On the Tradeoff between Economic Efficiency and Strategyproofness in Randomized Social Choice

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ABSTRACT

Two fundamental notions in microeconomic theory are *efficiency*—no agent can be made better off without making another one worse off—and *strategyproofness*—no agent can obtain a more preferred outcome by misrepresenting his preferences. When social outcomes are probability distributions (or lotteries) over alternatives, there are varying degrees of these notions depending on how preferences over alternatives are extended to preference over lotteries. We show that efficiency and strategyproofness are incompatible to some extent when preferences are defined using stochastic dominance (SD) and therefore introduce a natural weakening of SD based on Savage's sure-thing principle (ST). While *random serial dictatorship* is SD-strategyproof, it only satisfies ST-efficiency. Our main result is that *strict maximal lotteries*—an appealing class of social decision schemes due to Kreweras and Fishburn—satisfy SD-efficiency and ST-strategyproofness.

Categories and Subject Descriptors

I.2.11 [**Distributed Artificial Intelligence**]: Multiagent Systems; J.4 [**Computer Applications**]: Social and Behavioral Sciences - Economics

Keywords

Social choice theory; game theory; social decision schemes

1. INTRODUCTION

Two fundamental notions in microeconomic theory are *efficiency*—no agent can be made better off without making another one worse off—and *strategyproofness*—no agent can obtain a more preferred outcome by misrepresenting his preferences. The conflict between these two notions is already apparent in Gibbard and Satterthwaite's seminal theorem, which states that the only single-valued social choice functions that satisfy non-imposition—a weakening of efficiency—and strategyproofness are dictatorships [15, 25]. In this paper, we study efficiency and strategyproofness in the context of *social decision schemes (SDSs)*, i.e., functions that map a preference profile to a probability distribution (or lottery) over a fixed set of alternatives [e.g., 16, 2]. Randomized voting methods have a surprisingly long tradition going back to ancient Greece and have recently gained increased attention in political science [see, e.g., 27]. Within computer science, randomization has become a very successful technique for designing (computationally) efficient algorithms and has also been analyzed in the context of voting [e.g., 9, 23, 31].

In a probabilistic framework, the meaning of the concepts of efficiency and strategyproofness depends on how preferences over alternatives are extended to preferences over lotteries. We will refer to these extensions as *lottery extensions*. One of the most studied lottery extensions is *stochastic dominance (SD)*, which states that one lottery is preferred to another iff the former first-order stochastically dominates the latter. This extension is of particular importance because it coincides with the extension in which one lottery is preferred to another iff, for any utility representation consistent with the ordinal preferences, the former yields at least as much expected utility as the latter [see, e.g., 8]. Settings in which the existence of an underlying utility function cannot be assumed may call for weaker lottery extensions. A natural example is *deterministic dominance*, which merely states that one lottery is preferred to another iff all alternatives in the support of the former are strictly preferred to all alternatives in the support of the latter. Clearly, each of these lottery extensions gives to rise to different variants or degrees of efficiency and strategyproofness.

Perhaps the most well-known SDS is *random dictatorship (RD)*, in which one of the agents is uniformly chosen at random and then picks his most preferred alternative. Note that RD is only well-defined for strict preferences. Gibbard [16] has shown that RD is the only (strongly) SD-strategyproof SDS that never puts positive probability on Pareto dominated alternatives. It is easily seen that RD even satisfies the stronger condition of SD-efficiency. A drawback of Gibbard's beautiful result, however, is that it strongly relies on the non-existence of ties in the agents' preferences. In many important domains of social choice such as house allocation, matching, and coalition formation, ties are unavoidable because agents are indifferent among all outcomes in which their allocation, match, or coalition is the same [e.g., 26]. In the presence of ties, RD is typically extended to *random serial dictatorship (RSD)*, where dictators are invoked sequentially and ties between most-preferred alternatives are broken by subsequent dictators.

While RSD still satisfies SD-strategyproofness, it violates SD-efficiency. This was first observed by Bogomolnaia and

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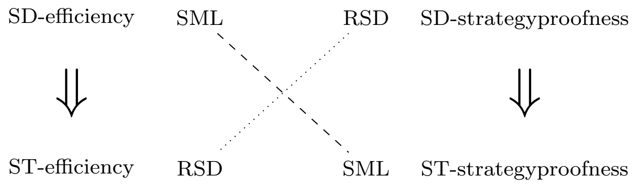


Figure 1: SML is SD-efficient and ST-strategyproof, but not SD-strategyproof. On the other hand, RSD is SD-strategyproof and ST-efficient, but not SD-efficient.

Moulin [4] in the restricted domain of house allocation. The example by Bogomolnaia and Moulin [4] can be translated to a preference profile with 24 alternatives in the general social choice setting. We give an independent example with four alternatives and show that this example is minimal by proving that RSD does satisfy SD-efficiency for three alternatives in Section 4. We doubt that there exists an anonymous SDS that satisfies both SD-efficiency and SD-strategyproofness and prove the incompatibility of these two properties for the subclass of majoritarian SDSs. In order to obtain positive results we then introduce a new lottery extension that lies in between stochastic dominance and deterministic dominance and is based on Savage's sure-thing principle (ST). SD and ST are then used to demonstrate an interesting tradeoff (see Figure 1). Random serial dictatorship is SD-strategyproof, but only satisfies ST-efficiency. On the other hand, schemes from a little known class of SDSs due to Kreweras and Fishburn called *strict maximal lotteries (SML)*, satisfy SD-efficiency and ST-strategyproofness. Strict maximal lotteries correspond to the mixed quasistrict Nash equilibria of the plurality game underlying the preferences of the voters. While ST-strategyproofness is quite weak, it is important to note that virtually all common Pareto optimal SDSs (except RSD) violate even much weaker strategyproofness notions. Moreover, SML satisfies a number of other desirable properties violated by RSD such as Condorcet-consistency and composition-consistency [18, 20].

2. PRELIMINARIES

Let $N = \{1, \dots, n\}$ be a set of voters with preferences over a finite set A of alternatives. The preferences of voter $i \in N$ are represented by a complete and transitive *preference relation* $R_i \subseteq A \times A$. The set of all preference relations will be denoted by $\mathcal{R}$. The interpretation of $(a, b) \in R_i$, usually denoted by $a \; R_i \; b$, is that voter i values alternative a at least as much as alternative b. In accordance with conventional notation, we write P_i for the strict part of R_i, i.e., $a \; P_i \; b$ if $a \; R_i \; b$ but not $b \; R_i \; a$. A *preference profile* $R = (R_1, \dots, R_n)$ is an n-tuple containing a preference relation R_i for every voter $i \in N$. The set of all preference profiles is thus given by $\mathcal{R}^n$.

Let furthermore $\Delta(A)$ denote the set of all *lotteries* (or *probability distributions*) over A, i.e.,

$$\Delta(A) = \left\{ \sum_{x \in A} p(x) \cdot x \colon \; p(x) \geq 0 \; \forall x \in A, \sum_{x \in A} p(x) = 1 \right\}.$$

The support of a lottery $p \in \Delta(A)$, denoted $\widehat{p}$, is the set of all alternatives to which p assigns positive probability, i.e.,

$$\widehat{p} = \{x \in A \colon p(x) > 0\}.$$

A lottery p is *degenerate* if $|\widehat{p}| = 1$, and we usually identify degenerate lotteries with the respective alternatives.

For any $\lambda \in [0, 1]$ and $p, q \in \Delta(A)$, the lottery

$$\lambda p + (1 - \lambda) q = \sum_{x \in A} (\lambda p(x) + (1 - \lambda) q(x)) \cdot x$$

is called a *convex combination* of p and q. Every lottery p can be written as the convex combination of $|\widehat{p}|$ degenerate lotteries. For a subset $B \subseteq A$ of alternatives, $\Delta(B)$ is the set of all lotteries that assign probability zero to all alternatives outside B, i.e., $\Delta(B) = \{p \in \Delta(A) \colon \widehat{p} \subseteq B\}$.

Our central object of study are social decision schemes, i.e., functions that map the individual preferences of the voters to a lottery over alternatives.

Definition 1. *A* social decision scheme (SDS) *is a function* $f : \mathcal{R}^n \to \Delta(A)$.

A minimal fairness condition for SDSs is *anonymity*, which requires that $f(R) = f(R')$ for all $R, R' \in \mathcal{R}^n$ and permutations $\pi : N \to N$ such that $R'_i = R_{\pi(i)}$ for all $i \in N$.

2.1 Lottery Extensions

In order to reason about the outcomes of SDSs, we need to make assumptions on how voters compare lotteries. A *lottery extension* maps preferences over alternatives to (possibly incomplete) preferences over lotteries.[1] We will now define the lottery extensions considered in this paper. For a more detailed account of lottery extensions and their properties, we refer to Cho [8].

Throughout this section, let $R_i \in \mathcal{R}$ be a preference relation and $p, q \in \Delta(A)$. The *trivial* lottery extension can only compare degenerate (or identical) lotteries. Since we identify degenerate lotteries with alternatives, the trivial extension is denoted simply by R_i. Formally, $p \; R_i \; q$ iff $p = q$ or

$$\exists x \neq y \in A \colon p(x) q(y) = 1 \text{ and } x \; P_i \; y.$$

The *deterministic dominance* extension, denoted $\widehat{R_i}$, prescribes that every alternative in the support of p is preferred to every alternative in the support of q, i.e., $p \; \widehat{R_i} \; q$ iff $p = q$ or

$$\forall x \neq y \in A \colon p(x) q(y) > 0 \Rightarrow x \; P_i \; y.$$

Next, we introduce a lottery extension that has not been considered before. The *sure thing (ST)* extension uses the same criterion as the deterministic dominance extension, but ignores all pairs of alternatives that are assigned the same probability in both lotteries. Let $\delta(p, q) = \{x \in A : p(x) \neq q(x)\}$ and define $p \; R_i^{ST} \; q$ iff $p = q$ or

$$\forall x \neq y \in A \colon \{x, y\} \cap \delta(p, q) \neq \emptyset \text{ and } p(x) q(y) > 0 \Rightarrow x \; P_i \; y.$$

The idea underlying ST is that the comparison of two lotteries should be independent of the part in which they coincide. This is strongly related to von Neumann and Morgenstern's independence axiom [30] and has also been used

[1]Since we are mainly interested in *strict* preferences over lotteries, we do not require that indifference between alternatives extends to indifference between the corresponding degenerate lotteries.

for defining preference extensions from alternatives to sets of alternatives [14].

Finally, *stochastic dominance (SD)* prescribes that for each alternative $x \in A$, the probability that p selects an alternative that is at least as good as x is greater or equal to the probability that q selects such an alternative. Formally, $p \ R_i^{SD} \ q$ iff

$$\sum_{x \in A : x R_i y} p(x) \geq \sum_{x \in A : x R_i y} q(x) \text{ for all } y \in A.$$

It is well-known that $p \ R_i^{SD} \ q$ iff the expected utility from p is higher than that from q for any utility function that is compatible with R_i.

It is straightforward to check that the four lottery extensions introduced above form an inclusion hierarchy.

Proposition 1. *$R_i \subseteq \widehat{R_i} \subseteq R_i^{ST} \subseteq R_i^{SD}$ for all $R_i \in \mathcal{R}$.*

Even though the SD extension is the finest lottery extension among the four, it is still incomplete. For example, the lotteries $\frac{1}{2}a + \frac{1}{2}c$ and b are incomparable for a voter preferring a to b to c.

We will mainly be concerned with the SD and ST extensions, and write P_i^{SD} and P_i^{ST} for the strict parts of the relations R_i^{SD} and R_i^{ST}, respectively.

We conclude the discussion of lottery extensions with a useful characterization of P_i^{ST}. The easy proof is omitted due to restricted space.

Proposition 2. *Let $R_i \in \mathcal{R}$ and $p, q \in \Delta(A)$. Then, $p \ P_i^{ST} \ q$ iff* [2]

(i) $\widehat{p} \setminus \widehat{q} \neq \emptyset$ and $\widehat{q} \setminus \widehat{p} \neq \emptyset$,

(ii) $(\widehat{p} \setminus \widehat{q}) \ P_i \ (\widehat{p} \cap \widehat{q}) \ P_i \ (\widehat{q} \setminus \widehat{p})$, and

(iii) $p(x) = q(x)$ for all $x \in \widehat{p} \cap \widehat{q}$.

2.2 Efficiency

Efficiency prescribes that there is no lottery that all voters prefer to the one returned by the SDS. Each lottery extension yields a corresponding notion of efficiency. Out of these, efficiency with respect to SD and ST will be of main interest in this paper.

Definition 2. *Let $\mathcal{E} \in \{SD, ST\}$. Given a preference profile R, a lottery p $\mathcal{E}$-*dominates *another lottery q if $p \ R_i^{\mathcal{E}} \ q$ for all $i \in N$ and $p \ P_i^{\mathcal{E}} \ q$ for some $i \in N$. An SDS f is $\mathcal{E}$-*efficient *if, for every $R \in \mathcal{R}^n$, there does not exist a lottery that $\mathcal{E}$-dominates $f(R)$.*

Since $R_i^{ST} \subseteq R_i^{SD}$, SD-efficiency implies ST-efficiency. A standard efficiency notion that cannot be phrased in terms of lottery extensions is (ex post) Pareto optimality. An SDS is *Pareto optimal* if it assigns probability zero to all Pareto-dominated alternatives. It can be shown that SD-efficiency implies Pareto optimality and that Pareto optimality implies ST-efficiency.

[2]For $B_1, B_2, B_3 \subseteq A$, we write $B_1 \ P_i \ B_2 \ P_i \ B_3$ if $b_1 \ P_i \ b_2$, $b_1 \ P_i \ b_3$, and $b_2 \ P_i \ b_3$ for all $b_j \in B_j$, $j \in \{1, 2, 3\}$.

2.3 Strategyproofness

Strategyproofness prescribes that no voter can obtain a more preferred outcome by misrepresenting his preferences. Again, we obtain varying degrees of this property depending on the underlying lottery extension.

Definition 3. *Let $\mathcal{E} \in \{SD, ST\}$. An SDS f is $\mathcal{E}$-*manipulable *if there exist preference profiles R and R' with $R_j = R'_j$ for all $j \neq i$ such that $f(R') \ P_i^{\mathcal{E}} \ f(R)$. An SDS is $\mathcal{E}$-*strategyproof *if it is not $\mathcal{E}$-manipulable.*

Since $R_i^{ST} \subseteq R_i^{SD}$, SD-strategyproofness implies ST-strategyproofness. Note that our definition of strategyproofness does *not* require that $f(R) \ R_i^{\mathcal{E}} \ f(R')$ for all R' with $R'_j = R_j$ for all $j \neq i$. This stronger strategyproofness notion, which is for instance used in Gibbard's [16] proof, seems unduly restrictive for lottery extensions in which most lotteries are incomparable. The weaker notion employed here is for example also used by Postlewaite and Schmeidler [22] for the SD extension.

Another strengthening of strategyproofness that is often considered is *group-strategyproofness*. It prescribes that no group of voters can jointly benefit from misrepresenting their preferences. All positive results in this paper concerning strategyproofness also hold for group-strategyproofness, while the negative ones even hold for individual strategyproofness.

3. AN IMPOSSIBILITY

Interestingly, it seems very difficult (if not impossible) to satisfy SD-efficiency and SD-strategyproofness simultaneously without violating anonymity. Randomizing over the winning sets of various commonly used social choice functions such as *Borda's rule*, *Copeland's rule*, *plurality with runoff*, *Hare's rule*, *Coombs's rule*, or *the weak Condorcet rule* fails to be SD-strategyproof because all these rules can even be manipulated with respect to the trivial lottery extension [28, Theorem 2.2]. Known SD-strategyproof SDSs that are assigning probabilities to alternatives in proportion to their Borda or Copeland scores [see e.g., 3, 9, 23], on the other hand, trivially fail to satisfy Pareto optimality (and therefore also SD-efficiency).

In this section, we prove a weak version of this incompatibility by showing that no majoritarian SDS is both SD-efficient and SD-strategyproof. An SDS f is said to be *majoritarian* if the lottery returned by f only depends on the unweighted majority graph induced by the majority comparisons between pairs of alternatives.

Theorem 1. *There exists no SD-efficient and SD-strategyproof majoritarian SDS on four or more alternatives.*

Proof. We prove the stronger statement that there exists no *Pareto optimal* and SD-strategyproof majoritarian SDS on four or more alternatives.

Let f be a Pareto optimal SDS. We will show that f is SD-manipulable. Consider the profile (R_1, R_2, R_3) given by[3]

$$1 : a, b, c, d$$
$$2 : b, c, d, a$$
$$3 : d, a, b, c$$

[3]Alternatives are listed in decreasing order of preference. For instance, the first voter strictly prefers a to b to c to d.

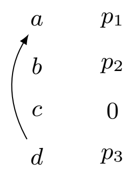
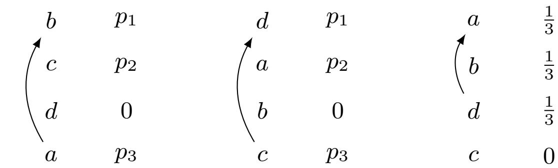


Figure 2: Majority graph G of the profile (R_1, R_2, R_3). An edge from node x to node y means that there is a majority of voters preferring x over y. All missing edges point downwards.



(i) (a, c) inverted (ii) (b, d) inverted (iii) (c, d) inverted

Figure 3: Majority graphs that result from G by inverting edges. All missing edges point downwards.

and let $p = f((R_1, R_2, R_3))$. Since b Pareto dominates c in this profile, $p(c) = 0$. The assumption that f is majoritarian can be utilized as follows. Figure 2 depicts the majority graph G of (R_1, R_2, R_3). Whenever a preference profile has a majority graph that is isomorphic to the one in Figure 2, then f assigns probability zero to the alternative that is third from the top (like c in the profile above). Majoritarianism also implies that the probabilities of the three remaining alternatives are fixed for all profiles that produce a majority graph isomorphic to G. Define $p_1 = p(a)$, $p_2 = p(b)$, and $p_3 = p(d)$. Thus, for every preference profile that produces a majority graph that looks like G, the SDS f assigns probability p_1 to the top alternative, probability p_2 to the second alternative, and probability p_3 to the bottom alternative.

Let us now introduce eight additional voters with preferences as follows.

$4 : a, c, b, d$	$6 : c, b, d, a$	$8 : c, d, a, b$	$10 : c, d, b, a$
$5 : d, b, c, a$	$7 : a, d, b, c$	$9 : b, a, d, c$	$11 : a, b, d, c$

Define $R = (R_1, \ldots, R_{11})$ and observe that the majority graph of R is identical to that of (R_1, R_2, R_3). The reason is that the new voters come in pairs which cancel each other out: the preferences of voter 4 are opposite to the preferences of voter 5, and so on. Therefore, we have $f(R) = p$. We will use a case distinction to show that f is SD-manipulable.

Case 1: $p_3 \geq p_1$. Consider the profile that is identical to R except that voter 4 has changed his preferences

$$\text{from} \quad R_4 : a, c, b, d \quad \text{to} \quad R'_4 : c, a, b, d.$$

The resulting majority graph is depicted in Figure 3(i). This graph is isomorphic to G, and consequently we know from the arguments above that f assigns probability p_1 to b, probability p_2 to c, probability 0 to d, and probability p_3 to a. This is summarized in the following table.

	a	c	b	d
$R_4 : a, c, b, d$	p_1	0	p_2	p_3
$R'_4 : c, a, b, d$	p_3	p_2	p_1	0

Since we have assumed that $p_3 \geq p_1$, it follows that voter 4 prefers the new lottery to the original lottery with respect to SD. Thus, voter 4 can benefit from misrepresenting his preferences.

Case 2: $p_1 > p_3 \geq p_2$. Voter 6 can manipulate as follows. By strengthening d versus b, the majority graph changes from G to the graph in Figure 3(ii).

	c	b	d	a
$R_6 : c, b, d, a$	0	p_2	p_3	p_1
$R'_6 : c, d, b, a$	p_3	0	p_1	p_2

Case 3: $p_3 < \frac{1}{3}$ and $p_2 \geq p_1$. Voter 8 can manipulate as follows. By strengthening d versus c, the majority graph changes from G to the graph in Figure 3(iii). For this graph, similar arguments as above imply that every majoritarian SDS yields the lottery $\frac{1}{3}a + \frac{1}{3}b + \frac{1}{3}d$.

	c	d	a	b
$R_8 : c, d, a, b$	0	p_3	p_1	p_2
$R'_8 : d, c, a, b$	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

Case 4: $p_3 < \frac{1}{3}$ and $p_1 \geq p_2$. Voter 10 can manipulate as follows. By strengthening d versus c, the majority graph changes from G to the graph in Figure 3(iii).

	c	d	b	a
$R_i : c, d, b, a$	0	p_3	p_2	p_1
$R'_i : d, c, b, a$	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

In each case, we have found a successful manipulation. Thus, f is SD-manipulable. □

No anonymous SD-efficient and SD-strategyproof SDS—majoritarian or not—is known. Clearly, it would be very desirable to strengthen Theorem 1 by extending the statement to the class of anonymous SDSs.

4. RANDOM SERIAL DICTATORSHIP

In this section, we examine random serial dictatorship (RSD)—an extension of random dictatorship to the case where voters may express indifference among alternatives. RSD is commonly used in house allocation, matching, and coalition formation domains where ties are ubiquitous [1, 7]. In these contexts, RSD is sometimes also referred to as the *random priority* mechanism.

In order to formally define RSD, we first introduce *serial dictatorships*, where dictators are invoked according to some fixed order and ties between most-preferred alternatives are broken by subsequent dictators. For a preference profile R and a permutation π of N, let

$$\sigma(R, \pi) = \Delta(\max_{R_{\pi(n)}}(\max_{R_{\pi(n-1)}}(\ldots(\max_{R_{\pi(1)}}(A))\ldots))).$$

That is, $\sigma(R, \pi)$ randomizes over the set of alternatives left when voters in order of π throw away those alternatives from the working set which are not maximally preferred. The set $\sigma(R, \pi)$ is almost always a singleton: if $\sigma(R, \pi)$ contains multiple lotteries, then *every* voter needs to be indifferent between all alternatives in the support of any of these lotteries.

RSD is the convex combination of $n!$ lotteries, each one of which is obtained by selecting $\sigma(R,\pi)$ for a distinct permutation π. Let $\pi_1,\dots,\pi_{n!}$ be an enumeration of the permutations over N and define

$$RSD(R) = \left\{ \sum_{j=1}^{n!} \frac{1}{n!} p_j : p_j \in \sigma(R,\pi_j)\ \forall j \in \{1,\dots,n!\} \right\}.$$

Clearly, the set $RSD(R)$ can only contain more than one lottery if there is some j such that $\sigma(R,\pi_j)$ contains more than one lottery. An SDS is called an RSD scheme if it always selects a lottery from the set $RSD(R)$.

Definition 4. *An SDS f is an* RSD scheme *if $f(R) \in RSD(R)$ for all $R \in \mathcal{R}^n$.*

If Φ is a property such as efficiency or strategyproofness, we write "RSD satisfies Φ" if every RSD scheme satisfies Φ.

It is well known that the serial dictator rule is SD-strategyproof. Any convex combination of serial dictator rules is also SD-strategyproof.

Proposition 3. *RSD is SD-strategyproof.*

It is furthermore well-known that RSD is Pareto optimal. Since Pareto optimality implies ST-efficiency, we immediately have the following.

Proposition 4. *RSD is ST-efficient.*

However, RSD is not SD-efficient.

Proposition 5. *For $|A| \geq 4$, RSD is not SD-efficient.*

Proof. We show that no RSD scheme is SD-efficient. Consider the following preference profile.[4]

$$\begin{aligned} 1 &: \{a,c\}, b, d \\ 2 &: \{b,d\}, a, c \\ 3 &: a, d, \{b,c\} \\ 4 &: b, c, \{a,d\} \end{aligned}$$

The unique RSD lottery is $\frac{5}{12}a + \frac{5}{12}b + \frac{1}{12}c + \frac{1}{12}d$, which is SD-dominated by $\frac{1}{2}a + \frac{1}{2}b$. In fact, it is even the case that all voters *strictly* prefer the latter lottery according to SD. Therefore RSD is not SD-efficient even for four voters and four alternatives. □

The lack of SD-efficiency of RSD was first discovered by Bogomolnaia and Moulin [4] in the context of house allocation.[5] What the proposition above actually shows is that for *any* utility representation consistent with the ordinal preferences of the voters, there exists a lottery which gives *strictly more* expected utility to *each* voter than the (unique) RSD lottery. We conclude this section by noting that Proposition 5 is tight. The proof is omitted due to restricted space.

Proposition 6. *For $|A| \leq 3$, RSD is SD-efficient.*

[4]We use curly braces to denote indifference. For instance, the first voter is indifferent between a and c.

[5]If the example by Bogomolnaia and Moulin [4] is translated to the general social choice setting, there are 24 alternatives.

5. STRICT MAXIMAL LOTTERIES

Maximal lotteries were first considered by Kreweras [17] and independently rediscovered and studied in detail by Fishburn [12]. Interestingly, maximal lotteries or variants thereof have been rediscovered again by economists, mathematicians, political scientists, and computer scientists [18, 11, 13, 24].

In order to define maximal lotteries, we need some notation. For a preference profile $R \in \mathcal{R}^n$ and two alternatives $x, y \in A$, the *majority margin* $g_R(x,y)$ is defined as the difference between the number of voters who prefer x to y and the number of voters who prefer y to x, i.e.,

$$g_R(x,y) = |\{i \in N \mid x\ R_i\ y\}| - |\{i \in N \mid y\ R_i\ x\}|.$$

Thus, $g_R(y,x) = -g_R(x,y)$ for all $x,y \in A$. A *maximal element*, a.k.a. *(weak) Condorcet winner*, is an alternative $x \in A$ with $g_R(x,y) \geq 0$ for all alternatives $y \in A$. It is well known that maximal elements may fail to exist. This drawback can however be remedied by considering lotteries over alternatives. The function g_R can be extended to pairs of lotteries by computing *expected* majority margins. For $p, q \in \Delta(A)$, define

$$g_R(p,q) = \sum_{(x,y)\in A\times A} p(x)q(y)g_R(x,y).$$

The set of maximal lotteries is then defined as

$$ML(R) = \{p \in \Delta(A) \mid g_R(p,q) \geq 0 \text{ for all } q \in \Delta(A)\}.$$

Von Neumann's minimax theorem [29] implies that $ML(R)$ is non-empty for all $R \in \mathcal{R}^n$. In fact, g_R can be interpreted as the payoff matrix of a symmetric zero-sum game—the so-called *plurality game*—and maximal lotteries as the mixed maximin strategies (or Nash equilibria) of this game. Interestingly, $ML(R)$ is a singleton in most cases. In particular, this holds if all voters have strict preferences *and* the number of voters is odd [19, 21].

A particularly interesting subclass of $ML(R)$ is given by the set of *strict* maximal lotteries, which corresponds to the set of *quasistrict* Nash equilibria of the plurality game. It can be shown that, within symmetric zero-sum games, these equilibria precisely correspond to the maximal lotteries with maximal support [10, 6].

Definition 5. *Let $R \in \mathcal{R}^n$. The set of strict maximal lotteries of R is given by*

$$SML(R) = \{p \in ML(R) \mid \widehat{q} \subseteq \widehat{p} \text{ for all } q \in ML(R)\}.$$

Convexity of the set $ML(R)$ implies that $SML(R)$ is non-empty and that all elements of $SML(R)$ have the same support, which we denote by $\widehat{SML}(R)$. This support can moreover be characterized as follows.

Lemma 1. *Let $p \in SML(R)$ and $x \in A$. Then,*

$$\begin{aligned} x \in \widehat{SML}(R) &\quad\Leftrightarrow\quad g_R(x,p) = 0, \text{ and} \\ x \notin \widehat{SML}(R) &\quad\Leftrightarrow\quad g_R(x,p) < 0. \end{aligned}$$

Proof. The statements follow from Lemma 4.2 by Dutta and Laslier [10]. □

An SDS is called an SML scheme if it always selects a strict maximal lottery and if the selection only depends on the set of strict maximal lotteries.

Definition 6. *An SDS f is an* SML scheme *if $f(R) \in SML(R)$ for all $R \in \mathcal{R}^n$ and $f(R) = f(R')$ whenever $SML(R) = SML(R')$.*

If Φ is a property such as efficiency or strategyproofness, we write "SML satisfies Φ" if every SML scheme satisfies Φ.

5.1 Efficiency

In this subsection, we prove that every SML scheme is SD-efficient. This result contrasts with our earlier observation that RSD fails SD-efficiency.

With each preference relation R_i on A we can associate a function $\phi_i : A \times A \to \{-1, 0, 1\}$ such that for all $x, y \in A$,

$$\phi_i(x, y) = \begin{cases} 1 & \text{if } x\ P_i\ y, \\ -1 & \text{if } y\ P_i\ x, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 2. *Let $p, q \in \Delta(A)$ with $p\ R_i^{SD}\ q$. Then,*

$$\sum_{(x,y)\in A\times A} p(x)q(y)\phi_i(x, y) \geq 0.$$

Proof. The following equivalences are easily verified.

$$p\ R_i^{SD}\ q \Leftrightarrow \sum_{x\in A:xP_iy} p(x) - \sum_{x\in A:xP_iy} q(x) \geq 0 \text{ for all } y \in A$$

$$\Leftrightarrow \sum_{x\in A:yP_ix} p(x) - \sum_{x\in A:yP_ix} q(x) \leq 0 \text{ for all } y \in A$$

Now, for any $i \in N$,

$$\sum_{(x,y)\in A\times A} p(x)q(y)\phi_i(x, y) = \sum_{(x,y)\in A\times A} p(x)q(y)\phi_i(x, y) - 0$$

$$= \sum_{(x,y)\in A\times A} p(x)q(y)\phi_i(x, y) - \sum_{(x,y)\in A\times A} q(x)q(y)\phi_i(x, y)$$

$$= \sum_{y\in A} \left(\sum_{x:xP_iy} p(x)q(y)(+1) + \sum_{x:yP_ix} p(x)q(y)(-1) \right)$$

$$- \sum_{y\in A} \left(\sum_{x:xP_iy} q(x)q(y)(+1) + \sum_{x:yP_ix} q(x)q(y)(-1) \right)$$

$$= \sum_{y\in A} q(y) \left(\sum_{x:xP_iy} p(x) - \sum_{x:xP_iy} q(x) \right)$$

$$- \sum_{y\in A} q(y) \left(\sum_{x:yP_ix} p(x) - \sum_{x:yP_ix} q(x) \right)$$

$$\geq 0.$$

The last step is easily seen in view of the equivalences stated at the beginning of the proof. ☐

Theorem 2. *SML is SD-efficient.*

Proof. Let $R \in \mathcal{R}^n$ and $q \in SML(R)$. Assume for contradiction that there exists a lottery p that SD-dominates q, i.e., $p\ R_i^{SD}\ q$ for all $i \in N$ and $p\ P_i^{SD}\ q$ for some $i \in N$. Lemma 2 yields

$$\sum_{(x,y)\in A\times A} p(x)q(y)\phi_i(x, y) \geq 0 \text{ for all } i \in N, \text{ and}$$

$$\sum_{(x,y)\in A\times A} p(x)q(y)\phi_i(x, y) > 0 \text{ for some } i \in N.$$

Summing up, we get

$$\sum_{i\in N} \sum_{(x,y)\in A\times A} p(x)q(y)\phi_i(x, y) > 0.$$

Since the sum on the left exactly equals $g_R(p, q)$, the latter inequality contradicts the assumption that q is a maximal lottery. ☐

5.2 Strategyproofness

Whereas SML fares well on the efficiency front, it does not do as well from the point of view of strategyproofness.

Proposition 7. *SML is not SD-strategyproof.*

Proof. We show that every SML scheme is SD-manipulable. Let $A = \{a, b, c\}$ and consider the following preference profile R.

$$1 : a, c, b$$
$$2 : a, b, c$$
$$3, 4 : b, c, a$$
$$5 : c, a, b$$

With the help of Lemma 1, it can be verified that $SML(R) = \{\frac{1}{3}a + \frac{1}{3}b + \frac{1}{3}c\}$. However, if voter 1 misrepresents his preferences between b and c by voting $a\ P'_i\ b\ P'_i\ c$, the outcome for the new preference profile R' is $SML(R') = \{\frac{3}{5}a + \frac{1}{5}b + \frac{1}{5}c\}$. Thus, $f(R')\ P_1^{SD}\ f(R)$ for any SML scheme f. ☐

On the other hand, we show that SML is at least ST-strategyproof.[6] We need the following lemma, which states that weakening alternatives outside $\widehat{SML}(R)$ does not alter the set $SML(R)$. The proof of this lemma utilizes Lemma 1 and is omitted due to restricted space.

Lemma 3. *Let $R \in \mathcal{R}^n$ and $a \in A$ with $a \notin \widehat{SML}(R)$. Let furthermore $R' \in \mathcal{R}^n$ be such that $g_{R'}(b, a) > g_R(b, a)$ for some $b \in A \setminus \{a\}$ and $g_{R'}(x, y) = g_R(x, y)$ for all $x, y \in A$ with $\{x, y\} \neq \{a, b\}$. Then, $SML(R') = SML(R)$.*

Theorem 3. *SML is ST-strategyproof.*

Proof. Assume for contradiction that there is an SML scheme f that is not ST-strategyproof. Then, there are two preference profiles R and R' such that $R_j = R'_j$ for all $j \neq i$ and $f(R')\ P_i^{ST}\ f(R)$.

For two alternatives $x, y \in A$ we say that x *is strengthened against* y if either (1) $y\ R_i\ x$ and $x\ P'_i\ y$, or (2) $y\ P_i\ x$ and $x\ R'_i\ y$. Define $\Delta(R, R') = \{(x, y) : x \text{ is strengthened against } y\}$. This set can be partitioned into the following four subsets.

$$\Delta_1 = \{(x, y) \in \Delta(R, R') : y \notin \widehat{p}\}$$
$$\Delta_2 = \{(x, y) \in \Delta(R, R') : x \notin \widehat{q}\} \setminus \Delta_1$$
$$\Delta_3 = \{(x, y) \in \Delta(R, R') : x \in \widehat{q},\ y \in \widehat{p}, \text{ and } \{x, y\} \not\subseteq \widehat{p} \cap \widehat{q}\}$$
$$\Delta_4 = \{(x, y) \in \Delta(R, R') : x \in \widehat{q},\ y \in \widehat{p}, \text{ and } \{x, y\} \subseteq \widehat{p} \cap \widehat{q}\}$$

We now construct two new preference profiles $\tilde{R}$ and $\tilde{R}'$ based on R and R'. The idea behind this construction is to

[6] The first part of this proof recycles an argument that was used by Brandt [5] to characterize set-valued social choice functions that are strategyproof with respect to deterministic dominance.

make R and R' agree on as many pairs as possible, while maintaining the invariant that the outcomes are p and q, respectively.

$\tilde{R}$ is identical to R except that for all pairs $(x,y) \in \Delta_1$, we strengthen x against y in the preferences of voter i such that $\tilde{R}_i$ agrees with R'_i on all such pairs.[7] Lemma 3 implies that $f(\tilde{R}) = f(R) = p$. Analogously, $\tilde{R}'$ is identical to R' except that for all pairs $(x,y) \in \Delta_2$, we strengthen y against x in the preferences of voter i such that $\tilde{R}'_i$ agrees with R_i on all such pairs. Lemma 3 implies that $f(\tilde{R}') = f(R') = q$.

By definition, $\tilde{R}$ and $\tilde{R}'$ differ only on pairs that are contained in Δ_3 or Δ_4. Observe, however, that $\Delta_3 = \emptyset$. To see this, assume for contradiction that there is a pair $(x,y) \in \Delta(R,R')$ with $x \in \widehat{q}$, $y \in \widehat{p}$, and $\{x,y\} \not\subseteq \widehat{p} \cap \widehat{q}$. There are three cases: (1) $x \in \widehat{q} \setminus \widehat{p}$ and $y \in \widehat{p} \setminus \widehat{q}$, (2) $x \in \widehat{q} \setminus \widehat{p}$ and $y \in \widehat{p} \cap \widehat{q}$, and (3) $x \in \widehat{p} \cap \widehat{q}$ and $y \in \widehat{p} \setminus \widehat{q}$. In each case, $q\ P_i^{ST}\ p$ implies $x\ P_i\ y$ (see Proposition 2). Since $(x,y) \in \Delta(R,R')$ implies $y\ R_i\ x$, we have a contradiction.

We thus have that $\Delta_3 = \emptyset$, and, consequently, that $\tilde{R}$ and $\tilde{R}'$ only differ on pairs of alternatives that are contained in Δ_4. In particular, $g_{\tilde{R}}$ and $g_{\tilde{R}'}$ agree on all pairs of alternatives that do not lie in $\widehat{p} \cap \widehat{q}$, i.e.,

$$g_{\tilde{R}}(a,b) = g_{\tilde{R}'}(a,b) \text{ for all } a,b \text{ with } \{a,b\} \not\subseteq \widehat{p} \cap \widehat{q}.$$

For such pairs, we omit the subscript and write $g(a,b)$ instead of $g_{\tilde{R}}(a,b)$. Likewise, we write $g(a,p)$ for $g_{\tilde{R}}(a,p)$ whenever $a \notin \widehat{p} \cap \widehat{q}$ and $p \in \Delta(A)$.

Let $x \in A$ and define a function $s : A \to [0,1]$ via

$$s(x) = \begin{cases} p(x) & \text{if } x \in \widehat{p} \\ q(x) & \text{if } x \in \widehat{q} \\ 0 & \text{otherwise.} \end{cases}$$

Note that s is well-defined because $p(z) = q(z)$ for all $z \in \widehat{p} \cap \widehat{q}$, and that s does *not* correspond to a lottery because the individual probabilities do not add up to one.

For $a \notin \widehat{p} \cap \widehat{q}$ and a subset $B \subseteq A$ of alternatives, let furthermore $s(a,B) = \sum_{b \in B} s(b)g(a,b)$. If $B = \widehat{q}$, we have $s(a,\widehat{q}) = \sum_{b \in \widehat{q}} s(b)g(a,b) = \sum_{b \in \widehat{q}} q(b)g(a,b) = \sum_{b \in A} q(b)g(a,b) = g(a,q)$. Analogously, $s(a,\widehat{p})$ equals $g(a,p)$.

Lemma 1 implies that $g(x,p) = 0$ and $g(x,q) < 0$ for all $x \in \widehat{p} \setminus \widehat{q}$, as well as $g(y,p) < 0$ and $g(y,q) = 0$ for all $y \in \widehat{q} \setminus \widehat{p}$. Therefore, we get

$$s(x,\widehat{q}) = g(x,q) < 0 = g(x,p) = s(x,\widehat{p}) \text{ for all } x \in \widehat{p} \setminus \widehat{q}, \text{ and}$$
$$s(y,\widehat{p}) = g(y,p) < 0 = g(y,q) = s(y,\widehat{q}) \text{ for all } y \in \widehat{q} \setminus \widehat{p}.$$

The inequality $s(x,\widehat{q}) < s(x,\widehat{p})$ remains valid if $s(x,\widehat{p} \cap \widehat{q})$ is subtracted from both sides. Since $s(x,\widehat{q}) - s(x,\widehat{p} \cap \widehat{q}) = s(\widehat{q} \setminus \widehat{p})$, we obtain

$$s(x,\widehat{q} \setminus \widehat{p}) < s(x,\widehat{p} \setminus \widehat{q}) \text{ for all } x \in \widehat{p} \setminus \widehat{q}, \text{ and}$$
$$s(y,\widehat{p} \setminus \widehat{q}) < s(y,\widehat{q} \setminus \widehat{p}) \text{ for all } y \in \widehat{q} \setminus \widehat{p}.$$

Multiplying both sides of these inequalities with a positive number, and writing $s'(a,B)$ for $s(a) \cdot s(a,B)$ results in

$$s'(x,\widehat{q} \setminus \widehat{p}) < s'(x,\widehat{p} \setminus \widehat{q}) \text{ for all } x \in \widehat{p} \setminus \widehat{q}, \text{ and}$$
$$s'(y,\widehat{p} \setminus \widehat{q}) < s'(y,\widehat{q} \setminus \widehat{p}) \text{ for all } y \in \widehat{q} \setminus \widehat{p}.$$

[7]Note that $\tilde{R}_i$ might not be transitive. Therefore, we do not assume transitivity of preferences in this proof. In fact, the statement of Theorem 3 becomes stronger but is easier to prove for general—possibly intransitive—preferences.

We finally summarize over $\widehat{q} \setminus \widehat{p}$ and $\widehat{q} \setminus \widehat{p}$, respectively, and get

$$\sum_{x \in \widehat{p} \setminus \widehat{q}} s'(x,\widehat{q} \setminus \widehat{p}) < \sum_{x \in \widehat{p} \setminus \widehat{q}} s'(x,\widehat{p} \setminus \widehat{q}), \text{ and} \quad (1)$$

$$\sum_{y \in \widehat{q} \setminus \widehat{p}} s'(y,\widehat{p} \setminus \widehat{q}) < \sum_{y \in \widehat{q} \setminus \widehat{p}} s'(y,\widehat{q} \setminus \widehat{p}). \quad (2)$$

In order to arrive at a contradiction, we state two straightforward identities that are based on the skew-symmetry of g.

$$\sum_{a \in B} s'(a,B) = 0 \text{ for all } B \subseteq A \setminus (\widehat{p} \cap \widehat{q}), \text{ and} \quad (3)$$

$$\sum_{b \in B} s'(b,C) + \sum_{c \in C} s'(c,B) = 0 \text{ for all } B,C \subseteq A \setminus (\widehat{p} \cap \widehat{q}). \quad (4)$$

Now (3) implies that the right hand side of both (1) and (2) is zero, and therefore

$$\sum_{x \in \widehat{p} \setminus \widehat{q}} s'(x,\widehat{q} \setminus \widehat{p}) < 0 \quad \text{and} \quad \sum_{y \in \widehat{q} \setminus \widehat{p}} s'(y,\widehat{p} \setminus \widehat{q}) < 0.$$

However, (4) implies that

$$\sum_{x \in \widehat{p} \setminus \widehat{q}} s'(x,\widehat{q} \setminus \widehat{p}) + \sum_{y \in \widehat{q} \setminus \widehat{p}} s'(y,\widehat{p} \setminus \widehat{q}) = 0,$$

a contradiction. □

ST-strategyproofness is rather weak, but it seems as if this is one of the highest degrees of strategyproofness one can hope for when also insisting on SD-efficiency. While ST-strategyproofness does allow manipulators to skew the resulting distribution, gross manipulative attacks such as distorting the outcome from one degenerate lottery to another—an attack that many common SDSs suffer from (see Section 3)—or from one support to another disjoint one are futile.

6. CONCLUSION

We pointed out an interesting tradeoff between efficiency and strategyproofness in randomized social choice, exemplified by two social decision schemes: random serial dictatorship and strict maximal lotteries. While the former satisfies the strong notion of SD-strategyproofness, it only satisfies ST-efficiency. For strict maximal lotteries, this is exactly the other way round (see Figure 1).

An important open question is whether SD-efficiency, SD-strategyproofness, and anonymity are incompatible in general. We only proved this statement for the rather limited class of majoritarian social decision schemes. Another interesting question concerns other schemes that satisfy SD-strategyproofness and ST-efficiency. Most of the schemes captured by Gibbard's characterization of strongly SD-strategyproof schemes [16, 2] also satisfy ST-efficiency. However, in contrast to RSD, they fail to satisfy Pareto optimality. The complementary question—schemes that satisfy SD-efficiency and ST-strategyproofness—is perhaps even more interesting. It can be shown that both properties are satisfied by all strict maximal lottery schemes, even when applying odd monotonic mappings to the majority margin as described by Fishburn [12]. On the other hand, we have counter-examples showing that other attractive schemes known to be strategyproof with respect to deterministic dominance [5] either fail SD-efficiency or ST-

strategyproofness. Perhaps, strict maximal lotteries can even be characterized using these properties.

There are also challenging computational problems associated with RSD. Of course, RSD can be implemented efficiently so as to return a single alternative, but we are not aware of a polynomial-time algorithm for computing the probabilities of an actual RSD lottery.[8] SMLs, on the other hand, can be easily computed in polynomial time via linear programming.

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8. REFERENCES

[1] A. Abdulkadiroğlu and T. Sönmez. Random serial dictatorship and the core from random endowments in house allocation problems. *Econometrica*, 66(3):689–702, 1998.

[2] S. Barberà. A note on group strategy-proof decision schemes. *Econometrica*, 47(3):637–640, 1979.

[3] S. Barberà. Majority and positional voting in a probabilistic framework. *Review of Economic Studies*, 46(2): 379–389, 1979.

[4] A. Bogomolnaia and H. Moulin. A new solution to the random assignment problem. *Journal of Economic Theory*, 100(2):295–328, 2001.

[5] F. Brandt. Group-strategyproof irresolute social choice functions. In T. Walsh, editor, *Proc. of 22nd IJCAI*, pages 79–84. AAAI Press, 2011.

[6] F. Brandt and F. Fischer. On the hardness and existence of quasi-strict equilibria. In B. Monien and U.-P. Schroeder, editors, *Proc. of 1st International Symposium on Algorithmic Game Theory (SAGT)*, volume 4997 of *LNCS*, pages 291–302. Springer, 2008.

[7] Y.-K. Che and F. Kojima. Asymptotic equivalence of probabilistic serial and random priority mechanisms. *Econometrica*, 78(5):1625—1672, 2010.

[8] W. J. Cho. Probabilistic assignment: A two-fold axiomatic approach. Unpublished manuscript, 2012.

[9] V. Conitzer and T. Sandholm. Nonexistence of voting rules that are usually hard to manipulate. In *Proc. of 21st AAAI Conference*, pages 627–634. AAAI Press, 2006.

[10] B. Dutta and J.-F. Laslier. Comparison functions and choice correspondences. *Social Choice and Welfare*, 16 (4):513–532, 1999.

[11] D. S. Felsenthal and M. Machover. After two centuries should Condorcet's voting procedure be implemented? *Behavioral Science*, 37(4):250–274, 1992.

[12] P. C. Fishburn. Probabilistic social choice based on simple voting comparisons. *Review of Economic Studies*, 51(167):683–692, 1984.

[8] We do have a non-trivial algorithm for finding alternatives that are selected with positive probability in some RSD lottery.

[13] D. C. Fisher and J. Ryan. Tournament games and positive tournaments. *Journal of Graph Theory*, 19(2):217–236, 1995.

[14] P. Gärdenfors. On definitions of manipulation of social choice functions. In J. J. Laffont, editor, *Aggregation and Revelation of Preferences*. North-Holland, 1979.

[15] A. Gibbard. Manipulation of voting schemes. *Econometrica*, 41:587–602, 1973.

[16] A. Gibbard. Manipulation of schemes that mix voting with chance. *Econometrica*, 45(3):665–681, 1977.

[17] G. Kreweras. Aggregation of preference orderings. In S. Sternberg, V. Capecchi, T. Kloek, and C. Leenders, editors, *Mathematics and Social Sciences I: Proceedings of the seminars of Menthon-Saint-Bernard, France (1–27 July 1960) and of Gösing, Austria (3–27 July 1962)*, pages 73–79, 1965.

[18] G. Laffond, J.-F. Laslier, and M. Le Breton. The bipartisan set of a tournament game. *Games and Economic Behavior*, 5:182–201, 1993.

[19] G. Laffond, J.-F. Laslier, and M. Le Breton. A theorem on symmetric two-player zero-sum games. *Journal of Economic Theory*, 72(2):426–431, 1997.

[20] J.-F. Laslier. Aggregation of preferences with a variable set of alternatives. *Social Choice and Welfare*, 17:269–282, 2000.

[21] M. Le Breton. On the uniqueness of equilibrium in symmetric two-player zero-sum games with integer payoffs. *Économie publique*, 17(2):187–195, 2005.

[22] A. Postlewaite and D. Schmeidler. Strategic behaviour and a notion of ex ante efficiency in a voting model. *Social Choice and Welfare*, 3:37–49, 1986.

[23] A. Procaccia. Can approximation circumvent Gibbard-Satterthwaite? In *Proc. of 24th AAAI Conference*, pages 836–841. AAAI Press, 2010.

[24] R. L. Rivest and E. Shen. An optimal single-winner preferential voting system based on game theory. In *Proc. of 3rd International Workshop on Computational Social Choice*, pages 399–410, 2010.

[25] M. A. Satterthwaite. Strategy-proofness and Arrow's conditions: Existence and correspondence theorems for voting procedures and social welfare functions. *Journal of Economic Theory*, 10:187–217, 1975.

[26] T. Sönmez and M. U. Ünver. Matching, allocation, and exchange of discrete resources. In J. Benhabib, M. O. Jackson, and A. Bisin, editors, *Handbook of Social Economics*, volume 1, chapter 17, pages 781–852. Elsevier, 2011.

[27] P. Stone. *The Luck of the Draw: The Role of Lotteries in Decision Making*. Oxford University Press, 2011.

[28] A. D. Taylor. *Social Choice and the Mathematics of Manipulation*. Cambridge University Press, 2005.

[29] J. von Neumann. Zur Theorie der Gesellschaftspiele. *Mathematische Annalen*, 100:295–320, 1928.

[30] J. von Neumann and O. Morgenstern. *Theory of Games and Economic Behavior*. Princeton University Press, 2nd edition, 1947.

[31] T. Walsh and L. Xia. Lot-based voting rules. In V. Conitzer and M. Winikoff, editors, *Proc. of 11th AAMAS Conference*, pages 603–610, 2012.