Proving the Incompatibility of Efficiency and Strategyproofness via SMT Solving

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Two important requirements when aggregating the preferences of multiple agents are that the outcome should be economically efficient and the aggregation mechanism should not be manipulable. In this paper, we provide a computer-aided proof of a sweeping impossibility using these two conditions for randomized aggregation mechanisms. More precisely, we show that every efficient aggregation mechanism can be manipulated for *all* expected utility representations of the agents' preferences. This settles an open problem and strengthens a number of existing theorems, including statements that were shown within the special domain of assignment. Our proof is obtained by formulating the claim as a satisfiability problem over predicates from real-valued arithmetic, which is then checked using an SMT (satisfiability modulo theories) solver. In order to verify the correctness of the result, a minimal unsatisfiable set of constraints returned by the SMT solver was then translated back into a proof in higher-order logic, which was automatically verified by an interactive theorem prover. To the best of our knowledge, this is the first application of SMT solvers in computational social choice.

1. Introduction

Models and results from microeconomic theory, in particular from game theory and social choice, have proven to be very valuable when reasoning about computational multiagent systems (see, e.g., Nisan et al., 2007; Shoham and Leyton-Brown, 2009; Rothe, 2015; Brandt et al., 2016a). Two fundamental notions in this context are efficiency—no agent can be made better off without making another one worse off—and strategyproofness—no agent can obtain a more preferred outcome by manipulating his preferences. Gibbard (1973) and Satterthwaite (1975) have shown that every strategyproof social choice function is either dictatorial or imposing. Hence, strategyproofness can only be achieved at the cost of discriminating among the agents or among the alternatives. One natural possibility to restore fairness, which is particularly popular in computer science, is to allow for randomization. Functions that map a profile of individual preferences to a probability distribution over alternatives (a so-called *lottery*) are known as *social decision schemes (SDSs)*.

Generalizing his previous result, Gibbard (1977) proved that the only strategyproof and *ex post* efficient social decision schemes are randomizations over dictatorships. Gibbard's notion of strategyproofness requires that no agent is better off by manipulating his preferences for *some* expected utility representation of the agents' ordinal preferences. This condition is quite demanding because an SDS may be deemed manipulable just because it can be manipulated for a contrived and highly unlikely utility representation. In this paper, we adopt a weaker notion of strategyproofness, first used by Postlewaite and Schmeidler (1986) and popularized by Bogomolnaia and Moulin (2001). This notion requires that no agent should be better off by manipulating his preferences for *all* expected utility representations of the agents' preferences. At the same time, we use a stronger notion of efficiency than Gibbard (1977). This notion is defined in analogy to our notion of strategyproofness and requires that no agent can be made better off for *all* utility representations of the agents' preferences, without making another one worse off for *some* utility representation. This type of efficiency was introduced by Bogomolnaia and Moulin (2001) and is also known as ordinal efficiency or *SD*-efficiency where *SD* stands for stochastic dominance.

Our main result establishes that no anonymous and neutral SDS satisfies efficiency and strategyproofness. This settles a conjecture by Aziz et al. (2013b) and generalizes theorems by Aziz et al. (2013b), Aziz et al. (2014), and Brandl et al. (2016b). It also strengthens related statements by Zhou (1990), Bogomolnaia and Moulin (2001), Katta and Sethuraman (2006), and Nesterov (2016), which were shown within the special domain of assignment.

Our proof of this theorem heavily relies on computer-aided solving techniques. Some of these have already been applied in computational social choice, where, due to the rigorous axiomatic foundation, computer-aided theorem proving appears to be a particularly promising line of research. Perhaps the best known result in this context stems from Tang and Lin (2009), who reduce well-known impossibility results, such as Arrow's theorem, to finite instances, which can then be checked by a Boolean satisfiability (SAT) solver. Their work has sparked a number of papers which, besides using this general idea for more complex settings or axioms, focus on proving *novel* results (Geist and Endriss, 2011; Brandl et al., 2015; Brandt et al., 2016b; Brandt and Geist, 2016).

In this paper, we go beyond the SAT-based techniques of previous contributions by designing an SMT (satisfiability modulo theories) encoding that captures axioms for *randomized* social choice. SMT can be viewed as an enriched form of the satisfiability problem (SAT) where Boolean variables are replaced by statements from a *theory*, such as specific data types or arithmetics. Similar to SAT, there is a range of SMT solvers developed by an active community that runs annual competitions (Barrett et al., 2013). Typically, SMT solvers are used as backends for verification tasks such as the verification of software. To capture axioms about lotteries, we use the theory of (quantifier-free) linear real arithmetic. Solving this version of SMT can be seen as an extension to *linear programming* in which arbitrary Boolean operators are allowed to connect (in-)equalities.

Following the idea of Brandt and Geist (2016), we extracted a *minimal unsatisfiable set (MUS)* of constraints in order to verify our result. Despite its relatively complex 94 (non-trivial) constraints, which operate on 47 canonical preference profiles, the MUS has been translated back into a proof in higher-order logic, which has been verified via the interactive theorem prover Isabelle/HOL, and, hence, releases any need to verify our program for generating it. We transformed this proof into a human-readable—but tedious to check—proof, which is given in the Appendix.

2. The Model

Let A be a finite set of m alternatives and $N = \{1, \dots, n\}$ a set of agents. A *(weak) preference relation* is a complete and transitive binary relation on A. The preference relation reported by agent i is denoted by $\succsim_i$, and the set of all preference relations by $\mathcal{R}$. In accordance with conventional notation, we write $\succ_i$ for the strict part of $\succsim_i$, i.e., $x \succ_i y$ if $x \succsim_i y$ but not $y \succsim_i x$, and $\sim_i$ for the indifference part of $\succsim_i$, i.e., $x \sim_i y$ if $x \succsim_i y$ and $y \succsim_i x$. A preference relation $\succsim_i$ is *linear* if $x \succ_i y$ or $y \succ_i x$ for all distinct alternatives $x, y \in A$. We will compactly represent a preference relation as a comma-separated list with all alternatives among which an agent is indifferent placed in a set. For example, $x \succ_i y \sim_i z$ is represented by $\succsim_i\colon x, \{y, z\}$. A *preference profile* $R = (\succsim_1, \dots, \succsim_n)$ is an n-tuple containing a preference relation $\succsim_i$ for each agent $i \in N$. The set of all preference profiles is thus given by $\mathcal{R}^N$. For a given $R \in \mathcal{R}^N$ and $\succsim \in \mathcal{R}$, $R^{i\mapsto\succsim}$ denotes a preference profile identical to R except that $\succsim_i$ is replaced with $\succsim$, i.e., $R^{i\mapsto\succsim} = R \setminus \{(i, \succsim_i)\} \cup \{(i, \succsim)\}$.

2.1. Social Decision Schemes

Our central objects of study are social decision schemes: functions that map a preference profile to a *lottery (or probability distribution)* over the alternatives. The set of all lotteries over A is denoted by $\Delta(A)$, i.e., $\Delta(A) = \{p \in \mathbb{R}^A_{\geq 0}\colon \sum_{x\in A} p(x) = 1\}$, where $p(x)$ is the probability that p assigns to x. Then, formally, a *social decision scheme (SDS)* is a function $f\colon \mathcal{R}^N \to \Delta(A)$. By $\operatorname{supp}(p)$ we denote the *support* of a lottery $p \in \Delta(A)$, i.e., the set of all alternatives to which p assigns positive probability. Two common minimal fairness conditions for SDSs are anonymity and neutrality, i.e., symmetry with respect to agents and alternatives, respectively. Formally, *anonymity* requires that $f(R) = f(R \circ \sigma)$ for all $R \in \mathcal{R}^N$ and permutations $\sigma\colon N \to N$ over agents. *Neutrality*, on the other hand, is defined via permutations over alternatives. An SDS f is *neutral* if $f(R)(x) = f(\pi(R))(\pi(x))$ for all $R \in \mathcal{R}^N$, permutations $\pi\colon A \to A$, and $x \in A$.[1]

[1] $\pi(R)$ is the preference profile obtained from π by replacing $\succsim_i$ with $\succsim_i^\pi$ for every $i \in N$, where $\pi(x) \succsim_i^\pi \pi(y)$ if and only if $x \succsim_i y$.

2.2. Efficiency and Strategyproofness

Many important properties of SDSs, such as efficiency and strategyproofness, require us to reason about the preferences that agents have over lotteries. This is commonly achieved by assuming that in a preference profile R every agent i, in addition to this preference relation $\succsim_i$, is equipped with a von Neumann-Morgenstern (vNM) *utility function* $u_i^R \colon A \to \mathbb{R}$. By definition, a utility function u_i^R has to be consistent with the ordinal preferences, i.e., for all $x, y \in A$, $u_i^R(x) \geq u_i^R(y)$ iff $x \succsim_i y$. A *utility representation* u then associates with each preference profile R an n-tuple $(u_1^R, \ldots, u_n^R)$ of such utility functions. Whenever the preference profile R is clear from the context, the superscript will be omitted and we write u_i instead of the more cumbersome u_i^R.

Given a utility function u_i, agent i prefers lottery p to lottery q iff the expected utility for p is at least as high as that of q. With slight abuse of notation the domain of utility functions can be extended to a linear function on $\Delta(A)$ by letting

$$u_i(p) = \sum_{x \in A} p(x) u_i(x).$$

It is straightforward to define efficiency and strategyproofness using expected utility. For a given utility representation u and a preference profile R, a lottery p *u-(Pareto-)dominates* another lottery q if

$$\begin{aligned} u_i(p) &\geq u_i(q) \text{ for all } i \in N \text{, and} \\ u_i(p) &> u_i(q) \text{ for some } i \in N. \end{aligned}$$

An SDS f is *u-efficient* if it never returns u-dominated lotteries, i.e., for all $R \in \mathcal{R}^N$, $f(R)$ is not u-dominated. The notion of u-strategyproofness can be defined analogously: for a given utility representation u, an SDS can be *u-manipulated* if there are $R \in \mathcal{R}^N$, $i \in N$, and $\succsim \in \mathcal{R}$ such that

$$u_i^R(f(R^{i \mapsto \succsim})) > u_i^R(f(R)).$$

An SDS is *u-strategyproof* if it cannot be u-manipulated.

The assumption that the vNM utility functions of all agents (and thus their complete preferences *over lotteries*) are known is quite unrealistic. Often even the agents themselves are uncertain about their preferences over lotteries and only know their ordinal preferences over alternatives.[2] A natural way to model this uncertainty is to leave the utility functions unspecified and instead *quantify over all utility functions* that are consistent with the agents' ordinal preferences. This model leads to much weaker notions of efficiency and strategyproofness.

[2] When assuming that all agents possess vNM utility functions, these utility functions could be taken as inputs for the aggregation function. Such aggregation functions are called *cardinal decision schemes* (see, e.g., Dutta et al., 2007). In addition to the fact that concrete vNM utility functions are typically unavailable, their representation may require infinite space.

Definition 1. An SDS is *efficient* if it never returns a lottery that is u-dominated for all utility representations u.

As mentioned in the introduction, this notion of efficiency is also known as *ordinal efficiency* or *SD-efficiency* (see, e.g., Bogomolnaia and Moulin, 2001; Aziz et al., 2014, 2015). The relationship to stochastic dominance will be discussed in more detail in Section 4.2.

Example 1. For illustration consider $A = \{a, b, c, d\}$ and the preference profile $R = (\succsim_1, \dots, \succsim_4)$,

$$\begin{aligned} &\succsim_1 \colon \{a, c\}, \{b, d\}, &\qquad &\succsim_2 \colon \{b, d\}, \{a, c\}, \\ &\succsim_3 \colon \{a, d\}, b, c, & &\succsim_4 \colon \{b, c\}, a, d \end{aligned}$$

Observe that the lottery $\nicefrac{7}{24}\, a + \nicefrac{7}{24}\, b + \nicefrac{5}{24}\, c + \nicefrac{5}{24}\, d$, which is returned by the well-known SDS *random serial dictatorship (RSD)*, is u-dominated by $\nicefrac{1}{2}\, a + \nicefrac{1}{2}\, b$ for every utility representation u. Hence, any SDS that returns this lottery for the profile R would not be efficient. On the other hand, the lottery $\nicefrac{1}{2}\, a + \nicefrac{1}{2}\, b$ is not u-dominated, which can, for instance, be checked via linear programming (see Lemma 4).

We can also define a weak notion of strategyproofness in analogy to our notion of efficiency.

Definition 2. An SDS is *strategyproof* if there is no utility representation u for which it can be u-manipulated.

Alternatively, there is a stronger version of strategyproofness by Gibbard (1977), in which an SDS should not be u-manipulable for *some* utility representation u.

For more information concerning the relationship between sets of possible utility functions and preference extensions, such as stochastic dominance, the reader is referred to Aziz et al. (2015).

3. The Result

Our main result shows that efficiency and strategyproofness are incompatible with basic fairness properties. Aziz et al. (2013b) raised the question whether there exists an anonymous, efficient, and strategyproof SDS. When additionally requiring neutrality, we can answer this question in the negative.

Theorem 1. *If $m \geq 4$ and $n \geq 4$, there is no anonymous and neutral SDS that satisfies efficiency and strategyproofness.*

The proof of Theorem 1, which heavily relies on computer-aided solving techniques, is discussed in Section 4. Let us first discuss the independence of the axioms and relate the result to existing theorems. *RSD* satisfies all axioms *except efficiency*; another SDS known as *maximal lotteries* satisfies all axioms *except strategyproofness* (cf. Aziz et al., 2013b). Serial dictatorship, the deterministic version of *RSD*, satisfies neutrality, efficiency, and strategyproofness *but violates anonymity*. It is unknown whether Theorem 1 still holds when dropping the assumption of neutrality. Our proof, however, only requires a technical weakening of neutrality (cf. Section 4.1).

3.1. Related Results for Social Choice

Our result generalizes several existing results and is closely related to a number of results in subdomains of social choice. Aziz et al. (2013b) proved a weak version of Theorem 1 for the rather restricted class of majoritarian SDSs, i.e., SDSs whose outcome may only depend on the pairwise majority relation. This statement has later been generalized by Aziz et al. (2014) to all SDSs whose outcome only depends on the *weighted* majority relation. More recently, Brandl et al. (2016b) have shown that while random dictatorship (is anonymous and neutral and) satisfies efficiency and strategyproofness on the domain of linear preferences, it cannot be extended to the full domain of weak preferences without violating at least one of these properties. Their theorem is a direct consequence of Theorem 1. Other impossibility results have been obtained for stronger notions of efficiency and strategyproofness, which weakens the corresponding statements. Aziz et al. (2014) have shown that there is no anonymous and neutral SDS that satisfies efficiency and strategyproofness with respect to the *pairwise comparison* lottery extension and with respect to the *upward lexicographic* extension.[3] Both of these notions of efficiency and strategyproofness are stronger than the ones used in Theorem 1.

3.2. Related Results for Assignment

A subdomain of social choice that has been thoroughly studied in the literature is the assignment (aka house allocation or two-sided matching with one-sided preferences) domain. An assignment problem can be associated with a social choice problem by letting the set of alternatives be the set of deterministic allocations and postulating that agents are indifferent among all allocations in which they receive the same object (see, e.g., Aziz et al., 2013a).[4] Thus, impossibility results for the assignment setting can be interpreted as impossibility results for the social choice setting because they even hold in a smaller domain and satisfying the axioms in the social choice domain implies satisfying them in any smaller domain.

In the following we discuss impossibility results in the assignment domain which, if interpreted for the social choice domain, can be seen as weaker versions of Theorem 1 because

[3] The statement for the pairwise comparison extension holds for at least three agents and three alternatives, whereas Theorem 1 does not hold for less then four alternatives since *RSD* satisfies all properties for up to three alternatives. In contrast to Theorem 1, the statement for the upward lexicographic extension does not require neutrality and also holds for linear preferences.

[4] Note that this transformation turns assignment problems with linear preferences over k objects into social choice problems with weak preferences over $k!$ allocations.

they are based on stronger notions of efficiency or strategyproofness or require additional properties. In a very influential paper, Bogomolnaia and Moulin (2001) have shown that no randomized assignment mechanism satisfies both efficiency and a strong notion of strategyproofness while treating all agents equally. The underlying notion of strategyproofness is identical to the one used by Gibbard (1977) and prescribes that the SDS cannot be u-manipulated for *any* utility representation u. The result by Bogomolnaia and Moulin even holds when preferences over objects are single-peaked (Kasajima, 2013). (Nevertheless, when transferred to the social choice domain, the preferences over allocations will not be single-peaked anymore.) In a related paper, Katta and Sethuraman (2006) proved that no assignment mechanism satisfies efficiency, strategyproofness, and envy-freeness for the full domain of preferences.[5] Nesterov (2016) proved three related impossibility theorems for varying notions of envy-freeness.

Settling a conjecture by Gale (1987), Zhou (1990) showed that no cardinal assignment mechanism satisfies u-efficiency and u-strategyproofness while treating all agents equally.[6] The relationship between Zhou's result and Theorem 1 is not obvious because Zhou's theorem concerns cardinal mechanisms, i.e., functions that take a utility profile rather than a preference profile as input. However, every cardinal assignment mechanism can be associated with an ordinal assignment mechanism by choosing the outcome for some consistent utility profile for every preference profile. This transformation turns a u-efficient and u-strategyproof cardinal mechanism into an efficient and strategyproof ordinal mechanism as these properties are purely ordinal. Hence, Theorem 1 implies that there is no anonymous, neutral, u-efficient, and u-strategyproof cardinal decision scheme.

4. Proving the Result

In this section, we first reduce the statement of Theorem 1 to the case of $m = 4$ and $n = 4$, which we then prove via SMT solving. We present an encoding for any finite instance of Theorem 1 as an SMT problem in the logic of (quantifier-free) linear real arithmetic (`QF_LRA`). For compatibility with different SMT solvers our encoding adheres to the SMT-LIB standard (Barrett et al., 2010). In total, we are going to design the following four types of SMT constraints:

- lottery definitions (Lottery),
- the orbit condition[7] (Orbit),
- strategyproofness (Strategyproofness), and

[5] Envy-freeness is a fairness property that is stronger than *equal treatment of equals* as used by Bogomolnaia and Moulin (2001).

[6] The theorem by Zhou only requires that agents with the same utility function receive the same amount of expected utility but not necessarily the same assignment. Gale's original conjecture assumed equal treatment of equals.

[7] The orbit condition models a part of neutrality.

- efficiency (Efficiency).

Other conditions such as anonymity are taken care of by the representation of preference profiles.

We then, first, apply an SMT solver to show that this set of constraints for the case of $m = 4$ and $n = 4$ is unsatisfiable, i.e., no SDS f with the desired properties exists. Second, we explain how the output of the solver can be used to obtain a human-verifiable proof of this result.

But let us start with the reduction lemma before we turn to the concrete encoding in the following subsections.

Lemma 1. *If there is an anonymous and neutral SDS f that satisfies efficiency and strategyproofness for $|A| = m$ alternatives and $|N| = n$ agents then, for all $m' \leq m$ and $n' \leq n$, we can also find an SDS f' defined for m' alternatives and n' agents that satisfies the same properties.*

Proof. Let f be an anonymous and neutral SDS that satisfies efficiency and strategyproofness for m alternatives and n agents. We define a projection f' of f onto $A' \subseteq A, |A'| = m' \leq m$ and $N' = \{1, 2, \ldots, n'\} \subseteq N, n' \leq n$ that satisfies all required properties:

For every preference profile R' on A' and N', let $f'(R') = f(R)$, where R is defined by the following conditions:

$$\succsim_i \cap (A' \times A') = \succsim_i' \text{ for all } i \in N', \tag{1}$$

$$x \succ_i y \text{ for all } x \in A', y \in A \setminus A' \text{ and } i \in N', \tag{2}$$

$$y \sim_i z \text{ for all } y, z \in A \setminus A' \text{ and } i \in N', \text{ and} \tag{3}$$

$$y \sim_i z \text{ for all } y, z \in A \text{ and } i \in N \setminus N'. \tag{4}$$

Informally, by (1) agents in N' have the same preferences over alternatives from A' in R and R'. Moreover, by (2) they like every alternative in A' strictly better than every alternative not in A' and by (3) they are indifferent between all alternatives not in A'. Finally, by (4) all agents in $N \setminus N'$ are completely indifferent. With these conditions, R is uniquely specified given R', and only lotteries p with $\mathrm{supp}(p) \subseteq A'$ are efficient in R. Thus, f' is well-defined and it is left to show that f' inherits the relevant properties from f. The SDS f' is anonymous since f is anonymous and agents in N can only differ by their preferences over A'. Neutrality follows as f is neutral and all agents are indifferent between all alternatives not in A'. Efficiency is satisfied by f' since f is efficient and the same set of lotteries is efficient in R and R'. Finally, f' is strategyproof because f is strategyproof and the outcomes of f' under the two profiles R' and $(R')^{i \mapsto \succsim'}$ are equal to the outcomes of f under the two (extended) profiles R and $R^{i \mapsto \succsim}$, respectively. □

4.1. Framework, Anonymity, and Neutrality

For a given number of agents n and set of alternatives A, we encode an arbitrary SDS $f \colon \mathcal{R}^N \to \Delta(A)$ by a set of real-valued variables $p_{R,x}$ with $R \in \mathcal{R}^N$ and $x \in A$. Each

$p_{R,x}$ then represents the probability with which alternative x is selected for profile R, i.e., $p_{R,x} = f(R)(x)$.

This encoding of lotteries leads to the first simple constraints for our SMT encoding, which ensure that for each preference profile R the corresponding variables $p_{R,x}$, $x \in A$ indeed encode a lottery:

$$\begin{aligned} \sum_{x \in A} p_{R,x} = 1 \text{ for all } R \in \mathcal{R}^N, \text{ and} \\ p_{R,x} \geq 0 \text{ for all } R \in \mathcal{R}^N \text{ and } x \in A. \end{aligned} \tag{Lottery}$$

We are now going to argue that, in conjunction with anonymity and neutrality (see Section 2), it suffices to consider these constraints for a subset of preference profiles. This is because, in contrast to the other axioms, we directly incorporate anonymity and neutrality into the structure of the encoding rather than formulating them as actual constraints. Similar to the construction involving canonical tournament representations by Brandt and Geist (2016), we model anonymity and neutrality by computing for each preference profile $R \in \mathcal{R}^N$ a *canonical representation* $R_\mathfrak{c} \in \mathcal{R}^N$ with respect to these properties. In this representation, two preference profiles R and R' are equal (i.e., $R_\mathfrak{c} = R'_\mathfrak{c}$) iff one can be transformed into the other by renaming the agents and alternatives. Equivalently, $R_\mathfrak{c} = R'_\mathfrak{c}$ iff, for every anonymous and neutral SDS f, the lotteries $f(R)$ and $f(R')$ are equal (modulo the renaming of the alternatives).

The SMT constraints and SMT variables are then instantiated only for these canonical representations $\mathcal{R}^N_\mathfrak{c} \subseteq \mathcal{R}^N$. Apart from enabling an encoding of anonymous and neutral SDSs without any explicit reference to permutations, this also offers a substantial performance gain compared to considering the full domain $\mathcal{R}^N$ of (non-anonymous and non-neutral) preference profiles: the number of preference profiles for $m = 4$ and $n = 4$ is 31,640,625 , whereas the number of *canonical* preference profiles is merely 60,865.

Technically, we compute the canonical representation $R_\mathfrak{c}$ as follows: Let $R = (\succsim_1, \dots, \succsim_n) \in \mathcal{R}^N$ be a preference profile. First, we identify R with a function $r : \mathcal{R} \to \mathbb{N}$, which we call *anonymous preference profile*, and which counts the number of agents with a certain preference relation, i.e., $r(\succsim) = |\{i \in N \mid \succsim_i = \succsim\}|$, thereby ignoring the identity of the agents. This representation fully captures anonymity.

To additionally enforce neutrality, we had to resort to a computationally demanding, naive solution: given r, we compute all anonymous preference profiles $\pi(r)$ that can be achieved via a permutation $\pi\colon A \to A$, and, among those profiles, choose the one $\pi_{\text{lexmin}}(r)$ with lexicographically minimal values (for some fixed ordering of preference relations). For the canonical representation $R_\mathfrak{c}$ we then pick any preference profile $R \in \mathcal{R}^N$ which agrees with $\pi_{\text{lexmin}}(r)$, for instance, by again using the same fixed ordering of preference relations. Fortunately, this approach is still feasible for the small numbers of alternatives with which we are dealing.

While this representation of preference profiles does not completely capture neutrality—the *orbit condition* (see Brandt and Geist, 2016) is missing—this weaker version suffices to prove the impossibility. In favor of simpler proofs we, however, include the simple constraints corresponding to a randomized version of the orbit condition.

In our context, an *orbit* O of a preference profile R is an equivalence class of alternatives. Two alternatives $x, y \in A$ are considered equivalent if $\pi(x) = y$ for some permutation $\pi\colon A \to A$ that maps the anonymous preference profile associated with R to itself (i.e., π is an automorphism of the anonymous preference profile). In such a situation, every anonymous and neutral SDS has to assign equal probabilities to x and y. We hence require that, for each orbit $O \in \mathcal{O}_R$ of a (canonical) profile R, the probabilities $p_{R,x}$ are equal for all alternatives $x \in O$. As an SMT constraint, this reads

$$p_{R,x} = p_{R,y} \tag{Orbit}$$

for all $R \in \mathcal{R}^N_{\mathfrak{c}}$, $O \in \mathcal{O}_R$, and $x, y \in O$.

Example 2. Consider the anonymous preference profile r based on R from Example 1 and the permutation $\pi = (ab)(cd)$. As $\pi(r) = r$ (and since no other non-trivial permutation has this property) the set of orbits of R is $\mathcal{O}_R = \{\{a, b\}, \{c, d\}\}$.

4.2. Stochastic Dominance

In order to avoid quantifying over utility functions, we leverage well-known representations of efficiency and strategyproofness via *stochastic dominance (SD)* (cf. Bogomolnaia and Moulin, 2001; McLennan, 2002; Aziz et al., 2015). A lottery p *stochastically dominates* a lottery q for an agent i (short: $p \succsim_i^{SD} q$) if for every alternative x, lottery p is at least as likely as lottery q to yield an alternative at least as good as x. Formally,

$$p \succsim_i^{SD} q \text{ iff } \sum_{y \succsim_i x} p(y) \geq \sum_{y \succsim_i x} q(y) \text{ for all } x \in A.$$

When $p \succsim_i^{SD} q$ and not $q \succsim_i^{SD} p$ we write $p \succ_i^{SD} q$.

As an example, consider the preference relation $\succsim_i\colon a, b, c$. We then have that

$$({}^2\!/_3\, a + {}^1\!/_3\, c) \succ_i^{SD} ({}^1\!/_3\, a + {}^1\!/_3\, b + {}^1\!/_3\, c)$$

while ${}^2\!/_3\, a + {}^1\!/_3\, c$ and b are incomparable according to stochastic dominance.

Lemma 2. *Let $\succsim_i \in \mathcal{R}$. A lottery p SD-dominates another lottery q for an agent i iff $u_i(p) \geq u_i(q)$ for every utility function u_i consistent with $\succsim_i$. As a consequence,*

1. *an SDS f is efficient iff, for all $R \in \mathcal{R}^N$, there is no lottery p such that $p \succsim_i^{SD} f(R)$ for all $i \in N$ and $p \succ_i^{SD} f(R)$ for some $i \in N$, and*

2. *an SDS f is manipulable iff there exist a preference profile R, an agent i, and a preference relation $\succsim$ such that $f(R^{i \mapsto \succsim}) \succ_i^{SD} f(R)$.*

Proof. For the direction from left to right, assume that $p \succsim_i^{SD} q$. Without loss of generality, let $A = \{x_1, \dots, x_m\}$ and $x_j \succsim_i x_k$ if and only if $j \leq k$ for all $j, k \in \{1, \dots, m\}$. Then, by definition, for all $j \in \{1, \dots, m\}$, $\sum_{k=1}^{j} p(x_k) \geq \sum_{k=1}^{j} q(x_k)$. Let u_i be a utility function consistent with $\succsim_i$, i.e., $u_i(x_j) \geq u_i(x_k)$ if and only if $j \leq k$. Then,

$$u_i(p) - u_i(q) = \sum_{j=1}^{m} (p(x_j) - q(x_j))u_i(x_j) = \sum_{j=1}^{m} \underbrace{(u_i(x_j) - u_i(x_{j+1}))}_{\geq 0} \underbrace{\sum_{k=1}^{j} (p(x_k) - q(x_k))}_{\geq 0} \geq 0,$$

where $u_i(x_{m+1})$ is set to 0. Hence, $u_i(p) \geq u_i(q)$.

For the direction from right to left, assume that $u_i(p) \geq u_i(q)$ for all utility functions u_i consistent with $\succsim_i$. Assume for contradiction that $p \not\succsim_i^{SD} q$, i.e., there is $x \in A$ such that $\sum_{y \succsim_i x} q(x) - \sum_{y \succsim_i x} p(x) = \epsilon > 0$. Let u_i be a utility function consistent with $\succsim_i$ such that $u_i(y) \in [1 - \epsilon/2, 1]$ for all $y \succsim_i x$ and $u_i(y) \in [0, \epsilon/2]$ for all $x \succ_i y$. Such a u_i exists, since $\epsilon > 0$. Then,

$$u_i(q) \geq (1 - \epsilon/2) \sum_{y \succsim_i x} q(y) \geq \sum_{y \succsim_i x} q(y) - \epsilon/2 > \sum_{y \succsim_i x} p(y) + \epsilon/2 \geq u_i(p),$$

which contradicts the assumption. $\square$

In words, Lemma 2 shows that an SDS f is efficient if and only if $f(R)$ is Pareto-efficient with respect to stochastic dominance for all preference profiles R. Secondly, f is manipulable if and only if some agent can misrepresent his preferences to obtain a lottery that he prefers to the lottery obtained by sincere voting with respect to stochastic dominance.

4.2.1. Encoding Strategyproofness

Starting from the above equivalence, encoding strategyproofness as an SMT constraint is now a much simpler task. For each (canonical) preference profile $R \in \mathcal{R}_{\mathfrak{c}}^N$, agent $i \in N$,[8] and preference relation $\succsim \in \mathcal{R}$, we encode that the manipulated outcome $f(R^{i \mapsto \succsim})$ is not

[8] Note that, due to anonymity, it is not necessary to iterate over all agents i. Rather it suffices to pick one agent per unique preference relation contained in R.

SD-preferred to the truthful outcome $f(R)$ by agent i:

$$
\begin{aligned}
&\neg\Big(f(R^{i\mapsto\succsim}) \succ_i^{SD} f(R)\Big)\\
\equiv\, & f(R^{i\mapsto\succsim}) \not\succsim_i^{SD} f(R) \vee f(R) \succsim_i^{SD} f(R^{i\mapsto\succsim})\\
\equiv\, & \left((\exists x \in A) \sum_{y \succsim_i x} f(R^{i\mapsto\succsim})(y) < \sum_{y \succsim_i x} f(R)(y)\right) \vee\\
& \left((\forall x \in A) \sum_{y \succsim_i x} f(R^{i\mapsto\succsim})(y) \overset{(*)}{\leq} \sum_{y \succsim_i x} f(R)(y)\right)\\
\equiv\, & \left(\bigvee_{x \in A} \sum_{y \succsim_i x} p_{(R^{i\mapsto\succsim})_{\mathfrak{c}}, \pi_{\mathfrak{c}}^{R^{i\mapsto\succsim}}(y)} < \sum_{y \succsim_i x} p_{R,y}\right) \vee\\
& \left(\bigwedge_{x \in A} \sum_{y \succsim_i x} p_{(R^{i\mapsto\succsim})_{\mathfrak{c}}, \pi_{\mathfrak{c}}^{R^{i\mapsto\succsim}}(y)} \overset{(**)}{=} \sum_{y \succsim_i x} p_{R,y}\right),
\end{aligned}
\tag{Strategyproofness}
$$

where $\pi_{\mathfrak{c}}^{R^{i\mapsto\succsim}}$ stands for a permutation of alternatives that (together with a potential renaming of alternatives) leads from $R^{i\mapsto\succsim}$ to $(R^{i\mapsto\succsim})_{\mathfrak{c}}$. The inequality $(*)$ can be replaced by the equality $(**)$ since the case of at least one strict inequality is captured by the corresponding disjunctive condition one line above.

4.2.2. Encoding Efficiency

While Lemma 2 helps to formulate efficiency as an SMT axiom it is not yet sufficient since a quantification over the set of all lotteries $\Delta(A)$ remains. In order to get rid of this quantifier, we apply two lemmas by Aziz et al. (2015), for which we include (slightly simplified) proofs in favor of a self-contained presentation. The first lemma states that efficiency of a lottery only depends on its support. The second lemma shows that deciding whether a lottery is efficient reduces to solving a linear program.

Lemma 3 (Aziz et al., 2015)**.** *Let $R \in \mathcal{R}^N$. A lottery $p \in \Delta(A)$ is efficient iff every lottery $p' \in \Delta(A)$ with* $\operatorname{supp}(p') \subseteq \operatorname{supp}(p)$ *is efficient.*

Proof. We prove the statement by contraposition: if $p' \in \Delta(A)$ is not efficient, then no lottery p with $\operatorname{supp}(p') \subseteq \operatorname{supp}(p)$ is efficient. If p' is not efficient, there is $q' \in \Delta(A)$ such that q' u-dominates p' for all utility representations u^R, i.e., for all agents $i \in N$ and all utility functions u_i consistent with $\succsim_i$, $u_i(q') - u_i(p') \geq 0$ and $u_{i'}(q') - u_{i'}(p') > 0$ for some agent $i' \in N$ and all utility functions $u_{i'}$ consistent with $\succsim_{i'}$. Let $v = q' - p' \in \mathbb{R}^A$. Note that, for all $x \in A$, $v(x) < 0$ implies $x \in \operatorname{supp}(p')$. Now let $\epsilon > 0$ small enough such that $q = p + \epsilon v \in \Delta(A)$. This is possible because $\operatorname{supp}(p') \subseteq \operatorname{supp}(p)$. By definition of q and linearity of u_i, we have that, for all $i \in N$ and all u_i consistent with $\succsim_i$, $u_i(q) - u_i(p) =$

$\epsilon u_i(v) = \epsilon(u_i(q') - u_i(p')) \geq 0$ and $u_{i'}(q) - u_{i'}(p) > 0$ for all $u_{i'}$ consistent with $\succsim_{i'}$. Thus, p is not efficient, contradiction the assumption. □

Lemma 4 (Aziz et al., 2015)**.** *Whether a lottery $p \in \Delta(A)$ is efficient for a given preference profile R can be computed in polynomial time by solving a linear program.*

Proof. Given the equivalence from Lemma 2, a lottery p is easily seen to be efficient iff the optimal objective value of the following linear program is zero (since then there is no lottery q that *SD*-dominates p):

$$\begin{aligned} \max_{q,r} \sum_{i\in N}\sum_{x\in A} r_{i,x} \quad \text{subject to} \\ \sum_{y \succsim_i x} q_y - r_{i,x} = \sum_{y \succsim_i x} p_y \text{ for all } x \in A,\, i \in N, \\ \sum_{x\in A} q_x = 1,\ q_x \geq 0 \text{ for all } x \in A, \\ r_{i,x} \geq 0 \text{ for all } x \in A,\, i \in N. \end{aligned}$$

□

Recall that an SDS is efficient if it never returns a dominated lottery. By Lemma 3, this is equivalent to never returning a lottery with *inefficient support*. To capture this, we encode, for each (canonical) preference profile $R \in \mathcal{R}_{\mathfrak{c}}^N$, that the probability for at least one alternative in every (inclusion-minimal) inefficient support $I_R \subseteq A$ is zero:

$$\bigvee_{x\in I_R} p_{R,x} = 0. \tag{Efficiency}$$

4.3. Restricted Domains

Since *RSD* (cf. Section 3) is known to satisfy both strategyproofness as well as efficiency for up to 3 alternatives, the search for an impossibility has to start at $m = 4$ alternatives. For $n = 3$ agents, the encoding is solved as satisfiable; for $n = 4$, an encoding of the full domain, unfortunately, becomes prohibitively large. Hence, for $m = 4$ and $n = 4$, one has to carefully optimize the domain under consideration, on the one hand, to include a sufficient number of profiles for a successful proof, and, on the other hand, not to include too many profiles, which would prevent the solver from terminating within a reasonable amount of time.

The following incremental strategy was found to be successful. We start with a specific profile R, from which we only consider sequences of potential manipulations as long as (in each step) the manipulated individual preferences are not too distinct from the truthful preferences. To this end, we measure the magnitude of manipulations by the Kendall tau distance τ, which counts pairwise disagreements between R_i and R_i' (see also Sato, 2013). A change in the individual preferences of an agent will be called a *k-manipulation* if $\tau(R_i, R_i') \leq k$. Then, for example, strategically swapping two alternatives is a 2-manipulation, and breaking or introducing a tie between two alternatives is a 1-manipulation.

On the domain which starts from the preference profile R from Example 1 and from there allows sequences of $(1, 2, 1, 2)$-manipulations we were able to prove the result within a few minutes of running-time.[9,10] On smaller domains (e.g., considering $(1, 2, 2)$-manipulations from R) the axioms are still compatible.

4.4. Verification of Correctness

The main drawbacks of the SMT-based proof are that *(i)* one must trust the correctness of the SMT solver, *(ii)* one must trust the correctness of the program that performs the encoding into SMT-LIB, and *(iii)* the proof is unstructured and completely unlike a hand-written mathematical argument, which makes it virtually impossible to be checked by humans.

In order to tackle the first issue, we used z3 to generate a *minimal unsatisfiable set (MUS)* of constraints, i.e., an inclusion-minimal set of constraints such that this set is still unsatisfiable (see, also, Brandt and Geist, 2016). The MUS corresponding to Theorem 1 consists of 94 constraints, not counting the (trivial) lottery definitions. This MUS, annotated with e.g., the 47 required canonical preference profiles, is available as part of an arXiv version of this paper (Brandl et al., 2016a). The unsatisfiability of the MUS has been verified by the solvers CVC4, MathSAT, Yices2, and z3.

We addressed the second issue by performing several sanity checks such as running solvers on multiple variants of the encoding which represent known theorems. This way, we reproduced (amongst others) the results by Bogomolnaia and Moulin (2001) and Katta and Sethuraman (2006), as well as the possibility result for $m < 4$.

To finally remove any doubt about correctness and simultaneously address the third issue, we translated the MUS into an independent proof, which no longer relies on SMT, within the interactive theorem prover Isabelle/HOL (Nipkow et al., 2002; Nipkow and Klein, 2014). Isabelle is a generic interactive theorem prover where *interactive* means that the prover does not find the proof by itself like an automated theorem prover—the user must give it a sequence of steps to follow and the prover's automation fills in the gaps. This allows proofs of more complex theorems that are outside the scope of fully-automated theorem provers. The proof of Theorem 1 in Isabelle is rather large (more than 500 lines of code) and therefore tedious to verify by hand. However, all aspects of the proof including formal definitions of

[9] I.e., first we allow any 1-manipulation from R, then, from every resulting profile, any 2-manipulation is allowed (not necessarily by the same agent), and so forth. Showing the result on this domain implies a slightly stronger statement where strategyproofness only has to hold for "small" lies (of at most Kendall tau distance 2).

[10] The SMT solver MathSAT (Cimatti et al., 2013) terminates quickly within less than 3 minutes with the suggested competition settings, whereas z3 (de Moura and Bjørner, 2008) requires some additional configuration, but then also supports core extraction within the same time frame.

Statement	Number of canonical preference profiles
Theorem 1	47
Brandl et al. (2016b, Theorem 1)	13
Aziz et al. (2014, Theorem 3)	10
Aziz et al. (2014, Theorem 2)	7
Aziz et al. (2014, Theorem 4)	7
Aziz et al. (2013b, Theorem 1)	5
Bogomolnaia and Moulin (2001, Theorem 2)	11
Kasajima (2013, Theorem 1)	9
Nesterov (2016, Theorem 2)	8
Nesterov (2016, Theorem 1)	6
Zhou (1990, Theorem 1)	5
Katta and Sethuraman (2006, Section 4)	2
Nesterov (2016, Theorem 1)	2

Table 1: Proof complexity comparison of impossibility statements using efficiency and strategyproofness in terms of the number of canonical preference profiles used in the proof. The statements in the lower part of the table have been proven for the assignment domain.

the social-choice-theoretic concepts, the reduction of the general case to that of $m = 4$ and $n = 4$, the generation of the constraints arising from the 47 canonical preference profiles, and the proof of the inconsistency of these constraints (which corresponds to the SMT proof) have been verified by Isabelle/HOL. This proof is available in the *Archive of Formal Proofs*, which is a peer-reviewed online repository of Isabelle proofs. There is one entry for the main result (Eberl, 2016b) and one for the Randomized Social Choice Theory library that it builds upon (Eberl, 2016a). A human-readable version of this proof is given in Appendix A.

5. Conclusion

In this paper, we have leveraged computer-aided solving techniques to prove a sweeping impossibility for randomized aggregation mechanisms.

It seems unlikely that this proof would have been found without the help of computers because manual proofs of significantly weaker statements already turned out to be quite complex (see Table 1 for a comparison of the proof complexity of related statements). Nevertheless, now that the theorem has been established, our computer-aided methods may guide the search for related, perhaps even stronger statements that allow for more intuitive proofs and provide more insights into randomized social choice.

Generally speaking, we believe that SMT solving and subsequent verification via Isabelle is applicable to a wide range of problems in randomized social choice. In particular, extending our result to the special domain of assignment (see Section 3.2) is desirable as this would strengthen a number of existing theorems. Other interesting questions are whether the impossibility still holds when weakening strategyproofness even further to *BD*-strategyproofness (see, e.g., Aziz et al., 2014) or when omitting neutrality.

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A. Proof of Theorem 1

A.1. Main Proof

We will now give the complete human-readable proof of Theorem 1. This proof is essentially a paraphrased version of the formal Isabelle/HOL proof, which is available in the AFP entry (Eberl, 2016b).

Our general approach will be to attempt to "solve" preference profiles, i.e., determine the exact value of $f(R_i)(x)$ (which we write as $p_{i,x}$) for a profile R_i and an alternative x. Where this is not possible, we try to express $p_{i,x}$ in terms of other $p_{j,y}$ or at least find simple inequalities that the $p_{i,x}$ satisfy. We do this until we have gained enough knowledge about the SDS to derive a contradiction.

A typical step in the proofs will be to pick a strategyproofness condition (which usually consist of several disjunctions) and simplify it with all the knowledge that we have – substituting the $p_{i,x}$ whose values we already know, e.g., substituting $p_{i,d} = 1 - p_{i,a}$ if we know that $p_{i,b} = p_{i,c} = 0$. We will use the fact that all $p_{i,x}$ are non-negative and that $\sum_{x\in A} p_{i,x} = 1$ without mentioning it explicitly.

Every step of the proof (i.e., "Condition X simplifies to ..." or "Condition X implies ...") is elementary in the sense that it can by solved automatically by Isabelle's automation – in fact, the proof printed here is often considerably more verbose and with more intermediate steps than would be necessary in Isabelle. Still, for a human, most of these steps will require a few steps of reasoning on paper. We chose not to go into more detail on the individual steps, since it would only have made the proof even longer and less readable.

The proof will reference orbit equations, efficiency conditions, and strategyproofness conditions on the set of 47 preference profiles mentioned before. As an aid to the reader, the proof contains tables listing all the knowledge that we currently have about the probabilities of the lottery returned by the hypothetical SDS after every few steps.

We start by listing the 47 preference profiles used in the proof by giving the weak rankings of each agent.

Profile	Agent 1	Agent 2	Agent 3	Agent 4
R_1	$\{c,d\},\{a,b\}$	$\{b,d\},a,c$	$a,b,\{c,d\}$	$\{a,c\},\{b,d\}$
R_2	$\{a,c\},\{b,d\}$	$\{c,d\},a,b$	$\{b,d\},a,c$	$a,b,\{c,d\}$
R_3	$\{a,b\},\{c,d\}$	$\{c,d\},\{a,b\}$	$d,\{a,b\},c$	$c,a,\{b,d\}$
R_4	$\{a,b\},\{c,d\}$	$\{a,d\},\{b,c\}$	$c,\{a,b\},d$	$d,c,\{a,b\}$
R_5	$\{c,d\},\{a,b\}$	$\{a,b\},\{c,d\}$	$\{a,c\},d,b$	$d,\{a,b\},c$
R_6	$\{a,b\},\{c,d\}$	$\{c,d\},\{a,b\}$	$\{a,c\},\{b,d\}$	d,b,a,c
R_7	$\{a,b\},\{c,d\}$	$\{c,d\},\{a,b\}$	a,c,d,b	$d,\{a,b\},c$
R_8	$\{a,b\},\{c,d\}$	$\{a,c\},\{b,d\}$	$d,\{a,b\},c$	$d,c,\{a,b\}$
R_9	$\{a,b\},\{c,d\}$	$\{a,d\},c,b$	$d,c,\{a,b\}$	$\{a,b,c\},d$

R_{10}	$\{a,b\},\{c,d\}$	$\{c,d\},\{a,b\}$	$\{a,c\},d,b$	$\{b,d\},a,c$
R_{11}	$\{a,b\},\{c,d\}$	$\{c,d\},\{a,b\}$	$d,\{a,b\},c$	c,a,b,d
R_{12}	$\{c,d\},\{a,b\}$	$\{a,b\},\{c,d\}$	$\{a,c\},d,b$	$\{a,b,d\},c$
R_{13}	$\{a,c\},\{b,d\}$	$\{c,d\},a,b$	$\{b,d\},a,c$	a,b,d,c
R_{14}	$\{a,b\},\{c,d\}$	$d,c,\{a,b\}$	$\{a,b,c\},d$	a,d,c,b
R_{15}	$\{a,b\},\{c,d\}$	$\{c,d\},\{a,b\}$	$\{b,d\},a,c$	a,c,d,b
R_{16}	$\{a,b\},\{c,d\}$	$\{c,d\},\{a,b\}$	a,c,d,b	$\{a,b,d\},c$
R_{17}	$\{a,b\},\{c,d\}$	$\{c,d\},\{a,b\}$	$\{a,c\},\{b,d\}$	$d,\{a,b\},c$
R_{18}	$\{a,b\},\{c,d\}$	$\{a,d\},\{b,c\}$	$\{a,b,c\},d$	$d,c,\{a,b\}$
R_{19}	$\{a,b\},\{c,d\}$	$\{c,d\},\{a,b\}$	$\{b,d\},a,c$	$\{a,c\},\{b,d\}$
R_{20}	$\{b,d\},a,c$	$b,a,\{c,d\}$	$a,c,\{b,d\}$	$d,c,\{a,b\}$
R_{21}	$\{a,d\},c,b$	$d,c,\{a,b\}$	$c,\{a,b\},d$	$a,b,\{c,d\}$
R_{22}	$\{a,c\},d,b$	$d,c,\{a,b\}$	$d,\{a,b\},c$	$a,b,\{c,d\}$
R_{23}	$\{a,b\},\{c,d\}$	$\{c,d\},\{a,b\}$	$\{a,c\},\{b,d\}$	$\{a,b,d\},c$
R_{24}	$\{c,d\},\{a,b\}$	d,b,a,c	$c,a,\{b,d\}$	$b,a,\{c,d\}$
R_{25}	$\{c,d\},\{a,b\}$	$\{b,d\},a,c$	$a,b,\{c,d\}$	$a,c,\{b,d\}$
R_{26}	$\{b,d\},\{a,c\}$	$\{c,d\},\{a,b\}$	$a,b,\{c,d\}$	$a,c,\{b,d\}$
R_{27}	$\{a,b\},\{c,d\}$	$\{b,d\},a,c$	$\{a,c\},\{b,d\}$	$\{c,d\},a,b$
R_{28}	$\{c,d\},a,b$	$\{b,d\},a,c$	$a,b,\{c,d\}$	$a,c,\{b,d\}$
R_{29}	$\{a,c\},d,b$	$\{b,d\},a,c$	$a,b,\{c,d\}$	$d,c,\{a,b\}$
R_{30}	$\{a,d\},c,b$	$d,c,\{a,b\}$	$c,\{a,b\},d$	$\{a,b\},d,c$
R_{31}	$\{b,d\},a,c$	$\{a,c\},d,b$	$c,d,\{a,b\}$	$\{a,b\},c,d$
R_{32}	$\{a,c\},d,b$	$d,c,\{a,b\}$	$d,\{a,b\},c$	$\{a,b\},d,c$
R_{33}	$\{c,d\},\{a,b\}$	$\{a,c\},d,b$	$a,b,\{c,d\}$	$d,\{a,b\},c$
R_{34}	$\{a,b\},\{c,d\}$	a,c,d,b	$b,\{a,d\},c$	$c,d,\{a,b\}$
R_{35}	$\{a,d\},c,b$	$a,b,\{c,d\}$	$\{a,b,c\},d$	$d,c,\{a,b\}$
R_{36}	$\{c,d\},\{a,b\}$	$\{a,c\},d,b$	$\{b,d\},a,c$	$a,b,\{c,d\}$
R_{37}	$\{a,c\},\{b,d\}$	$\{b,d\},\{a,c\}$	$a,b,\{c,d\}$	$c,d,\{a,b\}$
R_{38}	$\{c,d\},a,b$	$\{b,d\},a,c$	$a,b,\{c,d\}$	$\{a,c\},b,d$
R_{39}	$\{a,c\},d,b$	$\{b,d\},a,c$	$a,b,\{c,d\}$	$\{c,d\},a,b$
R_{40}	$\{a,d\},c,b$	$\{a,b\},c,d$	$\{a,b,c\},d$	$d,c,\{a,b\}$
R_{41}	$\{a,d\},c,b$	$\{a,b\},d,c$	$\{a,b,c\},d$	$d,c,\{a,b\}$
R_{42}	$\{c,d\},\{a,b\}$	$\{a,b\},\{c,d\}$	d,b,a,c	$c,a,\{b,d\}$
R_{43}	$\{a,b\},\{c,d\}$	$\{c,d\},\{a,b\}$	$d,\{a,b\},c$	$a,\{c,d\},b$
R_{44}	$\{c,d\},\{a,b\}$	$\{a,c\},d,b$	$\{a,b\},d,c$	$\{a,b,d\},c$
R_{45}	$\{a,c\},d,b$	$\{b,d\},a,c$	$\{a,b\},c,d$	$\{c,d\},b,a$
R_{46}	$\{b,d\},a,c$	$d,c,\{a,b\}$	$\{a,c\},\{b,d\}$	$b,a,\{c,d\}$
R_{47}	$\{a,b\},\{c,d\}$	$\{a,d\},c,b$	$d,c,\{a,b\}$	$c,\{a,b\},d$

Table 2: The 47 preference profiles used in the proof.

Now, to begin with the proof, we shall first focus on those profiles that have rich symmetries (i.e., orbit conditions) and restrictive efficiency conditions (e.g., that contain Pareto losers).

Table 3 lists profile automorphisms, i.e., permutations of the alternatives such that applying the permutation to the profile yields a profile that is anonymity-equivalent to the original profile. Given such a profile, an anonymous and neutral SDS must return the same probability for each alternative on an orbit of the permutation. To increase readability, the permutations are already written as a product of their orbits; for instance, the first orbit condition states that $p_{10,a} = p_{10,d}$ and $p_{10,b} = p_{10,c}$.

Profile	Permutation
R_{10}	$(a\ d)(b\ c)$
R_{26}	$(a)(b\ c)(d)$
R_{27}	$(a)(b\ c)(d)$
R_{28}	$(a)(b\ c)(d)$
R_{29}	$(a\ d)(b\ c)$
R_{43}	$(a\ d)(b\ c)$
R_{45}	$(a\ b\ d\ c)$

Table 3: The relevant profile automorphisms, written as a product of their orbits.

There are efficiency conditions of two different types: those derived from *ex post* efficiency alone assert that Pareto dominated alternatives have to be assigned probability 0, whereas those derived from *SD*-efficiency (but not *ex post* efficiency) assert that at least one of two alternatives has to be assigned probability 0.

The alternative b is Pareto-dominated in the following profiles and must therefore be assigned probability 0 by any *ex post* efficient SDS (and thereby also by any *SD*-efficient SDS):

R_3, R_4, R_5, R_7, R_8, R_9, R_{11}, R_{12}, R_{14}, R_{16}, R_{17}, R_{18}, R_{21}, R_{22}, R_{23}, R_{30}, R_{32}, R_{33}, R_{35}, R_{40}, R_{41}, R_{43}, R_{44}, R_{47}

Moreover, $\{b, c\}$ is an *SD*-inefficient support in the following profiles (i.e., any *SD*-efficient SDS must assign probability 0 to at least one of b and c):

R_{10}, R_{15}, R_{19}, R_{25}, R_{26}, R_{27}, R_{28}, R_{29}, R_{39}

To see that this is true, note that the lottery $\frac{1}{2}\,a + \frac{1}{2}\,d$ strictly Pareto-*SD*-dominates the lottery $\frac{1}{2}\,b + \frac{1}{2}\,c$ for each of these profiles.

Using the orbit conditions and efficiency conditions we arrive at the following conclusions:

- The orbit conditions of R_{45} obviously imply $p_{45,a} = p_{45,b} = p_{45,c} = p_{45,d} = \frac{1}{4}$.
- The efficiency conditions for R_{10} state that at least one of $p_{10,b}$ and $p_{10,c}$ is 0, and since the orbit conditions state that $p_{10,b} = p_{10,c}$, we have $p_{10,b} = p_{10,c} = 0$.
- In the same fashion, we can show that $p_{i,x} = 0$ for $i \in \{26, 27, 28, 29\}$ and $x \in \{b, c\}$. For R_{29}, the orbit condition then additionally implies $p_{29,a} = p_{29,d} = \frac{1}{2}$.
- The efficiency conditions for R_{43} state $p_{43,b} = p_{43,c} = 0$, and with the orbit condition $p_{43,a} = p_{43,d}$ we have $p_{43,a} = p_{43,d} = \frac{1}{2}$.

In summary, we have now derived the following information about the profiles:

	R_{10}	R_{26}	R_{27}	R_{28}	R_{29}	R_{43}	R_{45}
a	$\frac{1}{2}$				$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$
b	0	0	0	0	0	0	$\frac{1}{4}$
c	0	0	0	0	0	0	$\frac{1}{4}$
d	$\frac{1}{2}$				$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$

- Suppose $p_{39,c} = 0$. Then $(S_{29,39})$ implies $p_{39,d} \le \frac{1}{2}$ and $(S_{39,29})$ then implies $p_{39,b} = 0$. Since the efficiency condition for R_{39} states that $p_{39,b} = 0 \vee p_{39,c} = 0$, we can conclude that, in any case, $p_{39,b} = 0$.
- Using this, $(S_{39,29})$ now simplifies to $p_{39,a} \le \frac{1}{2}$.
- $(S_{10,36})$ simplifies to $p_{36,a} + p_{36,b} \le \frac{1}{2}$. Using this, $(S_{36,10})$ simplifies to $p_{36,a} = \frac{1}{2} \wedge p_{36,b} = 0$.
- $(S_{36,39})$ simplifies to $p_{39,a} \ge \frac{1}{2}$. Using this, $(S_{39,36})$ simplifies to $p_{39,a} = \frac{1}{2}$.
- $(S_{12,10})$ simplifies to $p_{12,a} + p_{12,d} \ge 1$, which implies $p_{12,c} = 0$.
- $(S_{10,12})$ then simplifies to $p_{12,a} \ge \frac{1}{2}$.
- $(S_{12,44})$ simplifies to $p_{44,a} \le p_{12,a}$. Using this, $(S_{44,12})$ simplifies to $p_{44,a} = p_{12,a} \wedge p_{44,c} = 0$.
- $(S_{9,35})$ simplifies to $p_{35,a} \le p_{9,a}$, and then $(S_{35,9})$ simplifies to $p_{9,a} = p_{35,a}$.
- $(S_{9,18})$ states that $p_{9,a} + p_{9,d} \le p_{18,a} + p_{18,d}$, and then $(S_{9,18})$ simplifies to $p_{18,c} = p_{9,c}$.

To summarize:

	R_9	R_{10}	R_{12}	R_{18}	R_{26}	R_{27}	R_{28}	R_{29}	R_{36}	R_{39}	R_{43}	R_{44}	R_{45}
a	$p_{35,a}$	$\frac{1}{2}$	$\ge \frac{1}{2}$					$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$p_{12,a}$	$\frac{1}{4}$
b	0	0	0	0	0	0	0	0	0	0	0	0	$\frac{1}{4}$
c		0	0	$p_{9,c}$	0	0	0	0			0	0	$\frac{1}{4}$
d		$\frac{1}{2}$	$\le \frac{1}{2}$						$\frac{1}{2}$		$\frac{1}{2}$	$1 - p_{12,a}$	$\frac{1}{4}$

- $(S_{5,10})$ implies $p_{5,d} \geq \frac{1}{2}$.
- $(S_{5,17})$ implies $p_{5,d} \leq p_{17,d}$, and $(S_{17,7})$ simplifies to $p_{17,d} \leq p_{7,d}$. Combined with $p_{5,d} \geq \frac{1}{2}$ from above, we have $p_{7,d} \geq \frac{1}{2}$. Using this, $(S_{7,43})$ implies $p_{7,a} = \frac{1}{2}$ and $p_{7,c} = 0$, and therefore $p_{7,d} = \frac{1}{2}$.
- $(S_{5,7})$ now simplifies to $p_{5,d} \leq \frac{1}{2}$, and $p_{5,d} \geq \frac{1}{2}$ was already shown, so we have $p_{5,d} = \frac{1}{2}$.
- $(S_{5,10})$ now simplifies to $p_{5,c} = 0$, and it is then clear that $p_{5,a} = \frac{1}{2}$.
- Suppose $p_{15,b} = 0$. Then $(S_{10,15})$ simplifies to $p_{15,a} + p_{15,c} \leq \frac{1}{2}$ and, using that, $(S_{15,10})$ implies $p_{15,c} = 0$. Since the efficiency conditions for R_{15} tell us that $p_{15,b} = 0 \vee p_{15,c} = 0$, we can conclude $p_{15,c} = 0$.
- $(S_{15,5})$ then implies $p_{15,a} \geq \frac{1}{2}$ and $(S_{15,7})$ implies $p_{15,a} \leq \frac{1}{2}$. We can conclude that $p_{15,a} = \frac{1}{2}$.
- $(S_{15,5})$ now simplifies to $p_{15,d} = \frac{1}{2} \wedge p_{15,b} = 0$.
- $(S_{27,13})$ simplifies to $p_{13,a} + p_{13,b} \leq p_{27,a}$. Using that, $(S_{13,27})$ simplifies to $p_{13,b} = p_{13,c} = 0$ and $p_{27,a} = p_{13,a}$.
- $(S_{15,13})$ now implies $p_{13,a} \geq \frac{1}{2}$ and $(S_{13,15})$ simplifies to $p_{13,a} \leq \frac{1}{2}$, so that we can conclude $p_{13,a} = p_{13,d} = p_{27,a} = p_{27,d} = \frac{1}{2}$.

We summarize what we have learned so far:

	R_5	R_7	R_9	R_{10}	R_{12}	R_{13}	R_{15}	R_{18}	R_{26}	R_{27}	R_{28}	R_{29}
a	$\frac{1}{2}$	$\frac{1}{2}$	$p_{35,a}$	$\frac{1}{2}$	$\geq \frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$			$\frac{1}{2}$		$\frac{1}{2}$
b	0	0	0	0	0	0	0	0	0	0	0	0
c	0	0		0	0	0	0	$p_{9,c}$	0	0	0	0
d	$\frac{1}{2}$	$\frac{1}{2}$		$\frac{1}{2}$	$\leq \frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$			$\frac{1}{2}$		$\frac{1}{2}$

	R_{36}	R_{39}	R_{43}	R_{44}	R_{45}
a	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$p_{12,a}$	$\frac{1}{4}$
b	0	0	0	0	$\frac{1}{4}$
c			0	0	$\frac{1}{4}$
d			$\frac{1}{2}$	$1 - p_{12,a}$	$\frac{1}{4}$

- We will now determine the probabilities for R_{19}. The efficiency condition tells us that $p_{19,b} = 0 \vee p_{19,c} = 0$.
  - Suppose $p_{19,b} = 0$. Then $(S_{10,19})$ simplifies to $p_{19,a} + p_{19,c} \leq \frac{1}{2}$ and $(S_{19,10})$ simplifies to $p_{19,a} + p_{19,c} = \frac{1}{2}$. We can therefore conclude that $p_{19,d} = \frac{1}{2}$. Using this, $(S_{27,19})$ then simplifies to $p_{19,a} = \frac{1}{2} \wedge p_{19,c} = 0$ and therefore $p_{19,d} = \frac{1}{2}$.
  - Suppose $p_{19,c} = 0$. Then $(S_{19,10})$ simplifies to $p_{19,a} \geq \frac{1}{2}$ and $(S_{19,27})$ simplifies to $p_{19,d} \geq \frac{1}{2}$. This clearly implies $p_{19,a} = p_{19,d} = \frac{1}{2}$ and $p_{19,b} = 0$.

- Using this, ($S_{19,1}$) simplifies to $p_{1,a} + p_{1,b} \leq \frac{1}{2}$, and with that, ($S_{1,19}$) simplifies to $p_{1,a} = \frac{1}{2} \wedge p_{1,b} = 0$.
- ($S_{33,5}$) simplifies to $p_{33,a} \geq \frac{1}{2}$. Moreover, ($S_{33,22}$) simplifies to $p_{22,c} + p_{22,d} \leq p_{33,c} + p_{33,d}$, i.e., $p_{33,a} \leq p_{22,a}$. We therefore have $p_{22,a} \geq \frac{1}{2}$. Using this, ($S_{22,29}$) simplifies to $p_{22,a} = p_{22,d} = \frac{1}{2}$ and therefore also $p_{22,c} = 0$.
- ($S_{32,28}$) implies $p_{28,a} \leq p_{32,d}$. Then ($S_{28,32}$) implies $p_{32,d} = p_{28,a}$. Moreover, ($S_{22,32}$) simplifies to $p_{32,a} \leq 1$. Using these two facts, ($S_{32,22}$) implies $p_{32,d} = \frac{1}{2}$ and therefore also $p_{28,a} = p_{28,d} = \frac{1}{2}$.
- ($S_{28,39}$) now simplifies to $p_{39,c} = 0$, and since we have already determined $p_{39,a} = \frac{1}{2}$ and $p_{39,b} = 0$, we can conclude $p_{39,d} = \frac{1}{2}$.
- ($S_{1,2}$) states that $p_{2,c} + p_{2,d} \leq p_{1,c} + p_{2,d}$. Using this, ($S_{2,1}$) simplifies to $p_{2,a} = p_{2,c} + p_{2,d} = \frac{1}{2}$ and therefore also $p_{2,b} = 0$. Using this, ($S_{39,2}$) simplifies to $p_{2,c} = 0 \wedge p_{2,d} = \frac{1}{2}$.
- We will now determine R_{42}:
  - ($S_{17,5}$) simplifies to $p_{17,a} + p_{17,c} \geq \frac{1}{2}$ and ($S_{5,17}$) simplifies to $p_{17,a} + p_{17,c} \leq \frac{1}{2}$, so we can conclude $p_{17,d} = \frac{1}{2}$.
  - ($S_{6,42}$) states that $p_{42,a} + p_{42,c} \leq p_{6,a} + p_{6,c}$ and ($S_{6,19}$) implies $p_{6,a} + p_{6,c} \leq \frac{1}{2}$. We can therefore conclude that $p_{42,a} + p_{42,c} \leq \frac{1}{2}$.
  - ($S_{17,11}$) states that $p_{11,a} + p_{11,c} \leq p_{17,a} + p_{17,c}$. Since $p_{11,b} = p_{17,b} = 0$, this is equivalent to $p_{11,d} \geq p_{17,d} = \frac{1}{2} \geq p_{42,a} + p_{42,c}$. With this, ($S_{42,11}$) implies $p_{42,c} \geq p_{11,d} \geq \frac{1}{2}$.
  - ($S_{17,3}$) simplifies to $p_{3,a} + p_{3,c} \leq p_{17,a} + p_{17,c}$; i.e., $p_{3,d} \geq p_{17,d} = \frac{1}{2}$.
  - Finally, using $p_{42,c} \geq \frac{1}{2}$ and $p_{3,d} \geq \frac{1}{2}$, ($S_{42,3}$) simplifies to $p_{42,c} \geq \frac{1}{2} \wedge p_{42,d} \geq \frac{1}{2}$ and therefore $p_{42,a} = p_{42,b} = 0$ and $p_{42,c} = p_{42,d} = \frac{1}{2}$.
- Using these values for R_{42}, the two conditions ($S_{37,42}\,(1)$) and ($S_{37,42}\,(2)$) now simplify to $p_{37,a} = \frac{1}{2} \vee p_{37,a} + p_{37,b} > \frac{1}{2}$ and $p_{37,c} = \frac{1}{2} \vee p_{37,c} + p_{37,d} > \frac{1}{2}$. Together, these obviously imply $p_{37,a} = p_{37,c} = \frac{1}{2}$ and $p_{37,b} = p_{37,d} = 0$.
- Similarly, R_{24} simplifies to $p_{24,a} + p_{24,b} \leq 0$ and therefore $p_{24,a} = p_{24,b} = 0$.

	R_1	R_2	R_5	R_7	R_9	R_{10}	R_{12}	R_{13}	R_{15}	R_{18}	R_{19}	R_{22}	R_{24}
a	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$p_{35,a}$	$\frac{1}{2}$	$\geq \frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$		$\frac{1}{2}$	$\frac{1}{2}$	0
b	0	0	0	0	0	0	0	0	0	0	0	0	0
c		0	0	0		0	0	0	0	$p_{9,c}$	0	0	
d		$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$		$\frac{1}{2}$	$\leq \frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$		$\frac{1}{2}$	$\frac{1}{2}$	

	R_{26}	R_{27}	R_{28}	R_{29}	R_{36}	R_{37}	R_{39}	R_{42}	R_{43}	R_{44}	R_{45}
a		$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$p_{12,a}$	$\frac{1}{4}$
b	0	0	0	0	0	0	0	0	0	0	$\frac{1}{4}$
c	0	0	0	0		$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	$\frac{1}{4}$
d		$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$		0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$1 - p_{12,a}$	$\frac{1}{4}$

- ($S_{24,34}$) implies $p_{34,b} \leq p_{24,c}$ and ($S_{34,24}$) implies $p_{24,c} \leq p_{34,b}$; we therefore have $p_{34,b} = p_{24,c}$. Using this, ($S_{34,24}$) simplifies to $p_{34,c} = 0$ and ($S_{24,34}$) simplifies to $p_{34,d} = 0$.
- ($S_{14,34}$) now simplifies to $p_{14,a} + p_{14,c} \geq 1$, so we have $p_{14,b} = p_{14,d} = 0$.
- ($S_{46,37}$) simplifies to $p_{46,a} = p_{46,c} = 0$.
- ($S_{46,20}$) now simplifies to $p_{20,a} + p_{20,c} \leq 0$, so we have $p_{20,a} = p_{20,c} = 0$.
- ($S_{20,21}$) now simplifies to $p_{21,b} = p_{21,c} = 0$.
- ($S_{12,16}$) simplifies to $p_{16,a} + p_{16,c} \leq p_{12,a}$.
- We now determine the probabilities for $p_{16,c}$:
  - ($S_{44,40}$) simplifies to $p_{12,a} \leq p_{40,a}$. Moreover, ($S_{9,40}$) simplifies to $p_{40,a} \leq p_{9,a}$. Combined with $p_{16,a} + p_{16,c} \leq p_{12,a}$, this implies $p_{16,a} + p_{16,c} \leq p_{9,a}$.
  - ($S_{14,16}$) implies $p_{16,a} \geq p_{14,a}$.
  - Combining the last two facts, we obtain $p_{16,c} \leq p_{9,a} - p_{14,a}$. Moreover, ($S_{14,9}$) implies $p_{9,a} - p_{14,a} \leq 0$. Combining this, we have $p_{16,c} = 0$.
- Therefore, the fact $p_{16,a} + p_{16,c} \leq p_{12,a}$, which we have shown before, now simplifies to $p_{16,a} \leq p_{12,a}$.
- Since ($S_{14,16}$) simplifies to $p_{14,a} \leq p_{16,a}$, we then have $p_{14,a} \leq p_{12,a}$.

	R_1	R_2	R_5	R_7	R_9	R_{10}	R_{12}	R_{13}	R_{14}	R_{15}	R_{16}	R_{18}	R_{19}
a	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$p_{35,a}$	$\frac{1}{2}$	$\geq \frac{1}{2}$	$\frac{1}{2}$	$\leq p_{12,a}$	$\frac{1}{2}$			$\frac{1}{2}$
b	0	0	0	0	0	0	0	0	0	0	0	0	0
c		0	0	0		0	0	0		0	0	$p_{9,c}$	0
d		$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$		$\frac{1}{2}$	$\leq \frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$			$\frac{1}{2}$

	R_{20}	R_{21}	R_{22}	R_{24}	R_{26}	R_{27}	R_{28}	R_{29}	R_{34}	R_{36}	R_{37}	R_{39}
a	0		$\frac{1}{2}$	0		$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$1 - p_{24,c}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
b		0	0	0	0	0	0	0	$p_{24,c}$	0	0	0
c	0	0	0		0	0	0	0	0		$\frac{1}{2}$	0
d			$\frac{1}{2}$			$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0		0	$\frac{1}{2}$

	R_{42}	R_{43}	R_{44}	R_{45}	R_{46}
a	0	$\frac{1}{2}$	$p_{12,a}$	$\frac{1}{4}$	0
b	0	0	0	$\frac{1}{4}$	
c	$\frac{1}{2}$	0	0	$\frac{1}{4}$	0
d	$\frac{1}{2}$	$\frac{1}{2}$	$1 - p_{12,a}$	$\frac{1}{4}$	

- We now show that $p_{12,a} = p_{9,a} = p_{35,a}$:
  - $(S_{14,9})$ implies $p_{9,a} \leq p_{14,a}$. Since $p_{14,a} \leq p_{12,a}$, we have $p_{9,a} \leq p_{12,a}$.
  - $(S_{44,40})$ simplifies to $p_{12,a} \leq p_{40,a}$. Moreover, $(S_{9,40})$ simplifies to $p_{40,a} \leq p_{9,a}$; therefore, we have $p_{12,a} \leq p_{9,a}$.
  - Combining these two inequalities yields $p_{12,a} = p_{9,a}$.
- Recall that $p_{14,a} \leq p_{12,a} = p_{9,a}$. Then $(S_{14,9})$ simplifies to $p_{9,a} = p_{14,a} \wedge p_{9,d} = 0$.
- $(S_{23,19})$ simplifies to $p_{23,a} + p_{23,d} \geq 1$ and therefore $p_{23,b} = p_{23,c} = 0$.
- $(S_{35,21})$ simplifies to $p_{21,a} \leq p_{35,a} + p_{35,c}$. Then $(S_{21,35})$ simplifies to $p_{35,c} = 0 \wedge p_{35,a} = p_{21,a}$.
- Next, we derive the probabilities for R_{18}:
  - $(S_{23,12})$ simplifies to $p_{21,a} \leq p_{23,a}$.
  - $(S_{23,18})$ simplifies to $p_{18,c} + p_{18,d} \leq 1 - p_{23,a}$. Since $p_{18,c} = p_{9,c} = 1 - p_{9,a} = 1 - p_{35,a} = 1 - p_{21,a}$, this is equivalent to $p_{18,d} \leq p_{21,a} - p_{23,a}$. Recall that $p_{9,b} = p_{9,c} = 0$, i.e., $p_{18,c} = p_{9,c} = 1 - p_{9,a} = 1 - p_{35,a} = 1 - p_{21,a}$. Substituting this in the inequality we have just derived and rearranging yields $p_{18,d} \leq p_{21,a} - p_{23,a}$.
  - Since $p_{21,a} \leq p_{23,a}$, the right-hand side of the above inequality is 0 and therefore $p_{18,d} = 0$.

  Now we can derive the probabilities for R_4:

- $(S_{47,30})$ simplifies to $p_{30,a} \leq p_{47,a}$.
- $(S_{4,47})$ simplifies to $p_{47,a} + p_{47,d} \leq p_{4,a} + p_{4,d}$, i.e., $p_{4,c} \leq p_{47,c}$.
- Adding these two inequalities, we obtain $p_{4,c} + p_{30,a} \leq 1 - p_{47,d}$.
- $(S_{30,21})$ simplifies to $p_{21,a} \leq p_{30,a}$, and with the previous inequality, we obtain $p_{4,c} + p_{21,a} \leq 1 - p_{47,d} \leq 1$. Substituting $p_{21,a} = p_{14,a}$ yields $p_{4,c} + p_{14,a} \leq 1$.
- $(S_{4,18})$ now simplifies to $p_{4,d} = 0 \land p_{4,c} = p_{21,d}$.

- $(S_{8,26})$ implies $p_{26,a} \leq p_{8,d}$. Using this, $(S_{26,8})$ simplifies to $p_{26,a} = p_{8,d}$. Using this, we look at $(S_{8,26})$ again and find that it now simplifies to $p_{8,a} + p_{8,d} = 1$, i.e., $p_{8,c} = p_{8,b} = 0$ and $p_{26,a} = 1 - p_{8,a}$.

	R_1	R_2	R_4	R_5	R_7	R_8	R_9	R_{10}	R_{12}	R_{13}	R_{14}
a	$\frac{1}{2}$	$\frac{1}{2}$	$p_{21,a}$	$\frac{1}{2}$	$\frac{1}{2}$		$p_{21,a}$	$\frac{1}{2}$	$p_{21,a}$	$\frac{1}{2}$	$p_{21,a}$
b	0	0	0	0	0	0	0	0	0	0	0
c		0	$1 - p_{21,a}$	0	0	0	$1 - p_{21,a}$	0	0	0	$1 - p_{21,a}$
d		$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$		0	$\frac{1}{2}$	$1 - p_{21,a}$	$\frac{1}{2}$	0

	R_{15}	R_{16}	R_{18}	R_{19}	R_{20}	R_{21}	R_{22}	R_{23}	R_{24}	R_{26}	R_{27}	R_{28}
a	$\frac{1}{2}$		$p_{21,a}$	$\frac{1}{2}$	0		$\frac{1}{2}$		0	$1 - p_{8,a}$	$\frac{1}{2}$	$\frac{1}{2}$
b	0	0	0	0		0	0	0	0	0	0	0
c	0	0	$1 - p_{21,a}$	0	0	0	0	0		0	0	0
d	$\frac{1}{2}$		0	$\frac{1}{2}$			$\frac{1}{2}$			$p_{8,a}$	$\frac{1}{2}$	$\frac{1}{2}$

	R_{29}	R_{34}	R_{35}	R_{36}	R_{37}	R_{39}	R_{42}	R_{43}	R_{44}	R_{45}	R_{46}
a	$\frac{1}{2}$	$1 - p_{24,c}$	$p_{21,a}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$p_{12,a}$	$\frac{1}{4}$	0
b	0	$p_{24,c}$	0	0	0	0	0	0	0	$\frac{1}{4}$	
c	0	0	0		$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	$\frac{1}{4}$	0
d	$\frac{1}{2}$	0	$1 - p_{21,a}$		0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$1 - p_{12,a}$	$\frac{1}{4}$	

- $(S_{4,47})$ simplifies to $p_{21,d} \leq p_{47,c}$.
- $(S_{47,30})$ simplifies to $p_{30,a} \leq p_{47,a}$. With this and the previous inequality, $(S_{30,21})$ simplifies to $p_{30,b} = p_{30,c} = 0$ and $p_{30,a} = p_{47,a}$.
- The last big and crucial step is to show that $p_{31,c} \geq \frac{1}{2}$:
  - The efficiency conditions for R_{25} tell us that $p_{25,b} = 0 \lor p_{25,c} = 0$. If $p_{25,c} = 0$, then $(S_{25,36})$ immediately implies $p_{25,a} \geq \frac{1}{2}$. If, on the other hand, $p_{25,b} = 0$, then $(S_{36,25})$ implies $p_{25,a} + p_{25,c} \leq p_{36,c} + \frac{1}{2}$, with which $(S_{25,36})$ then also implies $p_{25,a} \geq \frac{1}{2}$.
  - Using $p_{25,a} \geq \frac{1}{2}$, the condition $(S_{25,26})$ implies $p_{26,a} \geq \frac{1}{2}$, and therefore also $\frac{1}{2} \leq p_{26,a} + p_{47,d} = 1 - p_{8,a} + p_{47,d}$.

  - Now observe that $(S_{4,8})$ simplifies to $p_{21,a} \leq p_{8,a}$, which is equivalent to $1-p_{8,a} \leq p_{21,d}$. Combined with $p_{21,d} \leq p_{47,c}$, which we have shown before, we now have $\frac{1}{2} \leq p_{47,c} + p_{47,d}$.
  - $(S_{30,41})$ implies $p_{41,a} + p_{41,c} \leq p_{47,a}$, which is equivalent to $p_{47,c} + p_{47,d} \leq p_{41,d}$.
  - $(S_{41,31})$ simplifies to $p_{31,a} + p_{31,b} + p_{31,d} \leq p_{41,a} + p_{41,c}$, which is equivalent to $p_{41,d} \leq p_{31,c}$.
  - Combining this chain of inequalities, we finally have $p_{31,c} \geq \frac{1}{2}$.

- $(S_{2,38})$ simplifies to $p_{38,a} + p_{38,c} \leq \frac{1}{2}$, i.e., $p_{38,b} + p_{38,d} \geq \frac{1}{2}$. Using this and $p_{31,c} \geq \frac{1}{2}$, the condition $(S_{31,38})$ simplifies to $p_{38,b} + p_{38,d} = p_{31,b} + p_{31,d}$. This means that $p_{31,b} + p_{31,d} \geq \frac{1}{2}$, and since $p_{31,c} \geq \frac{1}{2}$, we can conclude $p_{31,b} + p_{31,d} = p_{31,c} = \frac{1}{2}$ and $p_{31,a} = 0$.

It is now easy to see that each of the three cases in $(S_{45,31})$ is a contradiction. We have thus shown that the conditions are inconsistent, and therefore, there is no anonymous and neutral SDS for four agents and alternatives that satisfies both *SD*-strategyproofness and *SD*-Efficiency. □

A.2. Strategyproofness Conditions

Table 4 lists the strategyproofness conditions that were used in the impossibility proof. They are a subset of the conditions derived by the *derive_strategyproofness_conditions* command with a distance threshold of 2, i.e., the required manipulations all have a size ≤ 2.

The first number in the name of the condition indicates the original profile and the second one is the manipulated profile (possibly with a permutation applied to the alternatives).

$$p_{2,d} + p_{2,c} \leq p_{1,d} + p_{1,c} \tag{$S_{1,2}$}$$

$$p_{19,a} < p_{1,a} \lor p_{19,a} + p_{19,b} < p_{1,a} + p_{1,b} \lor (p_{19,a} = p_{1,a} \land p_{19,a} + p_{19,b} = p_{1,a} + p_{1,b}) \tag{$S_{1,19}$}$$

$$\begin{aligned} &p_{1,d} + p_{1,c} < p_{2,d} + p_{2,c} \lor p_{1,d} + p_{1,c} + p_{1,a} < p_{2,d} + p_{2,c} + p_{2,a} \\ &\quad \lor (p_{1,d} + p_{1,c} = p_{2,d} + p_{2,c} \land p_{1,d} + p_{1,c} + p_{1,a} = p_{2,d} + p_{2,c} + p_{2,a}) \end{aligned} \tag{$S_{2,1}$}$$

$$p_{38,c} + p_{38,a} \leq p_{2,c} + p_{2,a} \tag{$S_{2,38}$}$$

$$p_{8,c} < p_{4,d} \lor p_{8,c} + p_{8,d} < p_{4,d} + p_{4,c} \lor (p_{8,c} = p_{4,d} \land p_{8,c} + p_{8,d} = p_{4,d} + p_{4,c}) \tag{$S_{4,8}$}$$

$$\begin{aligned} &p_{18,c} < p_{4,c} \lor p_{18,c} + p_{18,b} + p_{18,a} < p_{4,c} + p_{4,b} + p_{4,a} \\ &\quad \lor (p_{18,c} = p_{4,c} \land p_{18,c} + p_{18,b} + p_{18,a} = p_{4,c} + p_{4,b} + p_{4,a}) \end{aligned} \tag{$S_{4,18}$}$$

$$p_{47,d} + p_{47,a} \leq p_{4,d} + p_{4,a} \tag{$S_{4,47}$}$$

$$\begin{aligned} &p_{7,c} + p_{7,a} < p_{5,c} + p_{5,a} \lor p_{7,c} + p_{7,a} + p_{7,d} < p_{5,c} + p_{5,a} + p_{5,d} \\ &\quad \lor (p_{7,c} + p_{7,a} = p_{5,c} + p_{5,a} \land p_{7,c} + p_{7,a} + p_{7,d} = p_{5,c} + p_{5,a} + p_{5,d}) \end{aligned} \tag{$S_{5,7}$}$$

$$\begin{aligned} &p_{10,a} < p_{5,d} \lor p_{10,a} + p_{10,c} + p_{10,d} < p_{5,d} + p_{5,b} + p_{5,a} \\ &\quad \lor (p_{10,a} = p_{5,d} \land p_{10,a} + p_{10,c} + p_{10,d} = p_{5,d} + p_{5,b} + p_{5,a}) \end{aligned} \tag{$S_{5,10}$}$$

$$\begin{aligned} &p_{17,c} + p_{17,a} < p_{5,c} + p_{5,a} \lor p_{17,c} + p_{17,a} + p_{17,d} < p_{5,c} + p_{5,a} + p_{5,d} \\ &\quad \lor (p_{17,c} + p_{17,a} = p_{5,c} + p_{5,a} \land p_{17,c} + p_{17,a} + p_{17,d} = p_{5,c} + p_{5,a} + p_{5,d}) \end{aligned} \tag{$S_{5,17}$}$$

$$p_{19,d} < p_{6,d} \lor p_{19,d} + p_{19,b} < p_{6,d} + p_{6,b} \lor p_{19,d} + p_{19,b} + p_{19,a} < p_{6,d} + p_{6,b} + p_{6,a} \lor (p_{19,d} = p_{6,d} \land p_{19,d} + p_{19,b} = p_{6,d} + p_{6,b} \land p_{19,d} + p_{19,b} + p_{19,a} = p_{6,d} + p_{6,b} + p_{6,a}) \tag{$S_{6,19}$}$$

$$p_{42,c} + p_{42,a} \leq p_{6,c} + p_{6,a} \tag{$S_{6,42}$}$$

$$p_{43,d} < p_{7,a} \lor p_{43,d} + p_{43,b} < p_{7,a} + p_{7,c} \lor p_{43,d} + p_{43,b} + p_{43,a} < p_{7,a} + p_{7,c} + p_{7,d} \lor (p_{43,d} = p_{7,a} \land p_{43,d} + p_{43,b} = p_{7,a} + p_{7,c} \land p_{43,d} + p_{43,b} + p_{43,a} = p_{7,a} + p_{7,c} + p_{7,d}) \tag{$S_{7,43}$}$$

$$p_{26,a} < p_{8,d} \lor p_{26,a} + p_{26,b} + p_{26,d} < p_{8,d} + p_{8,b} + p_{8,a} \lor (p_{26,a} = p_{8,d} \land p_{26,a} + p_{26,b} + p_{26,d} = p_{8,d} + p_{8,b} + p_{8,a}) \tag{$S_{8,26}$}$$

$$p_{18,d} + p_{18,a} < p_{9,d} + p_{9,a} \lor p_{18,d} + p_{18,a} + p_{18,c} < p_{9,d} + p_{9,a} + p_{9,c} \lor (p_{18,d} + p_{18,a} = p_{9,d} + p_{9,a} \land p_{18,d} + p_{18,a} + p_{18,c} = p_{9,d} + p_{9,a} + p_{9,c}) \tag{$S_{9,18}$}$$

$$p_{35,b} + p_{35,a} \leq p_{9,b} + p_{9,a} \tag{$S_{9,35}$}$$

$$p_{40,b} + p_{40,a} \leq p_{9,b} + p_{9,a} \tag{$S_{9,40}$}$$

$$p_{12,b} + p_{12,d} < p_{10,c} + p_{10,a} \lor p_{12,b} + p_{12,d} + p_{12,a} < p_{10,c} + p_{10,a} + p_{10,d} \lor (p_{12,b} + p_{12,d} = p_{10,c} + p_{10,a} \land p_{12,b} + p_{12,d} + p_{12,a} = p_{10,c} + p_{10,a} + p_{10,d}) \tag{$S_{10,12}$}$$

$$p_{15,a} + p_{15,c} < p_{10,d} + p_{10,b} \lor p_{15,a} + p_{15,c} + p_{15,d} < p_{10,d} + p_{10,b} + p_{10,a} \lor (p_{15,a} + p_{15,c} = p_{10,d} + p_{10,b} \land p_{15,a} + p_{15,c} + p_{15,d} = p_{10,d} + p_{10,b} + p_{10,a}) \tag{$S_{10,15}$}$$

$$p_{19,a} + p_{19,c} < p_{10,d} + p_{10,b} \lor p_{19,a} + p_{19,c} + p_{19,d} < p_{10,d} + p_{10,b} + p_{10,a} \lor (p_{19,a} + p_{19,c} = p_{10,d} + p_{10,b} \land p_{19,a} + p_{19,c} + p_{19,d} = p_{10,d} + p_{10,b} + p_{10,a}) \tag{$S_{10,19}$}$$

$$p_{36,a} + p_{36,b} \leq p_{10,d} + p_{10,c} \tag{$S_{10,36}$}$$

$$p_{10,a} + p_{10,c} + p_{10,d} \leq p_{12,d} + p_{12,b} + p_{12,a} \tag{$S_{12,10}$}$$

$$p_{16,c} + p_{16,a} < p_{12,c} + p_{12,a} \lor p_{16,c} + p_{16,a} + p_{16,d} < p_{12,c} + p_{12,a} + p_{12,d} \lor (p_{16,c} + p_{16,a} = p_{12,c} + p_{12,a} \land p_{16,c} + p_{16,a} + p_{16,d} = p_{12,c} + p_{12,a} + p_{12,d}) \tag{$S_{12,16}$}$$

$$p_{44,b} + p_{44,a} \leq p_{12,b} + p_{12,a} \tag{$S_{12,44}$}$$

$$p_{15,d} + p_{15,c} < p_{13,d} + p_{13,b} \lor p_{15,d} + p_{15,c} + p_{15,a} < p_{13,d} + p_{13,b} + p_{13,a} \lor (p_{15,d} + p_{15,c} = p_{13,d} + p_{13,b} \land p_{15,d} + p_{15,c} + p_{15,a} = p_{13,d} + p_{13,b} + p_{13,a}) \tag{$S_{13,15}$}$$

$$p_{27,a} < p_{13,a} \lor p_{27,a} + p_{27,c} < p_{13,a} + p_{13,b} \lor p_{27,a} + p_{27,c} + p_{27,d} < p_{13,a} + p_{13,b} + p_{13,d} \lor (p_{27,a} = p_{13,a} \land p_{27,a} + p_{27,c} = p_{13,a} + p_{13,b} \land p_{27,a} + p_{27,c} + p_{27,d} = p_{13,a} + p_{13,b} + p_{13,d}) \tag{$S_{13,27}$}$$

$$p_{9,a} < p_{14,a} \lor p_{9,a} + p_{9,d} < p_{14,a} + p_{14,d} \lor p_{9,a} + p_{9,d} + p_{9,c} < p_{14,a} + p_{14,d} + p_{14,c} \lor (p_{9,a} = p_{14,a} \land p_{9,a} + p_{9,d} = p_{14,a} + p_{14,d} \land p_{9,a} + p_{9,d} + p_{9,c} = p_{14,a} + p_{14,d} + p_{14,c}) \tag{$S_{14,9}$}$$

$$p_{16,c} < p_{14,d} \lor p_{16,c} + p_{16,d} < p_{14,d} + p_{14,c} \lor (p_{16,c} = p_{14,d} \land p_{16,c} + p_{16,d} = p_{14,d} + p_{14,c}) \tag{$S_{14,16}$}$$

$$p_{34,d} + p_{34,b} + p_{34,a} \leq p_{14,c} + p_{14,b} + p_{14,a} \tag{$S_{14,34}$}$$

$$p_{5,d} < p_{15,a} \lor p_{5,d} + p_{5,b} < p_{15,a} + p_{15,c} \lor p_{5,d} + p_{5,b} + p_{5,a} < p_{15,a} + p_{15,c} + p_{15,d} \lor (p_{5,d} = p_{15,a} \land p_{5,d} + p_{5,b} = p_{15,a} + p_{15,c} \land p_{5,d} + p_{5,b} + p_{5,a} = p_{15,a} + p_{15,c} + p_{15,d}) \tag{$S_{15,5}$}$$

$$p_{7,d} + p_{7,b} < p_{15,d} + p_{15,b} \lor p_{7,d} + p_{7,b} + p_{7,a} < p_{15,d} + p_{15,b} + p_{15,a} \lor (p_{7,d} + p_{7,b} = p_{15,d} + p_{15,b} \land p_{7,d} + p_{7,b} + p_{7,a} = p_{15,d} + p_{15,b} + p_{15,a}) \tag{$S_{15,7}$}$$

$$p_{10,d} < p_{15,a} \lor p_{10,d} + p_{10,b} < p_{15,a} + p_{15,c} \lor p_{10,d} + p_{10,b} + p_{10,a} < p_{15,a} + p_{15,c} + p_{15,d} \lor (p_{10,d} = p_{15,a} \land p_{10,d} + p_{10,b} = p_{15,a} + p_{15,c} \land p_{10,d} + p_{10,b} + p_{10,a} = p_{15,a} + p_{15,c} + p_{15,d}) \quad (S_{15,10})$$

$$p_{13,d} + p_{13,b} \leq p_{15,d} + p_{15,c} \quad (S_{15,13})$$

$$p_{3,c} + p_{3,a} \leq p_{17,c} + p_{17,a} \quad (S_{17,3})$$

$$p_{5,c} + p_{5,a} \leq p_{17,c} + p_{17,a} \quad (S_{17,5})$$

$$p_{7,c} + p_{7,a} \leq p_{17,c} + p_{17,a} \quad (S_{17,7})$$

$$p_{11,c} + p_{11,a} \leq p_{17,c} + p_{17,a} \quad (S_{17,11})$$

$$p_{9,d} + p_{9,a} \leq p_{18,d} + p_{18,a} \quad (S_{18,9})$$

$$p_{1,b} + p_{1,a} \leq p_{19,b} + p_{19,a} \quad (S_{19,1})$$

$$p_{10,b} + p_{10,d} \leq p_{19,c} + p_{19,a} \quad (S_{19,10})$$

$$p_{27,d} + p_{27,b} \leq p_{19,d} + p_{19,c} \quad (S_{19,27})$$

$$p_{21,c} < p_{20,a} \lor p_{21,c} + p_{21,b} < p_{20,a} + p_{20,c} \lor (p_{21,c} = p_{20,a} \land p_{21,c} + p_{21,b} = p_{20,a} + p_{20,c}) \quad (S_{20,21})$$

$$p_{35,c} < p_{21,c} \lor p_{35,c} + p_{35,b} + p_{35,a} < p_{21,c} + p_{21,b} + p_{21,a} \lor (p_{35,c} = p_{21,c} \land p_{35,c} + p_{35,b} + p_{35,a} = p_{21,c} + p_{21,b} + p_{21,a}) \quad (S_{21,35})$$

$$p_{29,a} < p_{22,d} \lor p_{29,a} + p_{29,c} + p_{29,d} < p_{22,d} + p_{22,b} + p_{22,a} \lor (p_{29,a} = p_{22,d} \land p_{29,a} + p_{29,c} + p_{29,d} = p_{22,d} + p_{22,b} + p_{22,a}) \quad (S_{22,29})$$

$$p_{32,a} < p_{22,a} \lor p_{32,a} + p_{32,b} < p_{22,a} + p_{22,b} \lor (p_{32,a} = p_{22,a} \land p_{32,a} + p_{32,b} = p_{22,a} + p_{22,b}) \quad (S_{22,32})$$

$$p_{12,c} + p_{12,a} \leq p_{23,c} + p_{23,a} \quad (S_{23,12})$$

$$p_{18,c} + p_{18,d} \leq p_{23,d} + p_{23,c} \quad (S_{23,18})$$

$$p_{19,d} + p_{19,b} + p_{19,a} \leq p_{23,d} + p_{23,b} + p_{23,a} \quad (S_{23,19})$$

$$p_{34,b} < p_{24,c} \lor p_{34,b} + p_{34,d} < p_{24,c} + p_{24,a} \lor (p_{34,b} = p_{24,c} \land p_{34,b} + p_{34,d} = p_{24,c} + p_{24,a}) \quad (S_{24,34})$$

$$p_{26,d} + p_{26,c} < p_{25,d} + p_{25,b} \lor p_{26,d} + p_{26,c} + p_{26,a} < p_{25,d} + p_{25,b} + p_{25,a} \lor (p_{26,d} + p_{26,c} = p_{25,d} + p_{25,b} \land p_{26,d} + p_{26,c} + p_{26,a} = p_{25,d} + p_{25,b} + p_{25,a}) \quad (S_{25,26})$$

$$p_{36,a} < p_{25,a} \lor p_{36,a} + p_{36,c} < p_{25,a} + p_{25,c} \lor (p_{36,a} = p_{25,a} \land p_{36,a} + p_{36,c} = p_{25,a} + p_{25,c}) \quad (S_{25,36})$$

$$p_{8,d} < p_{26,a} \lor p_{8,d} + p_{8,b} < p_{26,a} + p_{26,c} \lor (p_{8,d} = p_{26,a} \land p_{8,d} + p_{8,b} = p_{26,a} + p_{26,c}) \quad (S_{26,8})$$

$$p_{13,b} + p_{13,a} \leq p_{27,c} + p_{27,a} \quad (S_{27,13})$$

$$p_{19,d} + p_{19,c} < p_{27,d} + p_{27,b} \lor p_{19,d} + p_{19,c} + p_{19,a} < p_{27,d} + p_{27,b} + p_{27,a} \lor (p_{19,d} + p_{19,c} = p_{27,d} + p_{27,b} \land p_{19,d} + p_{19,c} + p_{19,a} = p_{27,d} + p_{27,b} + p_{27,a}) \quad (S_{27,19})$$

$$p_{32,d} < p_{28,a} \lor p_{32,d} + p_{32,b} < p_{28,a} + p_{28,c} \lor (p_{32,d} = p_{28,a} \land p_{32,d} + p_{32,b} = p_{28,a} + p_{28,c}) \quad (S_{28,32})$$

$$p_{39,a} < p_{28,a} \lor p_{39,a} + p_{39,c} < p_{28,a} + p_{28,b} \lor (p_{39,a} = p_{28,a} \land p_{39,a} + p_{39,c} = p_{28,a} + p_{28,b}) \quad (S_{28,39})$$

$$p_{39,d} < p_{29,a} \lor p_{39,d} + p_{39,c} < p_{29,a} + p_{29,b} \lor (p_{39,d} = p_{29,a} \land p_{39,d} + p_{39,c} = p_{29,a} + p_{29,b}) \quad (S_{29,39})$$

$$p_{21,b} + p_{21,a} < p_{30,b} + p_{30,a} \lor p_{21,b} + p_{21,a} + p_{21,d} < p_{30,b} + p_{30,a} + p_{30,d} \lor (p_{21,b} + p_{21,a} = p_{30,b} + p_{30,a} \land p_{21,b} + p_{21,a} + p_{21,d} = p_{30,b} + p_{30,a} + p_{30,d}) \tag{$S_{30,21}$}$$

$$p_{41,c} < p_{30,c} \lor p_{41,c} + p_{41,b} + p_{41,a} < p_{30,c} + p_{30,b} + p_{30,a} \lor (p_{41,c} = p_{30,c} \land p_{41,c} + p_{41,b} + p_{41,a} = p_{30,c} + p_{30,b} + p_{30,a}) \tag{$S_{30,41}$}$$

$$p_{38,b} + p_{38,d} < p_{31,d} + p_{31,b} \lor p_{38,b} + p_{38,d} + p_{38,c} < p_{31,d} + p_{31,b} + p_{31,a} \lor (p_{38,b} + p_{38,d} = p_{31,d} + p_{31,b} \land p_{38,b} + p_{38,d} + p_{38,c} = p_{31,d} + p_{31,b} + p_{31,a}) \tag{$S_{31,38}$}$$

$$p_{22,b} + p_{22,a} < p_{32,b} + p_{32,a} \lor p_{22,b} + p_{22,a} + p_{22,d} < p_{32,b} + p_{32,a} + p_{32,d} \lor (p_{22,b} + p_{22,a} = p_{32,b} + p_{32,a} \land p_{22,b} + p_{22,a} + p_{22,d} = p_{32,b} + p_{32,a} + p_{32,d}) \tag{$S_{32,22}$}$$

$$p_{28,a} < p_{32,d} \lor p_{28,a} + p_{28,c} + p_{28,d} < p_{32,d} + p_{32,b} + p_{32,a} \lor (p_{28,a} = p_{32,d} \land p_{28,a} + p_{28,c} + p_{28,d} = p_{32,d} + p_{32,b} + p_{32,a}) \tag{$S_{32,28}$}$$

$$p_{5,a} < p_{33,a} \lor p_{5,a} + p_{5,b} < p_{33,a} + p_{33,b} \lor (p_{5,a} = p_{33,a} \land p_{5,a} + p_{5,b} = p_{33,a} + p_{33,b}) \tag{$S_{33,5}$}$$

$$p_{22,d} + p_{22,c} \leq p_{33,d} + p_{33,c} \tag{$S_{33,22}$}$$

$$p_{24,c} < p_{34,b} \lor p_{24,c} + p_{24,a} + p_{24,d} < p_{34,b} + p_{34,d} + p_{34,a} \lor (p_{24,c} = p_{34,b} \land p_{24,c} + p_{24,a} + p_{24,d} = p_{34,b} + p_{34,d} + p_{34,a}) \tag{$S_{34,24}$}$$

$$p_{9,a} < p_{35,a} \lor p_{9,a} + p_{9,b} < p_{35,a} + p_{35,b} \lor (p_{9,a} = p_{35,a} \land p_{9,a} + p_{9,b} = p_{35,a} + p_{35,b}) \tag{$S_{35,9}$}$$

$$p_{21,c} + p_{21,b} + p_{21,a} \leq p_{35,c} + p_{35,b} + p_{35,a} \tag{$S_{35,21}$}$$

$$p_{10,d} < p_{36,a} \lor p_{10,d} + p_{10,c} < p_{36,a} + p_{36,b} \lor (p_{10,d} = p_{36,a} \land p_{10,d} + p_{10,c} = p_{36,a} + p_{36,b}) \tag{$S_{36,10}$}$$

$$p_{25,c} + p_{25,a} < p_{36,c} + p_{36,a} \lor p_{25,c} + p_{25,a} + p_{25,d} < p_{36,c} + p_{36,a} + p_{36,d} \lor (p_{25,c} + p_{25,a} = p_{36,c} + p_{36,a} \land p_{25,c} + p_{25,a} + p_{25,d} = p_{36,c} + p_{36,a} + p_{36,d}) \tag{$S_{36,25}$}$$

$$p_{39,d} + p_{39,c} \leq p_{36,d} + p_{36,c} \tag{$S_{36,39}$}$$

$$p_{42,d} < p_{37,a} \lor p_{42,d} + p_{42,b} < p_{37,a} + p_{37,b} \lor (p_{42,d} = p_{37,a} \land p_{42,d} + p_{42,b} = p_{37,a} + p_{37,b}) \tag{$S_{37,42}\,(1)$}$$

$$p_{42,d} < p_{37,c} \lor p_{42,d} + p_{42,b} < p_{37,c} + p_{37,d} \lor (p_{42,d} = p_{37,c} \land p_{42,d} + p_{42,b} = p_{37,c} + p_{37,d}) \tag{$S_{37,42}\,(2)$}$$

$$p_{2,c} + p_{2,a} < p_{39,c} + p_{39,a} \lor p_{2,c} + p_{2,a} + p_{2,d} < p_{39,c} + p_{39,a} + p_{39,d} \lor (p_{2,c} + p_{2,a} = p_{39,c} + p_{39,a} \land p_{2,c} + p_{2,a} + p_{2,d} = p_{39,c} + p_{39,a} + p_{39,d}) \tag{$S_{39,2}$}$$

$$p_{29,a} + p_{29,b} < p_{39,d} + p_{39,c} \lor p_{29,a} + p_{29,b} + p_{29,d} < p_{39,d} + p_{39,c} + p_{39,a} \lor (p_{29,a} + p_{29,b} = p_{39,d} + p_{39,c} \land p_{29,a} + p_{29,b} + p_{29,d} = p_{39,d} + p_{39,c} + p_{39,a}) \tag{$S_{39,29}$}$$

$$p_{36,d} + p_{36,c} < p_{39,d} + p_{39,c} \lor p_{36,d} + p_{36,c} + p_{36,a} < p_{39,d} + p_{39,c} + p_{39,a} \lor (p_{36,d} + p_{36,c} = p_{39,d} + p_{39,c} \land p_{36,d} + p_{36,c} + p_{36,a} = p_{39,d} + p_{39,c} + p_{39,a}) \tag{$S_{39,36}$}$$

$$p_{31,d} + p_{31,b} + p_{31,a} \leq p_{41,c} + p_{41,b} + p_{41,a} \tag{$S_{41,31}$}$$

$$p_{3,d} < p_{42,d} \lor p_{3,d} + p_{3,b} < p_{42,d} + p_{42,b} \lor p_{3,d} + p_{3,b} + p_{3,a} < p_{42,d} + p_{42,b} + p_{42,a} \lor (p_{3,d} = p_{42,d} \land p_{3,d} + p_{3,b} = p_{42,d} + p_{42,b} \land p_{3,d} + p_{3,b} + p_{3,a} = p_{42,d} + p_{42,b} + p_{42,a}) \tag{$S_{42,3}$}$$

$$p_{11,d} < p_{42,c} \lor p_{11,d} + p_{11,b} < p_{42,c} + p_{42,a} \lor (p_{11,d} = p_{42,c} \land p_{11,d} + p_{11,b} = p_{42,c} + p_{42,a}) \tag{$S_{42,11}$}$$

$$p_{24,b} + p_{24,a} \leq p_{42,b} + p_{42,a} \tag{$S_{42,24}$}$$

$$p_{12,b} + p_{12,a} < p_{44,b} + p_{44,a} \lor p_{12,b} + p_{12,a} + p_{12,d} < p_{44,b} + p_{44,a} + p_{44,d} \lor (p_{12,b} + p_{12,a} = p_{44,b} + p_{44,a} \land p_{12,b} + p_{12,a} + p_{12,d} = p_{44,b} + p_{44,a} + p_{44,d}) \tag{$S_{44,12}$}$$

$p_{40,c} + p_{40,d} \leq p_{44,d} + p_{44,c}$	$(S_{44,40})$
$p_{31,c} + p_{31,d} < p_{45,b} + p_{45,a} \vee p_{31,c} + p_{31,d} + p_{31,b} < p_{45,b} + p_{45,a} + p_{45,c}$ $\vee (p_{31,c} + p_{31,d} = p_{45,b} + p_{45,a} \wedge p_{31,c} + p_{31,d} + p_{31,b} = p_{45,b} + p_{45,a} + p_{45,c})$	$(S_{45,31})$
$p_{20,c} + p_{20,a} \leq p_{46,c} + p_{46,a}$	$(S_{46,20})$
$p_{37,a} + p_{37,c} < p_{46,d} + p_{46,b} \vee p_{37,a} + p_{37,c} + p_{37,d} < p_{46,d} + p_{46,b} + p_{46,a}$ $\vee (p_{37,a} + p_{37,c} = p_{46,d} + p_{46,b} \wedge p_{37,a} + p_{37,c} + p_{37,d} = p_{46,d} + p_{46,b} + p_{46,a})$	$(S_{46,37})$
$p_{30,b} + p_{30,a} \leq p_{47,b} + p_{47,a}$	$(S_{47,30})$

Table 4: The strategyproofness conditions used in the impossibility proof.

Table 5 lists the manipulations that were used to obtain these strategyproofness conditions: the first column gives the name of the manipulation condition in the form $(S_{i,j})$, which also contains the information which two profiles are involved in the manipulation (R_i and R_j) The next columns contain the manipulating agent, his truthful preferences, and the false preferences that he needs to submit. The last column gives the permutation of the alternatives that yields R_j when applied to the manipulated instance of R_i.

Condition	Agent	Old Preferences	New Preferences	Permutation
$(S_{1,2})$	1	$\{c,d\},\{a,b\}$	$\{c,d\},a,b$	$(a)(b)(c)(d)$
$(S_{1,19})$	3	$a,b,\{c,d\}$	$\{a,b\},\{c,d\}$	$(a)(b)(c)(d)$
$(S_{2,1})$	2	$\{c,d\},a,b$	$\{c,d\},\{a,b\}$	$(a)(b)(c)(d)$
$(S_{2,38})$	1	$\{a,c\},\{b,d\}$	$\{a,c\},b,d$	$(a)(b)(c)(d)$
$(S_{4,8})$	4	$d,c,\{a,b\}$	$c,d,\{a,b\}$	$(a)(b)(c\ d)$
$(S_{4,18})$	3	$c,\{a,b\},d$	$\{a,b,c\},d$	$(a)(b)(c)(d)$
$(S_{4,47})$	2	$\{a,d\},\{b,c\}$	$\{a,d\},c,b$	$(a)(b)(c)(d)$
$(S_{5,7})$	3	$\{a,c\},d,b$	a,c,d,b	$(a)(b)(c)(d)$
$(S_{5,10})$	4	$d,\{a,b\},c$	$\{b,d\},a,c$	$(a\ d)(b\ c)$
$(S_{5,17})$	3	$\{a,c\},d,b$	$\{a,c\},\{b,d\}$	$(a)(b)(c)(d)$
$(S_{6,19})$	4	d,b,a,c	$\{b,d\},a,c$	$(a)(b)(c)(d)$
$(S_{6,42})$	3	$\{a,c\},\{b,d\}$	$c,a,\{b,d\}$	$(a)(b)(c)(d)$
$(S_{7,43})$	3	a,c,d,b	$a,\{c,d\},b$	$(a\ d)(b\ c)$
$(S_{8,26})$	3	$d,\{a,b\},c$	$d,b,\{a,c\}$	$(a\ d)(b\ c)$
$(S_{9,18})$	2	$\{a,d\},c,b$	$\{a,d\},\{b,c\}$	$(a)(b)(c)(d)$
$(S_{9,35})$	1	$\{a,b\},\{c,d\}$	$a,b,\{c,d\}$	$(a)(b)(c)(d)$
$(S_{9,40})$	1	$\{a,b\},\{c,d\}$	$\{a,b\},c,d$	$(a)(b)(c)(d)$
$(S_{10,12})$	3	$\{a,c\},d,b$	$\{a,c,d\},b$	$(a\ d)(b\ c)$
$(S_{10,15})$	4	$\{b,d\},a,c$	d,b,a,c	$(a\ d)(b\ c)$
$(S_{10,19})$	4	$\{b,d\},a,c$	$\{b,d\},\{a,c\}$	$(a\ d)(b\ c)$
$(S_{10,36})$	2	$\{c,d\},\{a,b\}$	$d,c,\{a,b\}$	$(a\ d)(b\ c)$
$(S_{12,10})$	4	$\{a,b,d\},c$	$\{b,d\},a,c$	$(a\ d)(b\ c)$

$(S_{12,16})$	3	$\{a,c\},d,b$	a,c,d,b	$(a)(b)(c)(d)$
$(S_{12,44})$	2	$\{a,b\},\{c,d\}$	$\{a,b\},d,c$	$(a)(b)(c)(d)$
$(S_{13,15})$	3	$\{b,d\},a,c$	$\{b,d\},\{a,c\}$	$(a)(b\ c)(d)$
$(S_{13,27})$	4	a,b,d,c	$\{a,b\},\{c,d\}$	$(a)(b\ c)(d)$
$(S_{14,9})$	4	a,d,c,b	$\{a,d\},c,b$	$(a)(b)(c)(d)$
$(S_{14,16})$	2	$d,c,\{a,b\}$	$\{c,d\},\{a,b\}$	$(a)(b)(c\ d)$
$(S_{14,34})$	3	$\{a,b,c\},d$	$b,\{a,c\},d$	$(a)(b)(c\ d)$
$(S_{15,5})$	4	a,c,d,b	$a,\{c,d\},b$	$(a\ d)(b\ c)$
$(S_{15,7})$	3	$\{b,d\},a,c$	$d,\{a,b\},c$	$(a)(b)(c)(d)$
$(S_{15,10})$	4	a,c,d,b	$\{a,c\},d,b$	$(a\ d)(b\ c)$
$(S_{15,13})$	2	$\{c,d\},\{a,b\}$	$\{c,d\},a,b$	$(a)(b\ c)(d)$
$(S_{17,3})$	3	$\{a,c\},\{b,d\}$	$c,a,\{b,d\}$	$(a)(b)(c)(d)$
$(S_{17,5})$	3	$\{a,c\},\{b,d\}$	$\{a,c\},d,b$	$(a)(b)(c)(d)$
$(S_{17,7})$	3	$\{a,c\},\{b,d\}$	a,c,d,b	$(a)(b)(c)(d)$
$(S_{17,11})$	3	$\{a,c\},\{b,d\}$	c,a,b,d	$(a)(b)(c)(d)$
$(S_{18,9})$	2	$\{a,d\},\{b,c\}$	$\{a,d\},c,b$	$(a)(b)(c)(d)$
$(S_{19,1})$	1	$\{a,b\},\{c,d\}$	$a,b,\{c,d\}$	$(a)(b)(c)(d)$
$(S_{19,10})$	4	$\{a,c\},\{b,d\}$	$\{a,c\},d,b$	$(a\ d)(b\ c)$
$(S_{19,27})$	2	$\{c,d\},\{a,b\}$	$\{c,d\},a,b$	$(a)(b\ c)(d)$
$(S_{20,21})$	3	$a,c,\{b,d\}$	$a,\{c,d\},b$	$(a\ c\ b\ d)$
$(S_{21,35})$	3	$c,\{a,b\},d$	$\{a,b,c\},d$	$(a)(b)(c)(d)$
$(S_{22,29})$	3	$d,\{a,b\},c$	$\{b,d\},a,c$	$(a\ d)(b\ c)$
$(S_{22,32})$	4	$a,b,\{c,d\}$	$\{a,b\},d,c$	$(a)(b)(c)(d)$
$(S_{23,12})$	3	$\{a,c\},\{b,d\}$	$\{a,c\},d,b$	$(a)(b)(c)(d)$
$(S_{23,18})$	2	$\{c,d\},\{a,b\}$	$c,d,\{a,b\}$	$(a)(b)(c\ d)$
$(S_{23,19})$	4	$\{a,b,d\},c$	$\{b,d\},a,c$	$(a)(b)(c)(d)$
$(S_{24,34})$	3	$c,a,\{b,d\}$	$c,\{a,d\},b$	$(a\ d)(b\ c)$
$(S_{25,26})$	2	$\{b,d\},a,c$	$\{b,d\},\{a,c\}$	$(a)(b\ c)(d)$
$(S_{25,36})$	4	$a,c,\{b,d\}$	$\{a,c\},d,b$	$(a)(b)(c)(d)$
$(S_{26,8})$	4	$a,c,\{b,d\}$	$a,\{c,d\},b$	$(a\ d)(b\ c)$
$(S_{27,13})$	3	$\{a,c\},\{b,d\}$	a,c,d,b	$(a)(b\ c)(d)$
$(S_{27,19})$	2	$\{b,d\},a,c$	$\{b,d\},\{a,c\}$	$(a)(b\ c)(d)$
$(S_{28,32})$	4	$a,c,\{b,d\}$	$a,\{c,d\},b$	$(a\ d)(b\ c)$
$(S_{28,39})$	3	$a,b,\{c,d\}$	$\{a,b\},d,c$	$(a)(b\ c)(d)$
$(S_{29,39})$	3	$a,b,\{c,d\}$	$\{a,b\},d,c$	$(a\ d)(b\ c)$
$(S_{30,21})$	4	$\{a,b\},d,c$	$a,b,\{c,d\}$	$(a)(b)(c)(d)$
$(S_{30,41})$	3	$c,\{a,b\},d$	$\{a,b,c\},d$	$(a)(b)(c)(d)$
$(S_{31,38})$	1	$\{b,d\},a,c$	$\{b,d\},c,a$	$(a\ c)(b\ d)$
$(S_{32,22})$	4	$\{a,b\},d,c$	$a,b,\{c,d\}$	$(a)(b)(c)(d)$
$(S_{32,28})$	3	$d,\{a,b\},c$	$d,b,\{a,c\}$	$(a\ d)(b\ c)$
$(S_{33,5})$	3	$a,b,\{c,d\}$	$\{a,b\},\{c,d\}$	$(a)(b)(c)(d)$
$(S_{33,22})$	1	$\{c,d\},\{a,b\}$	$d,c,\{a,b\}$	$(a)(b)(c)(d)$

$(S_{34,24})$	3	$b, \{a, d\}, c$	$b, d, \{a, c\}$	$(a\ d)(b\ c)$
$(S_{35,9})$	2	$a, b, \{c, d\}$	$\{a, b\}, \{c, d\}$	$(a)(b)(c)(d)$
$(S_{35,21})$	3	$\{a, b, c\}, d$	$c, \{a, b\}, d$	$(a)(b)(c)(d)$
$(S_{36,10})$	4	$a, b, \{c, d\}$	$\{a, b\}, \{c, d\}$	$(a\ d)(b\ c)$
$(S_{36,25})$	2	$\{a, c\}, d, b$	$a, c, \{b, d\}$	$(a)(b)(c)(d)$
$(S_{36,39})$	1	$\{c, d\}, \{a, b\}$	$\{c, d\}, a, b$	$(a)(b)(c)(d)$
$(S_{37,42}\,(1))$	3	$a, b, \{c, d\}$	a, b, d, c	$(a\ d)(b)(c)$
$(S_{37,42}\,(2))$	4	$c, d, \{a, b\}$	c, d, b, a	$(a\ c\ d\ b)$
$(S_{39,2})$	1	$\{a, c\}, d, b$	$\{a, c\}, \{b, d\}$	$(a)(b)(c)(d)$
$(S_{39,29})$	4	$\{c, d\}, a, b$	$d, c, \{a, b\}$	$(a\ d)(b\ c)$
$(S_{39,36})$	4	$\{c, d\}, a, b$	$\{c, d\}, \{a, b\}$	$(a)(b)(c)(d)$
$(S_{41,31})$	3	$\{a, b, c\}, d$	$\{b, c\}, a, d$	$(a)(b)(c\ d)$
$(S_{42,3})$	3	d, b, a, c	$d, \{a, b\}, c$	$(a)(b)(c)(d)$
$(S_{42,11})$	4	$c, a, \{b, d\}$	$c, \{a, b\}, d$	$(a\ b)(c\ d)$
$(S_{42,24})$	2	$\{a, b\}, \{c, d\}$	$b, a, \{c, d\}$	$(a)(b)(c)(d)$
$(S_{44,12})$	3	$\{a, b\}, d, c$	$\{a, b\}, \{c, d\}$	$(a)(b)(c)(d)$
$(S_{44,40})$	1	$\{c, d\}, \{a, b\}$	$c, d, \{a, b\}$	$(a)(b)(c\ d)$
$(S_{45,31})$	3	$\{a, b\}, c, d$	$b, a, \{c, d\}$	$(a\ d)(b\ c)$
$(S_{46,20})$	3	$\{a, c\}, \{b, d\}$	$a, c, \{b, d\}$	$(a)(b)(c)(d)$
$(S_{46,37})$	1	$\{b, d\}, a, c$	$\{b, d\}, \{a, c\}$	$(a\ d)(b\ c)$
$(S_{47,30})$	1	$\{a, b\}, \{c, d\}$	$\{a, b\}, d, c$	$(a)(b)(c)(d)$

Table 5: The manipulations required to obtain the strategyproofness conditions in Table 4