strict Maximal Lotteries

Theorem 3. *SML is PC-efficient.*

Proof. With each preference relation $R_i \in \mathcal{R}$ we can associate a function $\phi_i : A \times A \to \{-1, 0, 1\}$ such that for all $x, y \in A$,

$$
\phi_i(x, y) = \begin{cases} 1 & \text{if } x \; P_i \; y, \\ -1 & \text{if } y \; P_i \; x, \text{ and} \\ 0 & \text{otherwise.} \end{cases}
$$

With slight abuse of notation, we extend ϕ_i to a function from $\Delta(A) \times \Delta(A)$ to $[0,1]$. For $p, q \in \Delta(A)$, let

$$\phi_i(p,q) = \sum_{x,y \in A} p(x)q(y)\phi_i(x,y).$$

Observe that $p \; {R_i}^{PC} \; q$ iff $\phi_i(p,q) \geq 0$. Now, let $R \in \mathcal{R}^n$ and $q \in SML(R)$. If q is not PC-efficient, there are $p \in \Delta(A)$ and $j \in N$ such that $p \; {P_j}^{PC} \; q$ and $p \; {R_i}^{PC} \; q$ for all $i \in N$. Thus, $\phi_j(p,q) > 0$ and $\phi_i(p,q) \geq 0$ for all $i \in N$. This implies that

$$g_R(p,q) = \sum_{i \in N} \phi_i(p,q) > 0,$$

where the first inequality follows from the definition of SML. However, this contradicts the assumption that $q \in SML(R)$. Thus, q is PC-efficient. $\square$

For the proof of Theorem 4 we need the following lemma, which states that weakening alternatives outside $\widehat{SML}(R)$ does not affect the set of strict maximal lotteries.

Lemma 1. *Let $R \in \mathcal{R}^n$ and $a \in A$ with $a \notin \widehat{SML}(R)$. Let furthermore $R' \in \mathcal{R}^n$ be such that $g_{R'}(b,a) > g_R(b,a)$ for some $b \in A \setminus \{a\}$ and $g_{R'}(x,y) = g_R(x,y)$ for all $x, y \in A$ with $\{x,y\} \neq \{a,b\}$. Then, $SML(R') = SML(R)$.*

Proof. Let R, R', and a be as stated. First, assume for contradiction that $a \in \widehat{SML}(R')$. It follows from Dutta and Laslier (1999), Theorem 4.4, that $a \in \widehat{SML}(R)$ which is a contradiction. Now, by definition, $p \in SML(R)$ if $g_R(x,p) = 0$ for all $x \in \widehat{SML}(R)$ and $g_R(x,p) < 0$ for all $x \notin \widehat{SML}(R)$. Since for all $x \in A \setminus \{a\}$,

$$g_R(x,p) = \sum_{y \in A} p(y) g_R(x,y) = \sum_{y \in \widehat{p}} p(y) g_R(x,y) = \sum_{y \in \widehat{p}} p(y) g_{R'}(x,y) = g_{R'}(x,p),$$

it follows that $SML(R) = SML(R')$. $\square$

Theorem 4. *SML is ST-strategyproof.*

Proof. Assume for contradiction that there is an SML scheme f that is not ST-strategyproof. Then, there are two preference profiles R and R' and an agent i such that $R_j = R'_j$ for all $j \neq i$ and $q = f(R') \; P_i^{ST} \; f(R) = p$.

For two alternatives $x, y \in A$ we say that x *is strengthened against* y if either (1) $y \; R_i \; x$ and $x \; P'_i \; y$, or (2) $y \; P_i \; x$ and $x \; R'_i \; y$. Define $\Delta(R,R') = \{(x,y) \colon x \text{ is strengthened against } y\}$. This set can be partitioned into the following four subsets.

$$\begin{aligned}
\Delta_1 &= \{(x,y) \in \Delta(R,R') \colon y \notin \widehat{p}\} \\
\Delta_2 &= \{(x,y) \in \Delta(R,R') \colon x \notin \widehat{q}\} \setminus \Delta_1 \\
\Delta_3 &= \{(x,y) \in \Delta(R,R') \colon x \in \widehat{q},\, y \in \widehat{p}, \text{ and } \{x,y\} \not\subseteq \widehat{p} \cap \widehat{q}\} \\
\Delta_4 &= \{(x,y) \in \Delta(R,R') \colon x \in \widehat{q},\, y \in \widehat{p}, \text{ and } \{x,y\} \subseteq \widehat{p} \cap \widehat{q}\}
\end{aligned}$$

We now construct two new preference profiles $\tilde{R}$ and $\tilde{R}'$ based on R and R'. The idea behind this construction is to make R and R' agree on as many pairs as possible, while maintaining the invariant that the outcomes are p and q, respectively.

$\tilde{R}$ is identical to R except that for all pairs $(x, y) \in \Delta_1$, we strengthen x against y in the preferences of agent i such that $\tilde{R}_i$ agrees with R'_i on all such pairs.[20] Lemma 1 implies that $f(\tilde{R}) = f(R) = p$. Analogously, $\tilde{R}'$ is identical to R' except that for all pairs $(x, y) \in \Delta_2$, we strengthen y against x in the preferences of agent i such that $\tilde{R}'_i$ agrees with R_i on all such pairs. Lemma 1 implies that $f(\tilde{R}') = f(R') = q$.

By definition, $\tilde{R}$ and $\tilde{R}'$ differ only on pairs that are contained in Δ_3 or Δ_4. Observe, however, that $\Delta_3 = \emptyset$. To see this, assume for contradiction that there is a pair $(x, y) \in \Delta(R, R')$ with $x \in \widehat{q}$, $y \in \widehat{p}$, and $\{x, y\} \not\subseteq \widehat{p} \cap \widehat{q}$. There are three cases:

- $x \in \widehat{q} \setminus \widehat{p}$ and $y \in \widehat{p} \setminus \widehat{q}$,
- $x \in \widehat{q} \setminus \widehat{p}$ and $y \in \widehat{p} \cap \widehat{q}$, and
- $x \in \widehat{p} \cap \widehat{q}$ and $y \in \widehat{p} \setminus \widehat{q}$.

In each case, $q \; P_i^{ST} \; p$ implies $x \; P_i \; y$. Since $(x, y) \in \Delta(R, R')$ implies $y \; R_i \; x$, we have a contradiction.

We thus have that $\Delta_3 = \emptyset$, and, consequently, that $\tilde{R}$ and $\tilde{R}'$ only differ on pairs of alternatives that are contained in Δ_4. In particular, $g_{\tilde{R}}$ and $g_{\tilde{R}'}$ agree on all pairs of alternatives that do not lie in $\widehat{p} \cap \widehat{q}$, i.e.,

$$g_{\tilde{R}}(a, b) = g_{\tilde{R}'}(a, b) \text{ for all } a, b \text{ with } \{a, b\} \not\subseteq \widehat{p} \cap \widehat{q}.$$

For such pairs, we omit the subscript and write $g(a, b)$ instead of $g_{\tilde{R}}(a, b)$. Likewise, we write $g(a, p)$ for $g_{\tilde{R}}(a, p)$ whenever $a \notin \widehat{p} \cap \widehat{q}$ and $p \in \Delta(A)$.

Let $x \in A$ and define a function $s \colon A \to [0, 1]$ via

$$s(x) = \begin{cases} p(x) & \text{if } x \in \widehat{p} \\ q(x) & \text{if } x \in \widehat{q} \\ 0 & \text{otherwise.} \end{cases}$$

Note that s is well-defined because $p(z) = q(z)$ for all $z \in \widehat{p} \cap \widehat{q}$, and that s may *not* correspond to a lottery because the individual probabilities may not add up to one.

For $a \notin \widehat{p} \cap \widehat{q}$ and a subset $B \subseteq A$ of alternatives, let furthermore $s(a, B) = \sum_{b \in B} s(b) g(a, b)$. If $B = \widehat{q}$, we have $s(a, \widehat{q}) = \sum_{b \in \widehat{q}} s(b) g(a, b) = \sum_{b \in \widehat{q}} q(b) g(a, b) = \sum_{b \in A} q(b) g(a, b) = g(a, q)$. Analogously, $s(a, \widehat{p})$ equals $g(a, p)$.

[20] Note that $\tilde{R}_i$ might not be transitive. Therefore, we do not assume transitivity of preferences in this proof. In fact, the statement of Theorem 4 becomes stronger but is easier to prove for general—possibly intransitive—preferences.

By definition, $g(x,p)=0$ and $g(x,q)<0$ for all $x\in\widehat{p}\setminus\widehat{q}$, as well as $g(y,p)<0$ and $g(y,q)=0$ for all $y\in\widehat{q}\setminus\widehat{p}$. Therefore, we get

$$\begin{aligned}
s(x,\widehat{q}) &= g(x,q) < 0 = g(x,p) = s(x,\widehat{p}) \text{ for all } x\in\widehat{p}\setminus\widehat{q}, \text{ and}\\
s(y,\widehat{p}) &= g(y,p) < 0 = g(y,q) = s(y,\widehat{q}) \text{ for all } y\in\widehat{q}\setminus\widehat{p}.
\end{aligned}$$

The inequality $s(x,\widehat{q})<s(x,\widehat{p})$ remains valid if $s(x,\widehat{p}\cap\widehat{q})$ is subtracted from both sides. Since $s(x,\widehat{q})-s(x,\widehat{p}\cap\widehat{q})=s(\widehat{q}\setminus\widehat{p})$ and $s(x,\widehat{p})-s(x,\widehat{p}\cap\widehat{q})=s(\widehat{p}\setminus\widehat{q})$, we obtain

$$\begin{aligned}
s(x,\widehat{q}\setminus\widehat{p}) &< s(x,\widehat{p}\setminus\widehat{q}) \text{ for all } x\in\widehat{p}\setminus\widehat{q}, \text{ and}\\
s(y,\widehat{p}\setminus\widehat{q}) &< s(y,\widehat{q}\setminus\widehat{p}) \text{ for all } y\in\widehat{q}\setminus\widehat{p}.
\end{aligned}$$

Multiplying both sides of these inequalities with a positive number, and writing $s'(a,B)$ for $s(a)\cdot s(a,B)$ results in

$$\begin{aligned}
s'(x,\widehat{q}\setminus\widehat{p}) &< s'(x,\widehat{p}\setminus\widehat{q}) \text{ for all } x\in\widehat{p}\setminus\widehat{q}, \text{ and}\\
s'(y,\widehat{p}\setminus\widehat{q}) &< s'(y,\widehat{q}\setminus\widehat{p}) \text{ for all } y\in\widehat{q}\setminus\widehat{p}.
\end{aligned}$$

We finally summarize over $\widehat{q}\setminus\widehat{p}$ and $\widehat{q}\setminus\widehat{p}$, respectively, and get

$$\sum_{x\in\widehat{p}\setminus\widehat{q}} s'(x,\widehat{q}\setminus\widehat{p}) < \sum_{x\in\widehat{p}\setminus\widehat{q}} s'(x,\widehat{p}\setminus\widehat{q}), \text{ and} \tag{1}$$

$$\sum_{y\in\widehat{q}\setminus\widehat{p}} s'(y,\widehat{p}\setminus\widehat{q}) < \sum_{y\in\widehat{q}\setminus\widehat{p}} s'(y,\widehat{q}\setminus\widehat{p}). \tag{2}$$

In order to arrive at a contradiction, we state two straightforward identities that are based on the skew-symmetry of g.

$$\sum_{a\in B} s'(a,B) = 0 \text{ for all } B\subseteq A\setminus(\widehat{p}\cap\widehat{q}), \text{ and} \tag{3}$$

$$\sum_{b\in B} s'(b,C) + \sum_{c\in C} s'(c,B) = 0 \text{ for all } B,C\subseteq A\setminus(\widehat{p}\cap\widehat{q}). \tag{4}$$

Now (3) implies that the right hand side of both (1) and (2) is zero, and therefore

$$\begin{aligned}
\sum_{x\in\widehat{p}\setminus\widehat{q}} s'(x,\widehat{q}\setminus\widehat{p}) &< 0, \text{ and}\\
\sum_{y\in\widehat{q}\setminus\widehat{p}} s'(y,\widehat{p}\setminus\widehat{q}) &< 0.
\end{aligned}$$

However, (4) implies that

$$\sum_{x\in\widehat{p}\setminus\widehat{q}} s'(x,\widehat{q}\setminus\widehat{p}) + \sum_{y\in\widehat{q}\setminus\widehat{p}} s'(y,\widehat{p}\setminus\widehat{q}) = 0,$$

a contradiction. $\square$

A.3. Negative Results

Theorem 5. *There is no anonymous, neutral, PC-efficient, and PC-strategyproof SDS for $n \geq 3$ and $m \geq 3$.*

Proof. This result is established by reasoning about a set of preference profiles for a fixed number of agents and alternatives and deriving a contradiction. We prove the statement for $n = 3$ and $m = 3$. It can be generalized to any larger number of agents and alternatives as described in the proof of Theorem 2. To show a statement for more agents, we add agents that are indifferent between all alternatives. Both constructions do not affect the incentives of agents and the set of efficient lotteries. Hence, the proof carries through with the same arguments.

Let f be an SDS that satisfies anonymity, neutrality, PC-efficiency, and PC-strategyproofness. First, we consider the following preference profile.

$$R_1^1\colon a, \{b, c\} \qquad R_2^1\colon b, a, c \qquad R_3^1\colon c, a, b$$

Anonymity and neutrality imply that $f(R^1)(b) = f(R^1)(c)$. The only PC-efficient lottery which puts equal weight on b and c is the degenerate lottery a, since every other lottery of this form is dominated by a (agent 2 and 3 are indifferent while agent 1 is strictly better off). Hence, $f(R^1) = a$. Now, consider the following profile.

$$R_1^2\colon a, \{b, c\} \qquad R_2^2\colon b, a, c \qquad R_3^2\colon \{a, c\}, b$$

In this profile a Pareto-dominates c, hence $f(R^2)(c) = 0$. If agent 3 reports R_3^1 instead of R_3^2, he receives one of his most preferred alternatives, namely a, with probability 1. Therefore, PC-strategyproofness implies that $f(R^2) = a$. Now, consider the following preference profile.

$$R_1^3\colon a, \{b, c\} \qquad R_2^3\colon b, \{a, c\} \qquad R_3^3\colon \{a, c\}, b$$

Again, PC-efficiency implies that $f(R^3)(a) = 0$, since a Pareto-dominates c. If $f(R^3)(b) > 0$, agent 2 has an incentive to report R_2^3 instead of R_2^2 in R^2. Thus, $f(R^3) = a$.

Since we will need it later, we state an observation for the following preference profile.

$$R_1^4\colon c, a, b \qquad R_2^4\colon a, b, c \qquad R_3^4\colon b, c, a$$

Anonymity and neutrality imply that $f(R^4) = 1/3\, a + 1/3\, b + 1/3\, c$. Also notice that agent 1 prefers any lottery with higher probability on c than on b to the uniform lottery according to the PC-extension if his preferences are R_1^4. Now, consider another preference profile.

$$R_1^5\colon \{a, c\}, b \qquad R_2^5\colon a, b, c \qquad R_3^5\colon b, c, a$$

Here we distinguish two cases. First, assume $f(R^5) = a$ and consider a deviation by agent 3.

$$R_1^6\colon \{a, c\}, b \qquad R_2^6\colon a, b, c \qquad R_3^6\colon c, b, a$$

Anonymity and neutrality imply that $f(R^6)(a) = f(R^6)(c)$. Any lottery of this form other than $1/2\,a + 1/2\,c$ is PC-dominated by the latter. Thus, PC-efficiency implies that $f(R^6) = 1/2\,a + 1/2\,c$. But agent 3 prefers $1/2\,a + 1/2\,c$ to a if his preferences are R_3^5. This is a contradiction to PC-strategyproofness. The second case is $f(R^5) \neq a$. If $f(R^5)(c) > f(R^5)(b)$, then by the above observation, agent 1 prefers $f(R^5)$ to $f(R^4)$ if his preferences are R_1^4. This is a contradiction to PC-strategyproofness. Hence, $f(R^5)(c) \leq f(R^5)(b)$ and, since $f(R^5) \neq a$, $f(R^5)(b) > 0$.

$$R_1^7\colon \{a,c\}, b \qquad\qquad R_2^7\colon a, b, c \qquad\qquad R_3^7\colon b, \{a,c\}$$

It follows from $f(R^5)(b) > 0$ that $f(R^7)(b) > 0$, since otherwise agent 3 can benefit from reporting R_3^7 instead of R_3^5. In particular, we get $f(R^7) \neq a$. Finally, consider the following preference profile.

$$R_1^8\colon \{a,c\}, b \qquad\qquad R_2^8\colon a, \{b,c\} \qquad\qquad R_3^8\colon b, \{a,c\}$$

It follows from anonymity that $f(R^8) = f(R^3) = a$. But this implies that agent 2 can successfully deviate from R_2^7 to R_2^8, since he prefers a to any other lottery if his preferences are R_2^7. Hence, we obtain the desired contradiction. □

Theorem 6. *There is no pairwise,* ex post *efficient, and BD-strategyproof SDS for $n \geq 4$ and $m \geq 4$.*

Proof. We prove the statement for $n = 4$ and $m = 4$. It can be generalized to any larger number of agents and alternatives as described in the proof of Theorem 2. Let f be a pairwise, *ex post* efficient, and BD-strategyproof SDS. We first consider the preference profile R^1 and its weighted majority graph depicted in Figure 3 (i).

$$R_1^1\colon a, c, \{b,d\} \qquad\qquad R_2^1\colon b, d, \{a,c\}$$

Both, c and d are Pareto-dominated in R^1 and, thus, *ex post* efficiency implies $f(R^1)(c) = f(R^1(d) = 0$. Since f is pairwise, and in particular anonymous and neutral, it follows that $f(R^1) = 1/2\,a + 1/2\,b = p$. Now we consider the preference profile R^2 and its weighted majority graph as in Figure 3 (ii).

$$R_1^2\colon a, c, \{b,d\} \qquad\qquad R_2^2\colon \{b,d\}, \{a,c\}$$

Both agents are indifferent between b and d and again c is Pareto-dominated. Thus, pairwiseness and *ex post* efficiency imply that $f(R^2)(b) = f(R^2)(d)$ and $f(R^2)(c) = 0$. Hence, $f(R^2) = (1 - 2\lambda)\,a + \lambda\,b + \lambda\,d = q$ for some $\lambda \in [0, 1/2]$.

First, assume for a contradiction $\lambda > 1/3$. We consider the following preference profile and its weighted majority graph depicted in Figure 3 (iii).

$$R_1^3\colon a, \{b,c,d\} \qquad\qquad R_2^3\colon \{b,d\}, \{a,c\}$$

Pairwiseness implies that $f(R^3) = 1/3\, a + 1/3\, b + 1/3\, d = r$. But $r\ (P_1^2)^{BD}\ q$ if $\lambda > 1/3$, which contradicts BD-strategyproofness of f since agent 1 can manipulate in R^2 by reporting R_1^3 instead of R_1^2.

Now assume for a contradiction that $\lambda = 1/3$.

$R_1^4\colon a, c, b, d$ $\qquad$ $R_2^4\colon \{b, d\}, \{a, c\}$

$R_3^4\colon a, c, \{b, d\}$ $\qquad$ $R_4^4\colon \{b, d\}, c, a$

The weighted majority graph of R^4 is equal to that of R^1 and, thus, $f(R^4) = f(R^1) = p$.

$R_1^5\colon a, c, b, d$ $\qquad$ $R_2^5\colon \{b, d\}, \{a, c\}$

$R_3^5\colon a, c, \{b, d\}$ $\qquad$ $R_4^5\colon d, b, c, a$

The majority graph of R^5 is equal to that of R^2 and, hence, $f(R^5) = q$. But then, agent 4 in R^4 can manipulate by reporting R_4^5 instead of R_4^4 since $q\ (P_4^4)^{BD}\ p$. This again contradicts BD-strategyproofness of f.

Finally, we assume $\lambda < 1/3$ and consider.

$R_1^6\colon a, c, \{b, d\}$ $\qquad$ $R_2^6\colon \{b, d\}, \{a, c\}$

$R_3^6\colon \{b, c, d\}, a$ $\qquad$ $R_4^6\colon a, \{b, c, d\}$

The weighted majority graph of R^6 is equal to that of R^2 and, therefore, $f(R^6) = f(R^2) = q$. We consider one last preference profile.

$R_1^7\colon a, c, \{b, d\}$ $\qquad$ $R_2^7\colon \{b, d\}, \{a, c\}$

$R_3^7\colon \{b, d\}, c, a$ $\qquad$ $R_4^7\colon a, \{b, c, d\}$

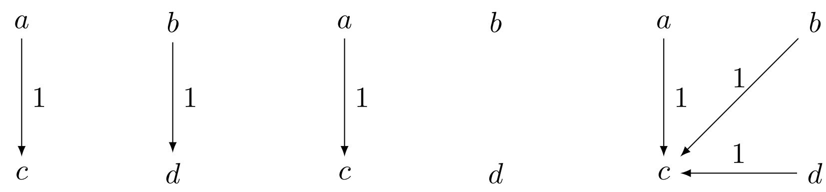
The majority graph of R^7 is equal to that of R^1, which implies that $f(R^7) = f(R^3) = r$. But $r\ (P_3^6)^{BD}\ q$ if $\lambda < 1/3$. Thus, agent 3 in R^6 can benefit from reporting R_3^7 instead of R_3^6. In any case, we found a successful manipulation, contradicting BD-strategyproofness of f. $\square$

A.4. Group-strategyproofness

Theorem 8. *For $n \geq 3$ and $m \geq 3$, there is no anonymous, neutral,* ex post *efficient, and BD-group-strategyproof SDS, even when preferences are dichotomous.*

Proof. We prove the statement for $n = 3$ and $m = 3$. It can be generalized to any larger number of agents and alternatives as described in the proof of Theorem 2. Assume for contradiction there is an SDS f with the properties as stated and consider the following preference profile.

$R_1^1\colon \{a, b\}, c$ $\qquad$ $R_2^1\colon \{a, c\}, b$ $\qquad$ $R_3^1\colon \{b, c\}, a$



(i) (ii) (iii)

Figure 3: Graphs depicting pairwise comparisons. An edge from x to y is labeled with $g_R(x, y)$, the number of agents preferring x to y minus the number of agents preferring y to x in preference profile R. All missing edges denote majority ties.

By neutrality and anonymity, $f(R^1) = {}^1\!/_3\, a + {}^1\!/_3\, b + {}^1\!/_3\, c$. Now let agents 1 and 2 change their preferences and consider the profile R^2.

$R_1^2\colon a, \{b, c\}$ $\qquad$ $R_2^2\colon a, \{b, c\}$ $\qquad$ $R_3^2\colon \{b, c\}, a$

Again by neutrality and anonymity, $f(R^2) = (1 - 2\lambda)a + \lambda b + \lambda c$. If $\lambda > {}^1\!/_3$, then agents 1 and 2 would rather report R_1^1 and R_2^1 respectively if their true preferences were R_1^2 and R_2^2. On the other hand, if $\lambda < {}^1\!/_3$ and their true preferences were R_1^1 and R_2^1, they would rather report R_1^2 and R_2^2. Hence, $\lambda = {}^1\!/_3$ and $f(R^2) = {}^1\!/_3\, a + {}^1\!/_3\, b + {}^1\!/_3\, c$.

$R_1^3\colon a, \{b, c\}$ $\qquad$ $R_2^3\colon \{a, b\}, c$ $\qquad$ $R_3^3\colon b, \{a, c\}$

In R^3, c is Pareto-dominated, thus by neutrality and anonymity, $f(R^3) = {}^1\!/_2\, a + {}^1\!/_2\, b$. To this end, we consider the following profile.

$R_1^4\colon a, \{b, c\}$ $\qquad$ $R_2^4\colon \{a, b\}, c$ $\qquad$ $R_3^4\colon \{b, c\}, a$

If agent 3 changes his preferences from R_3^3 to R_3^4, c is still Pareto-dominated and his preferences over a and b remain unchanged. Hence, by BD-strategyproofness, $f(R^4) = f(R^3)$. But then agent 2 in R^2 would have an incentive to report R_2^4 instead of R_2^2, a contradiction. $\square$

Let BD-strong-group-strategyproofness be the strengthening of BD-group-strategyproofness in which only one of the deviating agents has to be strictly better off. For this notion of group-stategyproofness, the statement of Theorem 8 holds even without requiring anonymity and neutrality. The proof of Theorem 9 is based on a construction by Bogomolnaia et al. (2005), but uses a weaker notion of strategyproofness.

Theorem 9. *There is no* ex post *efficient and BD-strong-group-strategyproof SDS for $n \geq 3$ and $m \geq 3$, even when preferences are dichotomous.*

Proof. We prove the statement for $n = 3$ and $m = 3$. It can be generalized to any larger number of agents and alternatives as described in the proof of Theorem 2. Assume for contradiction that f is an SDS with properties as stated and consider the following preference profile.

$$R_1^1\colon a, \{b, c\} \qquad R_2^1\colon b, \{a, c\} \qquad R_3^1\colon c, \{a, b\}$$

Let $f(R^1) = p$. We assume without loss of generality that $p(a) > 0$. Now consider a variation of the previous preference profile in which agent 1 is completely indifferent and let $f(R^2) = q$.

$$R_1^2\colon \{a, b, c\} \qquad R_2^2\colon b, \{a, c\} \qquad R_3^2\colon c, \{a, b\}$$

Clearly, a is Pareto-dominated by both b and c and therefore $q(a) = 0$. If agent 1 claims his preferences are as in R_1^3, alternative a remains Pareto-dominated by b.

$$R_1^3\colon \{a, b\}, \{c\} \qquad R_2^3\colon b, \{a, c\} \qquad R_3^3\colon c, \{a, b\}$$

Let $f(R^3) = r$ where $r(a) = \mathtt{0}$. If we assume that R_1^2 is the true preference relation of agent 1, group-strategyproofness requires that agent 2 should not prefer r to q because otherwise this may be seen as a beneficial group deviation of agents 1 and 2. As a consequence, $r(b) \leq q(b)$. Similarly, a group deviation by agents 1 and 3 implies that $r(c) \leq q(c)$ and consequently that $r = q$.

$$R_1^4\colon \{a, c\}, \{b\} \qquad R_2^4\colon b, \{a, c\} \qquad R_3^4\colon c, \{a, b\}$$

If we consider the profile R^4, a is Pareto-dominated by c and analogous arguments imply that $f(R^4) = q$.

Finally, consider the preference profile R^3 again. Strategyproofness implies that agent 1 should not benefit from deviating to R_1^1. It can be shown that

$$\neg(p\ (R_1^3)^{BD}\ q) \text{ iff } q(c) < \frac{p(c)}{p(b) + p(c)}.$$

Similarly, agent 1 should not benefit from deviating to R_1^1 in profile R^4 and

$$\neg(p\ (R_1^4)^{BD}\ q) \text{ iff } q(b) < \frac{p(b)}{p(b) + p(c)}.$$

Adding both inequalities yields that

$$q(b) + q(c) = 1 < \frac{p(b)}{p(b) + p(c)} + \frac{p(c)}{p(b) + p(c)} = 1,$$

a contradiction. □