Finding and Recognizing Popular Coalition Structures

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ABSTRACT

An important aspect of multi-agent systems concerns the formation of coalitions that are stable or optimal in some well-defined way. The notion of *popularity* has recently received a lot of attention in this context. A partition is popular if there is no other partition in which more agents are better off than worse off. In 2019, a long-standing open problem concerning popularity was solved by proving that computing popular partitions in roommate games is NP-hard, even when preferences are strict. We show that this result breaks down when allowing for randomization: *mixed* popular partitions can be found efficiently via linear programming and a separation oracle. Mixed popular partitions are particularly attractive because they are guaranteed to exist in any coalition formation game. Our result implies that one can efficiently verify whether a given partition in a roommate game is popular and that strongly popular partitions can be found in polynomial time (resolving an open problem). By contrast, we prove that both problems become computationally intractable when moving from coalitions of size 2 to coalitions of size 3, even when preferences are strict and globally ranked. Moreover, we give elaborate proofs showing the NP-hardness of finding popular, strongly popular, and mixed popular partitions in additively separable hedonic games and finding popular partitions in fractional hedonic games.

KEYWORDS

Coalition Formation; Social Choice Theory

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1 INTRODUCTION

Coalitions and coalition formation have been a central concern of game theory, ever since the publication of von Neumann and Morgenstern's *Theory of Games and Economic Behavior* in 1944. The traditional models of coalitional game theory, in particular TU (transferable utility) and NTU (non-transferable utility) coalitional games, involve a formal specification of what each group of agents can achieve on their own. Drèze and Greenberg [20] noted that in many situations this is not feasible, possible, or even relevant to the coalition formation process, as, e.g., in the formation of social clubs, teams, or societies. Instead, in coalition formation games, the agents' preferences are defined directly over the coalition structures, i.e., partitions of the agents in disjoint coalitions. Formally, coalition formation can thus be considered as a special case of the general voting setting, where the agents entertain preferences over a special type of alternatives, namely coalition partitions of themselves, from which one or more need to be selected. In most situations it is natural to assume that an agent's appreciation of a partition only depends on the coalition he is a member of and not on how the remaining agents are grouped. Popularized by Bogomolnaia and Jackson [10], much of the work on coalition formation now concentrates on these so-called *hedonic games*.

The main focus in hedonic games has been on finding and recognizing partitions that satisfy various notions of *stability*—such as Nash stability, individual stability, or core stability—or optimality—such as Pareto optimality, utilitarian welfare maximality, or egalitarian welfare maximality [see 7, for an overview]. In this paper, we focus on the notion of *popularity* [24], which has the flavor of both stability and optimality. A partition is popular if there is no other partition that is preferred by a majority of the agents. Moreover, a partition is strongly popular if it is preferred to every other partition by some majority of agents. Popularity thus corresponds to the notion of weak and strong Condorcet winners in voting theory, i.e., candidates that are at least as good as any other candidate in pairwise majority comparisons. Just like stability notions, popularity is based on the idea that a subset of agents breaks off in order to increase their well-being. However, since the new partition has to make at least as many agents better off than worse off, popularity also has the flavor of optimality. In the standard reference *Algorithmics of Matching Under Preferences*, Manlove [36, p. 333] writes that "popular matchings […] have been an exciting area of research in the last few years." A recent survey on popular matchings is provided by Cseh [17].

In contrast to Pareto optimal partitions, popular partitions are not guaranteed to exist. We therefore also consider *mixed* popular partitions, as proposed by Kavitha et al. [31] and whose existence follows from the Minimax Theorem. A mixed popular partition is a probability distribution over partitions p such that there is no other mixed partition q such that the expected number of agents who prefer the partition returned by p to q is at least as large as the other way round. Mixed popular partitions are a special case of *maximal lotteries*, a randomized voting rule that has recently gathered increased attention in social choice theory [11, 13, 14, 23].

We study the computational complexity of popular, strongly popular, and mixed popular partitions in a variety of hedonic coalition formation settings including additively separable hedonic games, fractional hedonic games as well as hedonic games where the coalition size is bounded. The latter includes flatmate games (which only allow coalitions of up to three agents) and roommate games (which only allow coalitions of up to two agents). Our main findings are as follows.

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- Generalizing earlier results by Kavitha et al. [31], we show how mixed popular partitions in roommate games can be computed in polynomial time via linear programming and a separation oracle on a subpolytope of the matching polytope for non-bipartite graphs.[1] This stands in contrast to a recent result showing that computing popular partitions in roommate games is NP-hard [22, 26].
- As corollaries we obtain that verifying popular partitions [9], finding Pareto optimal partitions [4], and finding strongly popular partitions can all be done in polynomial time in roommate games, even when preferences admit ties. The latter statement resolves an acknowledged open problem.[2]
- We provide the first negative computational results for mixed popular partitions and strongly popular partitions by showing that finding these partitions in flatmate games is NP-hard. Moreover, it turns out, that verifying whether a given partition is popular, strongly popular, or mixed popular in flatmate games is coNP-complete. All of these results hold for strict and globally ranked preferences, i.e., coalitions appear in the same order in each individual preference ranking. This is interesting because finding popular partitions in roommate games becomes tractable under the same restrictions.
- We prove that computing popular, strongly popular, and mixed popular partitions is NP-hard in symmetric additively separable hedonic games and that computing popular partitions is NP-hard in symmetric fractional hedonic games. Furthermore, we show coNP-completeness of all corresponding verification problems.

2 RELATED WORK

Gärdenfors [24] first proposed the notions of popularity and strong popularity in the context of marriage games. He showed that popular matchings (or "majority assignments" in his terminology) need not exist when preferences are weak, but that existence is guaranteed for strict preferences because every stable matching is popular. As a consequence, the Gale-Shapley algorithm efficiently identifies popular matchings in marriage games with strict preferences. Kavitha and Nasre [32], Huang and Kavitha [27], and Kavitha [30] provide efficient algorithms for computing popular matchings that satisfy additional properties such as rank maximality or maximum cardinality. For weak preferences, computing popular matchings is NP-hard, even when all agents belonging to one side have strict preferences [9, 18].

In the more restricted setting of house allocation (henceforth housing games), Abraham et al. [2] proposed efficient algorithms for finding popular allocations of maximum cardinality for both weak and strict preferences. Mahdian [35] proved an interesting threshold for the existence of popular allocations: if there are n agents and the number of houses exceeds αn with $\alpha \approx 1.42$, then the probability that there is a popular allocation converges to 1 as n goes to infinity.

For roommate games with weak preferences, NP-hardness of computing popular matchings follows from the above-mentioned hardness results for marriage games. It was recently shown that this problem is still NP-hard when preferences are strict [19, 22, 26]. Also, finding a maximum-cardinality popular matching in instances where popular matchings are guaranteed to exist is NP-hard [15].

There are less results on *strongly* popular matchings. It is known from Gärdenfors [24] that a strongly popular matching is a unique popular matching and that every strongly popular matching is stable in roommate and marriage games. Based on these insights, Biró et al. [9] showed that strongly popular matchings in roommate games and marriage games with strict preferences can be found efficiently by first computing an arbitrary stable matching and then checking whether it is strongly popular. The case of weak preferences was left open and little progress has been made since then. Király and Mészáros-Karkus [34] recently gave an algorithm for finding strongly popular matchings in marriage games where preferences are strict, except that agents belonging to one side may be completely indifferent. In housing games, a matching is strongly popular if and only if it is a unique perfect matching. Hence, strongly popular matchings in housing games can be found in polynomial time. All of the above mentioned results on strong popularity, including the open problem, follow from Corollary 4.8.

Mixed popular matchings were introduced by Kavitha et al. [31] who also showed how to compute a *fractional* popular matching in housing games and marriage games, which can then be translated into a mixed popular matching via a Birkhoff-von Neumann decomposition. This is possible in these bipartite settings because every fractional matching is implementable as a probability distribution over deterministic matchings. When moving from marriage markets to roommate markets, this does not hold anymore. For example, a matching involving three agents where every pair of agents is matched with probability $1/2$ is not implementable. Huang and Kavitha [28] have shown that in marriage games with strict preferences, the popular matching polytope is half-integral and that half-integral mixed popular matchings can be computed in polynomial time. No such matchings are guaranteed to exist when preferences are weak. They also apply the same techniques to roommate games in order to compute an optimal half-integral solution over the bipartite matching polytope in the case of strict preferences. However, the resulting solutions may again fail to be implementable. Apart from that, their methods heavily rely on computing stable matchings, which may be intractable when preferences are weak. By contrast, our results in Section 4.2.1 are based on the matching polytope for non-bipartite graphs via odd-set constraints and allow both to deal with ties and to efficiently compute a solution that is implementable using LP methods (Proposition 4.2). The axiomatic properties of mixed popular matchings such as efficiency and strategyproofness were investigated by Aziz et al. [6], Brandt et al. [16], and Brandl et al. [12].

To the best of our knowledge, popularity, strong popularity, and mixed popularity have not been studied for coalition formation settings that go beyond coalitions of size 2 except for a theorem by Aziz et al. [5, Th. 15] who claimed that checking whether a partition is popular in ASGHs is NP-hard and that verifying whether

[1]The results by Kavitha et al. [31] only hold for house allocation and marriage markets and cannot be straightforwardly extended to roommate markets. See Section 2 for more details.

[2]See, for example, Biró et al. [9] and Manlove [36]: "A third open problem is the complexity of finding a strongly popular matching (or reporting that none exists), for an instance of RPT [Roommate Problem with Ties]" [9, p. 107]; "Our last open problem concerns the complexity of the problem of finding a strongly popular matching, or reporting that none exists, given an instance of SRTI [Stable Roommates with Ties and Incomplete lists], which is unknown at the time of writing" [36, p. 380].

a partition is popular is coNP-complete. However, the proof of the first statement is incorrect.[3] We substantially modified the reduction to prove a stronger statement and independently proved a stronger statement for the verification problem.

3 PRELIMINARIES

Let N be a finite set of agents. A *coalition* is a non-empty subset of N. By $\mathcal{N}_i$ we denote the set of coalitions agent i belongs to, i.e., $\mathcal{N}_i = \{S \subseteq N : i \in S\}$. A *coalition structure*, or simply a *partition*, is a partition π of the agents N into coalitions, where $\pi(i)$ is the coalition agent i belongs to. A *hedonic game* is a pair $(N, \succsim)$, where $\succsim = (\succsim_i)_{i \in N}$ is a *preference profile* specifying the preferences of each agent i as a complete and transitive *preference relation* $\succsim_i$ over $\mathcal{N}_i$. If $\succsim_i$ is also anti-symmetric we say that i's preferences are *strict*. Otherwise, we say that preferences are *weak*. $S \succ_i T$ if $S \succsim_i T$ but not $T \succsim_i S$—i.e., i *strictly prefers* S to T—and $S \sim_i T$ if both $S \succsim_i T$ and $T \succsim_i S$—i.e., i is *indifferent* between S and T. In hedonic games, agents are only concerned about their own coalition. Accordingly, preferences over coalitions naturally extend to preferences over partitions as follows: $\pi \succsim_i \pi'$ if and only if $\pi(i) \succsim_i \pi'(i)$.

Two basic properties of partitions are Pareto optimality and individual rationality. Given a hedonic game $(N, \succsim)$, a partition π is *Pareto optimal* if there is no partition π' such that $\pi' \succsim_j \pi$ for all agents j and $\pi' \succ_i \pi$ for at least one agent i. Partition π is *individually rational* if $\pi(i) \succsim_i \{i\}$ for all $i \in N$, i.e., each agent i prefers $\pi(i)$ to staying alone. The rationale behind individual rationality is that agents cannot be forced into a coalition.

Individual rationality is also the crucial ingredient of a succinct representation of hedonic games where only the preferences over individual rational coalitions are considered [8]. A hedonic game $(N, \succsim)$ is represented by *Individually Rational Lists of Coalitions* (IRLC) via the game $(N, \succsim')$ where $\succsim'$ is a preference profile such that $\succsim'_i$ is the restriction of $\succsim_i$ to individually rational sets in $\mathcal{N}_i$. In this case, $(N, \succsim)$ is called a *completion* of $(N, \succsim')$. This representation of games is useful to obtain meaningful hardness results because the size of the naive representation of a hedonic game is exponential in the number of agents while the IRLC representation may only require polynomial space if the number of individually rational coalitions is small enough.

In order to define popularity and strong popularity, let $N(\pi, \pi')$ be the set of agents who prefer π over π', i.e., $N(\pi, \pi') = \{i \in N : \pi(i) \succ_i \pi'(i)\}$, where π, π' are two partitions of N. On top of that, we define the *popularity margin* of π and π' as $\phi(\pi, \pi') = |N(\pi, \pi')| - |N(\pi', \pi)|$. Then, π is called *more popular* than π' if $\phi(\pi, \pi') > 0$. Furthermore, π is called *popular* if, for all partitions π', $\phi(\pi, \pi') \geq 0$, i.e., no partition is more popular than π. π is called *strongly popular* if, for all partitions π', $\phi(\pi, \pi') > 0$, i.e., π is more popular than every other partition. Note that there can be at most one strongly popular partition in any hedonic game.

For a hedonic game $(N, \succsim)$ in IRLC representation, a partition π is called *popular* if it is popular in the completion of $(N, \succsim)$ where, for each agent, all coalitions that are not individually rational are gathered in a single indifference class that is less preferred than the singleton coalition. This definition of popularity generalizes the definition of popularity that is used for marriage games by Kavitha et al. [31], and adds the appropriate perspective on individual rationality.[4] Note that a popular partition need not be individually rational.

Many hedonic games do not admit a popular partition. However, existence can be guaranteed by introducing randomization via mixed partitions, i.e., probability distributions over partitions. Let two mixed partitions $p = \{(\pi_1, p_1), \dots, (\pi_k, p_k)\}$ and $q = \{(\sigma_1, q_1), \dots, (\sigma_l, q_l)\}$ be given, where $(p_1, \dots, p_k), (q_1, \dots. q_l)$ are probability distributions. We define the popularity margin of p and q as their expected popularity margin, i.e.,

$$\phi(p, q) = \sum_{i=1}^{k} \sum_{j=1}^{l} p_i q_j \phi(\pi_i, \sigma_j).$$

Clearly, the definition of popularity carries over to the extension of ϕ. As first observed by Kavitha et al. [31], mixed popular partitions always exist, because they can be interpreted as maximin strategies of a symmetric zero-sum game [see, also 6, 23].

Proposition 3.1. *Every hedonic game admits a mixed popular partition.*

Proof. Every hedonic game can be viewed as a two-player symmetric zero-sum game where the rows and columns of the two players are indexed by all possible partitions $\pi_1, \dots, \pi_{B_{|N|}}$ and the entry at position (i, j) of the game matrix is $\phi(\pi_i, \pi_j)$. By the Minimax Theorem [38], the value of this game is 0 and therefore, any maximin strategy, whose existence is guaranteed, is popular. □

4 RESULTS

4.1 Basic Relationships

Clearly, a strongly popular partition is also popular and a popular partition, interpreted as a probability distribution with singleton support, is mixed popular. Furthermore, every coalition structure in the support of a mixed popular partition is Pareto optimal. This already follows from a more general statement by Fishburn [23, Prop. 3]. We give a simple proof for completeness.

Proposition 4.1. *Let $p = \{(\pi_1, p_1), \dots, (\pi_k, p_k)\}$ be a mixed popular partition. Then, for every $i = 1, \dots, k$ with $p_i > 0$, π_i is Pareto optimal.*

Proof. Let $p = \{(\pi_1, p_1), \dots, (\pi_k, p_k)\}$ be a mixed popular partition and fix $i \in \{1, \dots, k\}$ such that $p_i > 0$. Assume for contradiction that π'_i is a Pareto improvement over π_i. Define $p' = \{(\pi_1, p_1), \dots, (\pi_{i-1}, p_{i-1}), (\pi'_i, p_i), (\pi_{i+1}, p_{i+1}), \dots, (\pi_k, p_k)\}$. Note that $\phi(\pi'_i, p) = \sum_{j=1, j\neq i}^{k} p_j \phi(\pi'_i, \pi_j) + p_i \phi(\pi'_i, \pi_i) \geq \sum_{j=1, j\neq i}^{k} p_j \phi(\pi_i, \pi_j) + p_i \phi(\pi'_i, \pi_i) > \sum_{j=1, j\neq i}^{k} p_j \phi(\pi_i, \pi_j) + p_i \phi(\pi_i, \pi_i) = \phi(\pi_i, p)$.

[3] The reduction fails because for a 'yes'-instance of Exact 3-Cover, the partition π claimed to be popular for the ASHG it maps to is not popular: the partition $\pi' = \{\{y^s, z_1^s, z_2^s\} : s \in S\} \cup \{\{b_1^r, a_2^r\} : r \in R\} \cup \{\{b_2^r, a_1^r, a_3^r\} : r \in R\}$ is more popular.

[4] The IRLC representation ignores preferences over coalitions that are not individually rational. However, in contrast to core stability or Nash stability, these preferences can affect whether a partition is popular or not. In order to circumvent this problem one could strengthen the definition of popularity by requiring that a coalition needs to be popular for *all* extensions of the IRLC represented preferences. All our results also hold for this notion, because we construct individually rational partitions for which the two notions of popularity coincide.

Then, $\phi(p', p) = \sum_{j=1, j\neq i}^{k} p_j \phi(\pi_j, p) + p_i \phi(\pi_i', p) > \sum_{j=1, j\neq i}^{k} p_j \phi(\pi_j, p) + p_i \phi(\pi_i, p) = \phi(p, p) = 0$.

Hence, p is not mixed popular, a contradiction. ◻

We thus have the following relationships between strong popularity (sPop), popularity (Pop), partitions in the support of any mixed popular partition (supp(mPop)), and Pareto optimality (PO):

sPop ⟹ Pop ⟹ **supp(mPop)** ⟹ **PO**.

The concepts printed in boldface are guaranteed to exist. As a consequence, hardness results for Pareto optimality imply hardness of mixed popular partitions (though not for popular partitions since they need not exist). The existence problems for popular and strongly popular partitions are naturally contained in the complexity class Σ_2^p. The verification problems are contained in coNP. The coNP-hardness of the verification problem of popular partitions implies coNP-completeness of the verification of mixed popular partitions. This is because every popular partition is a degenerate mixed popular partition and because a mixed popular partition is less popular than another mixed partition if and only if it is less popular than a deterministic partition. Conversely, polynomial-time algorithms for mixed popularity can be used to efficiently verify whether a partition is popular.

4.2 Ordinal Hedonic Games

In this section we investigate hedonic games in IRLC representation. Important subclasses of these games are defined by restricting the size of individually rational coalitions using a global constant. We thus obtain *flatmate games* as games in which only coalitions of up to three agents are individually rational and *roommate games* as games in which only coalitions of size 2 are individually rational. Further restrictions are obtained by bipartitioning the set of agents, say, into males and females and additionally demanding that one group of agents is completely indifferent. A *marriage game* is a roommate game where the agents can be partitioned in two sets such that the only individually rational partitions are formed with agents from the other set. A *housing game* is a marriage game where all agents belonging to one set of the partition are completely indifferent. All of these classes permit polynomially bounded IRLC representations and form the following inclusion relationship when preferences are weak:

Housing ⊂ Marriage ⊂ Roommates ⊂ Flatmates ⊂ IRLC.

In roommate games (and their subclasses), partitions are referred to as *matchings*.

4.2.1 *Roommate Games*.

We start by investigating mixed popularity in roommate games, which will turn out to have important consequences for popular and strongly popular matchings.

Kavitha et al. [31] showed that mixed popular matchings in housing games and marriage games can be found in polynomial time. However, as explained in Section 2, their algorithm cannot be applied to roommate games. In this section, we show how to obtain an algorithm for the more general class of roommate games.

To introduce our matching notation, we fix a graph $G = (N, E)$ where the vertex set is the set of agents and there is an edge between two vertices if the corresponding coalition of size 2 is individually rational for both agents. For technical reasons, it is useful to restrict attention to the case of perfect matchings. Similarly to the construction by Kavitha et al. [31], this can be achieved by introducing worst-case partners w_a for every agent a with $\{a, w_a\} \sim_a \{a\}$. These worst-case partners are not individually rational for all other original agents, and are indifferent among all other agents themselves. They mimic the case that an agent remains unmatched and do not affect the popularity of a partition. We now establish a relationship between mixed matchings and fractional matchings, where the latter are defined as points in the matching polytope $P_{Mat} \subseteq [0,1]^E$, defined as follows [21].

$$P_{Mat} = \{x \in \mathbb{R}^E \colon \sum_{e \in E, v \in e} x(e) = 1 \; \forall v \in N, \quad \sum_{e \in \{\{v,w\} \in E \colon v, w \in C\}} x(e) \leq \frac{|C| - 1}{2} \; \forall C \subseteq N, |C| \text{ odd}, \quad x(e) \geq 0 \; \forall e \in E\}$$

The main constraint ensures that for every odd set of agents C, the weight of the fractional matching restricted to these agents is at most $(|C| - 1)/2$, where this fraction denotes the maximum cardinality that any matching on the set C may have.

Given a matching M, denote by $\chi_M \in P_{Mat}$ its incidence vector. We obtain a correspondence of mixed matchings and fractional matchings by mapping a mixed matching $p = \{(M_1, p_1), \ldots, (M_k, p_k)\}$ to the fractional matching $x_p := \sum_{i=1}^{k} p_i \chi_{M_i}$. Note that $x_p \in P_{Mat}$ by convexity. Since we only want to operate on the more concise matching polytope, we need to ensure that we can recover a mixed matching efficiently. The following proposition, which is based on general LP theory, can be seen as an extension of the Birkhoff-von Neumann theorem to non-bipartite graphs.

Proposition 4.2. *Let $G = (N, E)$ be a graph and $x \in P_{Mat}$ a vector in the associated matching polytope. Then, a mixed matching $p = \{(M_1, p_1), \ldots, (M_k, p_k)\}$ such that $x_p = x$ can be found in polynomial time.*

Proof. The separation problem for the matching polytope P_{Mat} can be solved in polynomial time, i.e., the class of matching polytopes is solvable. Therefore, given a graph $G = (N, E)$ and a vector $x \in P_{Mat}$ we can find a convex combination of extreme points of P_{Mat} that yield x in polynomial time [25, Th. 3.9].

Since the extreme points of the matching polytope are the incidence vectors of matchings [21], this is a mixed matching whose corresponding fractional matching is x. ◻

It thus suffices to define popularity of fractional matchings equivalently to popularity of mixed matchings that induce them. Popular fractional matchings will be described as feasible points of a (non-empty) subpolytope of the matching polytope. The separation problem for the subpolytope will be tractable by a modification of the algorithm that determines the unpopularity margin of a matching given by McCutchen [37].

To this end, we need to define the popularity margin for fractional matchings. Given $x, y \in P_{Mat}$, we define their *popularity* *margin* as

$$\phi(x,y) = \sum_{a \in N} \sum_{i,j \in N_G(a)} x(a,i)y(a,j)\phi_a(i,j)$$

where $N_G(a) = \{v \in N : \{v,a\} \in E\}$ is the neighborhood of a in G and

$$\phi_a(i,j) = \begin{cases} 1 & \text{if } i \succ_a j \\ -1 & \text{if } i \prec_a j \\ 0 & \text{if } i \sim_a j \end{cases}.$$

The proof of the next property is identical to the corresponding statement for marriage games by Kavitha et al. [31].

Proposition 4.3. *Let p and q be mixed matchings. Then,*

$$\phi(p,q) = \phi(x_p, x_q).$$

In particular, p is popular if and only if for all matchings M, $\phi(x_p, x_M) \geq 0$, where $x_M := \chi_M$.

As a consequence, mixed popular matchings correspond precisely to the feasible points of the following polytope.

$$P_{Pop} = \{x \in P_{Mat} : \phi(x, x_M) \geq 0 \text{ for all matchings } M\}$$

It remains to find a feasible point of the popularity polytope P_{Pop}. By adopting the auxiliary graph in McCutchen's algorithm for non-bipartite graphs, we can find a matching M minimizing $\phi(x, x_M)$ by solving a maximum weight matching problem. This solves the separation problem for P_{Pop}.

Proposition 4.4. *The separation problem for P_{Pop} can be solved in polynomial time.*

All missing proofs can be found in the appendix.

We are now ready to prove the following theorem.

Theorem 4.5. *Mixed popular matchings in roommate games with weak preferences can be found in polynomial time.*

Proof. By Proposition 4.4 and by means of the Ellipsoid method [33], we can find a fractional popular matching in polynomial time. This can be translated into a mixed popular matching by Proposition 4.2. □

Theorem 4.5 has a number of interesting consequences. Since every mixed popular matching is Pareto optimal, we now have an LP-based algorithm to find Pareto optimal matchings for weak preferences as an alternative to combinatorial algorithms like the Preference Refinement Algorithm by Aziz et al. [4].

Corollary 4.6. *Pareto optimal matchings in roommate games with weak preferences can be found in polynomial time.*

Biró et al. [9] provided a sophisticated algorithm for verifying whether a given matching is popular. An efficient LP-based algorithm for this problem follows from Theorem 4.5.

Corollary 4.7. *It can be efficiently verified whether a given matching in a roommate game is popular.*

Finally, the linear programming approach allows us to resolve the open problem of finding strongly popular matchings when preferences are weak.

Corollary 4.8. *Finding a strongly popular matching or deciding that no such matching exists in roommate games with weak preferences can be done in polynomial time.*

Proof. If a strongly popular matching exists, it is unique. In particular, it is the unique mixed popular matching. Given a (deterministic) matching M, we can check in polynomial time if it is strongly popular. Simply apply the reduction of Proposition 4.4 and check whether the maximum weight matching amongst the matchings different to M on the auxiliary graph has negative weight (in which case the matching is strongly popular) or not. To this end, we compute a maximum weight matching for every (incomplete) graph that is obtained by deleting exactly one edge from the auxiliary graph. The maximum weight matching amongst these matchings has the highest weight amongst matchings different from M.

The algorithm to compute a strongly popular matching if one exists first computes a fractional popular matching. If it does not correspond to a deterministic matching, there exists no strongly popular matching. Otherwise, it is deterministic and, as described above, we can check if it is strongly popular. If this is the case, we return it. If not, there exists no strongly popular matching. □

Since there can be at most one strongly popular matching, the verification problem for strongly popular matchings in roommate games can also be solved efficiently.

4.2.2 ***Flatmate Games.*** It turns out that moving from coalitions of size 2 to size 3 renders all search problems related to popular partitions intractable. For mixed popular partitions, we can leverage the relationship to Pareto optimal partitions. Aziz et al. [4, Th. 5] have shown that finding Pareto optimal partitions in flatmate games with weak preferences is NP-hard. Since mixed popular partitions are guaranteed to exist (Proposition 3.1) and satisfy Pareto optimality (Proposition 4.1), this immediately implies the NP-hardness of mixed popular partitions by means of a Turing reduction.[5]

Theorem 4.9. *Computing a partition in the support of a mixed popular partition in flatmate games with weak preferences is NP-hard.*

For strict preferences, the same method does not work. Pareto optimal partitions can always be found efficiently by serial dictatorship. Therefore, we will give direct reductions that yield hardness for strong popularity and mixed popularity in flatmate games with strict preferences. These reductions are based on related graphs that correspond to instances of the NP-complete problem X3C [29]. An instance (R,S) of *Exact* 3*-Cover* (X3C) consists of a ground set R together with a set S of 3-element subsets of R. A 'yes'-instance is an instance such that there exists a subset $S' \subseteq S$ that partitions R. We will first describe the graph underlying our hardness constructions, then prove a key property of this graph, and finally give the actual reduction.

To this end, consider an instance (R,S) of X3C. Let $k = \min\{k \in \mathbb{N} : 2^k \geq |R|\}$ be the smallest power of 2 that is larger than the cardinality of R. We define a flatmate game on vertex set $N = \bigcup_{j=0}^{k} N_j$, where $N_j = \bigcup_{i=1}^{2^j} A_j^i$ consists of 2^j sets of agents A_j^i.

We define the sets of agents as

[5]Using the same argument, one can transfer further results on Pareto optimality [4], e.g., for room-roommate games or three-cyclic matching games.

- $A_k^i = \{a_k^i, b_k^i, c_k^i\}$ for $i = 1, \dots, 2^k$, and
- $A_j^i = \{a_j^i, b_j^i, c_j^i, \alpha_j^i, \beta_j^i, \gamma_j^i, \delta_j^i\}$ for $j = 0, \dots, k-1, i = 1, \dots, 2^j$.

Similar names of agents suggest that these agents are going to play the same role in the reduction. The preferences are designed in a way such that if there exists no 3-partition of R through sets in S, then there exists a unique best partition that assigns more than half of the agents a top-ranked coalition. Otherwise, there exists a partition that puts exactly all the other agents in one of their top coalitions. For the sets in the definition of the preferences, an arbitrary tie-breaking can be used to obtain strict preferences. We order the set R in an arbitrary but fixed way, say $R = \{r^1, \dots, r^{|R|}\}$ and for a better understanding of the proof and the preferences, we label the agents $b_k^i = r^i$ for $i = 1, \dots, |R|$. If we view the set of agents N as $k+1$ levels of agents, then the ground set R of the instance of X3C is identified with some specific agents in the top level k. Preferences of the agents are as follows.

- $\{a_k^i, b_k^i, c_k^i\} \succ_{a_k^i} \{a_k^i\}$, $i = 1, \dots, 2^k$
- $\{a_j^i, \beta_j^i, \gamma_j^i\} \succ_{a_j^i} \{a_j^i, b_j^i, c_j^i\} \succ_{a_j^i} \{a_j^i\}$, $j = 0, \dots, k-1, i = 1, \dots, 2^j$
- $\{\{b_k^i, b_k^v, b_k^w\} \colon \{r^i, r^v, r^w\} \in S \text{ for some } 1 \leq v, w \leq |R|\} \succ_{b_k^i} \{a_k^i, b_k^i, c_k^i\} \succ_{b_k^i} \{b_k^i\}$, $i = 1, \dots, |R|$
- $\{b_k^i\}$, $i = |R| + 1, \dots, 2^k$
- $\{b_j^i, c_{j+1}^{2i-1}, c_{j+1}^{2i}\} \succ_{b_j^i} \{a_j^i, b_j^i, c_j^i\} \succ_{b_j^i} \{b_j^i\}$, $j = 0, \dots, k-1, i = 1, \dots, 2^j$
- $\{a_j^i, b_j^i, c_j^i\} \succ_{c_j^i} \{c_j^i\}$, $j = 0, \dots, k, i = 1, \dots, 2^j$
- $\{\alpha_j^i, \beta_j^i\} \succ_{\alpha_j^i} \{\alpha_j^i\}$, $j = 0, \dots, k-1, i = 1, \dots, 2^j$
- $\{\beta_j^i, \gamma_j^i, a_j^i\} \succ_{\beta_j^i} \{\beta_j^i, \alpha_j^i\} \succ_{\beta_j^i} \{\beta_j^i\}$, $j = 0, \dots, k-1, i = 1, \dots, 2^j$
- $\{\gamma_j^i, \delta_j^i\} \succ_{\gamma_j^i} \{\gamma_j^i\}$, $j = 0, \dots, k-1, i = 1, \dots, 2^j$
- $\{\delta_j^i, \alpha_{j+1}^{2i-1}, \alpha_{j+1}^{2i}\} \succ_{\delta_j^i} \{\delta_j^i, \gamma_j^i\} \succ_{\delta_j^i} \{\delta_j^i\}$, $j = 0, \dots, k-1, i = 1, \dots, 2^j$

The structure of the flatmate game is illustrated in Figure 1 for the case $k = 3$. We will be particularly interested in coalitions of the types $\{a_j^i, b_j^i, c_j^i\}$, $\{\alpha_j^i, \beta_j^i\}$, and $\{\gamma_j^i, \delta_j^i\}$, which are indicated by undirected edges. These coalitions form the partition π^* of Lemma 4.10 that we need later to investigate for strong and mixed popularity in the respective reductions. The directed edges indicate that an agent at the tail of the arrow needs to form a coalition with the agent at the tip of the arrow in order to improve from her coalition of the above type. The set of agents consists of a binary tree of triangles depicted by the circular-shaped vertices. The important property of this tree is that whenever a coalition of the type $\{a_j^i, b_j^i, c_j^i\}$ gets dissolved, there can only be an improvement in popularity for the agents in A_j^i if they propagate changes in the partition upwards within this tree. This is achieved for agents b_j^i directly through the binary tree and for agents a_j^i with help of the auxiliary agents $\{\alpha_j^i, \beta_j^i, \gamma_j^i, \delta_j^i\}$ that are depicted as diamond-shaped vertices.

In the following lemma and theorem, we denote for any subset $M \subseteq N$ of agents and partitions π, π' of N, $\phi_M(\pi, \pi') = |N(\pi, \pi') \cap M| - |N(\pi', \pi) \cap M|$, that is, the popularity margin on a the subset M with respect to π and π'.

Lemma 4.10. *Let an instance (R, S) of X3C be given and define the corresponding flatmate game as above. Consider the partition $\pi^* = \{\{a_j^i, b_j^i, c_j^i\} \colon j = 0, \dots, k, i = 1, \dots, 2^j\} \cup \{\{\alpha_j^i, \beta_j^i\}, \{\gamma_j^i, \delta_j^i\} \colon j = 0, \dots, k-1, i = 1, \dots, 2^j\}$. Let $\pi \neq \pi^*$ be an arbitrary partition of agents distinct from π^*. Then $\phi(\pi^*, \pi) \geq 1$. In addition, if $c_0^1 \in N(\pi^*, \pi)$, then $\phi(\pi^*, \pi) \geq 3$ or $\{b_k^i \colon i = 1, \dots, 2^k\} \subseteq N(\pi, \pi^*)$.*

Proof sketch. Let an instance (R, S) of X3C be given and define the corresponding flatmate game as above. Let π^* be defined as in the lemma and $\pi \neq \pi^*$ an other partition. We recursively define the following sets of agents: for $i = 1, \dots, 2^k$, $T_k^i = A_k^i$ and for $j = k-1, \dots, 0, i = 1, \dots, 2^j$, $T_j^i = A_j^i \cup T_{j+1}^{2i-1} \cup T_{j+1}^{2i}$. The core of the proof is the following claim that can be proved by induction over $j = k, \dots, 0$.

For every $i = 1, \dots, 2^j$ holds: Assume there exists an agent $x \in T_j^i$ with $\pi(x) \neq \pi^*(x)$. Then $\phi_{T_j^i}(\pi^*, \pi) \geq 1$. If even $\pi(a_j^i) \neq \pi^*(a_j^i)$, then $\phi_{T_j^i}(\pi^*, \pi) \geq 3$ or $\{b_k^i \colon i = 1, \dots, 2^k\} \cap T_j^i \subseteq N(\pi, \pi^*)$.

For the induction step, one essentially proves that changing the coalitions in A_j^i causes severe loss in popularity, unless we propagate changes to substructures via b_j^i or β_j^i. Clearly, the assertion of the lemma follows from the case $j = 0$. □

We are now ready to apply the lemma for the desired reductions.

Theorem 4.11. *Deciding whether there exists a strongly popular partition in flatmate games is coNP-hard, even if preferences are strict.*

Proof. The reduction is from X3C. Given an instance (R, S) of X3C, we define a hedonic game on agent set $N' = N \cup \{z\}$ where the agents N are as in the above construction with the identical preferences and $\{c_0^1, z\} \succ_z \{z\}$. Note that $|N'| = 3\sum_{j=0}^{k} 2^j + 4\sum_{j=0}^{k-1} 2^j + 1 = 10 \cdot 2^k - 6 = O(|R|)$ and the reduction is in polynomial time.

Consider the partition $\pi^* = \{\{a_j^i, b_j^i, c_j^i\} \colon j = 0, \dots, k, i = 1, \dots, 2^j\} \cup \{\{\alpha_j^i, \beta_j^i\}, \{\gamma_j^i, \delta_j^i\} \colon j = 0, \dots, k-1, i = 1, \dots, 2^j\} \cup \{\{z\}\}$. It follows directly from Lemma 4.10 that π^* is popular and hence there exists a strongly popular partition if and only if π^* is strongly popular. We will prove that this is the case if and only if the instance of X3C is a 'no'-instance.

Assume that there exists no 3-partition of R through sets in S and let an arbitrary partition $\pi \neq \pi^*$ be given. Then there exists an agent $x \in N$ with $\pi(x) \neq \pi^*(x)$ and it follows from Lemma 4.10 that $\phi(\pi^*, \pi) \geq \phi_N(\pi^*, \pi) - 1 \geq 3 - 1 = 2$. Hence, π^* is strongly popular.

Conversely, assume that there exists a 3-partition $S' \subseteq S$ of R. Define $\pi = \{\{b_k^v, b_k^w, b_k^x\} \colon \{v, w, x\} \in S'\} \cup \{\{b_k^i\} \colon i = |R| + 1 \dots, 2^k\} \cup \{\{\delta_{k-1}^i, a_k^{2i-1}, a_k^{2i}\} \colon i = 1, \dots, 2^{k-1}\} \cup \{\{b_j^i, c_{j+1}^{2i-1}, c_{j+1}^{2i}\}, \{\delta_j^i, \alpha_{j+1}^{2i-1}, \alpha_{j+1}^{2i}\}, \{a_j^i, \beta_j^i, \gamma_j^i\} \colon j = 1, \dots, k-1, i = 1, \dots, 2^j\} \cup \{\{\alpha_0^1\}, \{z, c_0^1\}\}$. It is easily checked that $\phi(\pi, \pi^*) = 0$.

Indeed, $N(\pi, \pi^*) = \{b_k^i \colon i = 1, \dots, 2^k\} \cup \{\beta_j^i, \delta_j^i, a_j^i \colon j = 0, \dots, k-1, i = 1, \dots, 2^j\} \cup \{z\}$. Therefore, $|N(\pi, \pi^*)| = 2^k + 4\sum_{j=1}^{k-1} 2^j + 1 = 5 \cdot 2^k - 3 = \frac{1}{2}|N'|$. Hence, $\phi(\pi, \pi^*) \geq 0$ and equality follows from popularity of π^*. Therefore, there exists no strongly popular partition. □

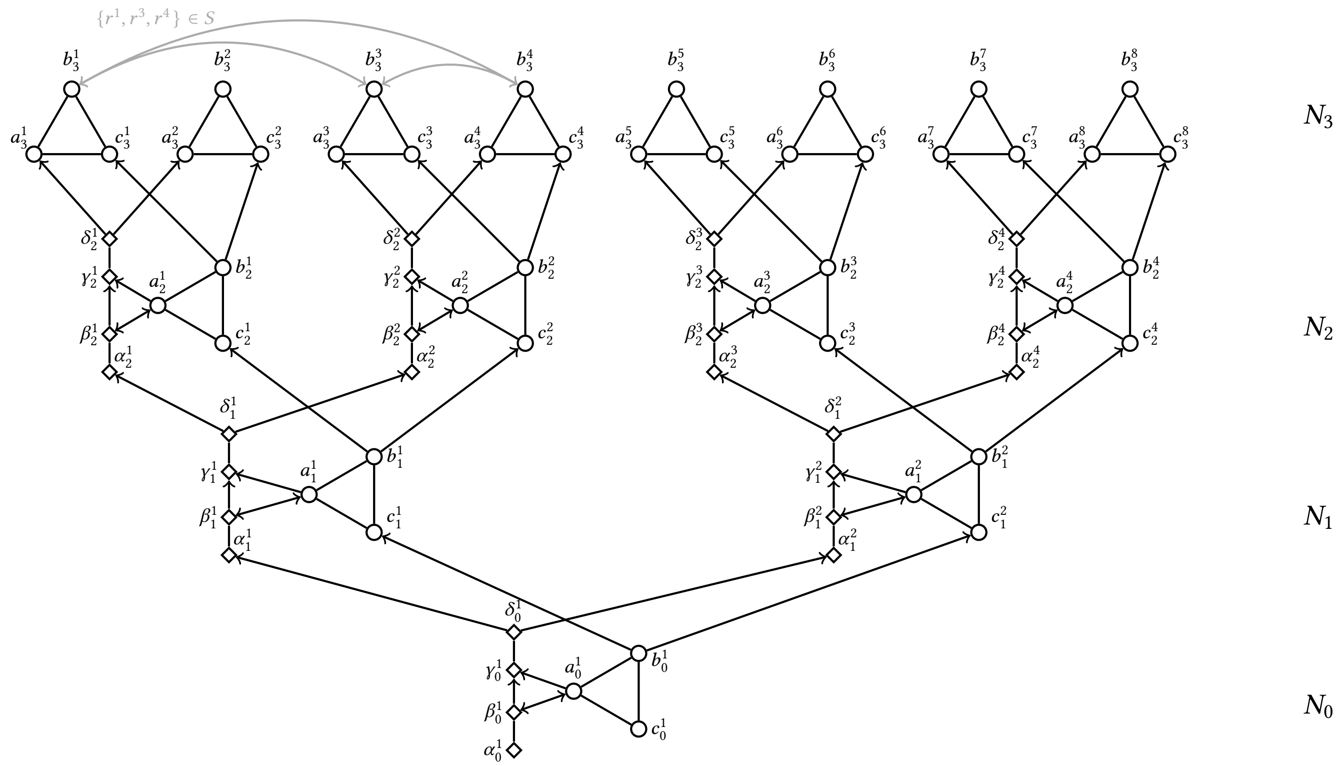

Figure 1: Schematic of the reduction for flatmate games with strict preferences. There is an edge between two agents if they are in the coalition π^* defined in Lemma 4.10. Directed edges indicate improvements from π^*. The gray edges suggest a 3-elementary set in S.

A similar reduction as in Theorem 4.11 works also for mixed popularity. However, we need two auxiliary agents to control the switch between a strongly popular and non-popular partition.

Theorem 4.12. *Computing a mixed popular partition in flatmate games is NP-hard, even if preferences are strict.*

To conclude the section, we deal with the problem of verifying whether a given partition is popular or strongly popular. Hardness of verifying popular partitions in flatmate games is shown by a complicated reduction from E3C. We have gadgets for elements in S and control the switch between 'yes' and 'no' instances by means of a binary tree. For a simpler, yet weaker hardness result, this tree could be contracted into a single vertex, but only at the expense of having to allow for coalitions of more than three agents.

Theorem 4.13. *Verifying whether a given partition in a flatmate game with strict preferences is popular is coNP-complete.*

For strong popularity, we obtain the same result.

Theorem 4.14. *Verifying whether a given partition in a flatmate game is strongly popular is coNP-complete, even if preferences are strict.*

Proof. In the proof of Theorem 4.11, the partition π^* is strongly popular if, and only if, (R, S) is a 'no'-instance of X3C. □

4.2.3 ***Globally Ranked Preferences.*** A natural question that arises after hardness results have been established is whether there are meaningful preference restrictions under which these results do not hold. In many cases, hardness breaks down when assuming that preference profiles adhere to certain structural restrictions. One such preference restriction that has been considered in the domain of coalition formation is that there exists one common global ranking $\succsim$ of all coalitions in $2^N \setminus \{\emptyset\}$ and each individual preference relation $\succsim_i$ is the restriction of $\succsim$ to $\mathcal{N}_i$. It is known that under these *globally ranked preferences*, every roommate game admits a stable matching, which can furthermore be efficiently computed [1]. Since every stable matching also happens to be popular (see Section 2), this implies that computing popular matchings in roommates games, which was recently shown to be NP-hard [19, 22, 26], becomes tractable under globally ranked preferences.

By contrast, all hardness results shown in Section 4.2.2 hold even when preferences are globally ranked. This confirms the robustness of these results and underlines the crucial difference between settings with coalitions of size 2 and coalitions of size 3.

4.3 Cardinal Hedonic Games

Important subclasses of hedonic games that admit succinct representations are based on cardinal utility functions. For one, there are *additively separable hedonic games* [10], where the utility that an

	weak preferences				*strict preferences*			
	PO	mPop	sPop	Pop	PO	mPop	sPop	Pop
IRLC	↑	↑	↑	↑	in P	↑	↑	↑
Flatmates	NP-h.[a]	NP-h. (Th. 4.9)	NP-h. (Th. 4.11)	↑	↓	NP-h. (Th. 4.12)	NP-h. (Th. 4.11)	↑
Roommates	in P[b]	in P (Th. 4.5)	in P (Cor. 4.8)	↑	↓	in P (Th. 4.5)	in P[d]	NP-h.[g]
Marriage	↓	↓	↓	NP-h.[e]	↓	↓	↓	in P[f]
Housing	↓	↓	↓	in P[c]	in P	in P[h]	in P	in P[c]

Table 1: Complexity of finding partitions in ordinal hedonic games. New results are highlighted in gray and implications are marked by gray arrows. NP-hardness of computing a popular or strongly popular partition always follows by a Turing reduction from the existence problem. Whenever computing a mixed popular partition is NP-hard, then verifying a deterministic partition is coNP-complete.

[a]: Aziz et al. [4, Th. 5], [b]: Aziz et al. [4, Th. 7], [c]: Abraham et al. [2, Th. 3.9], [d]: Biró et al. [9, Th. 6], [e]: Biró et al. [9, Th. 11], Cseh et al. [18, Th. 2], [f]: Gärdenfors [24, Th. 3], [g]: Gupta et al. [26, Th. 1.1], Faenza et al. [22, Th. 4.6], Cseh and Kavitha [19, Th. 2], [h]: Kavitha et al. [31, Th. 2]

agent associates with a coalition is the sum of utilities he ascribes to each member of the coalition. On the other hand, there are *fractional hedonic games* [3], where the sum of utilities is divided by the number of agents contained in the coalition.

In the following, let $v_i(j)$ denote the utility that agent i associates with agent j. A hedonic game $(N, \succsim)$ is an *additively separable hedonic game* (ASHG) if there is $(v_i(j))_{i,j \in N}$ that for every agent i, the preferences $\succsim_i$ are induced by the cardinal utilities given by $v(S) = \sum_{j \in S} v_i(j)$, for $S \subseteq N$. The hedonic game $(N, \succsim)$ is a *fractional hedonic game* (FHG) if there exists $(v_i(j))_{i,j \in N}$ such that for every agent i, the preferences $\succsim_i$ are induced by the cardinal utilities given by $v(S) = (\sum_{j \in S} v_i(j))/|S|$, for $S \subseteq N$. We focus on *symmetric* ASHGs and FHGs, i.e., games for which $v_i(j) = v_j(i)$ for all $i, j \in N$.

All hardness results in this section are obtained by rather involved reductions from E3C.

Theorem 4.15. *Checking whether there exists a popular partition in a symmetric ASHG is NP-hard.*

The verification problem for ASHGs turns out to be coNP-complete. The proof of Theorem 4.16 is simpler and holds for a more restricted class of games than the proof by Aziz et al. [5].

Theorem 4.16. *Checking whether a given partition in a symmetric ASHG is popular is coNP-complete.*

The reductions for mixed and strong popularity on ASHGs rely on a similar idea as for flatmate games with strict preferences. We find a graph that satisfies similar properties as the flatmate game considered in Lemma 4.10. This graph is used for the next four results.

Theorem 4.17. *Checking whether there exists a strongly popular partition in a symmetric ASHG is coNP-hard.*

Theorem 4.18. *Verifying whether a given partition in a symmetric ASHG is strongly popular is coNP-complete.*

Theorem 4.19. *Computing a mixed popular partition in a symmetric ASHG is NP-hard.*

We even obtain coNP-hardness of the existence of popular partitions.

Theorem 4.20. *Checking whether there exists a popular partition in a symmetric ASHG is coNP-hard.*

Theorem 4.21. *Checking whether there exists a popular partition in a symmetric FHG is NP-hard, even if all weights are non-negative.*

The hardness proof for the verification problem for FHGs is a more involved version of the proof for ASHGs.

Theorem 4.22. *Checking whether a given partition in a symmetric FHG is popular is coNP-complete, even if all weights are non-negative and the underlying graph is bipartite.*

The graphs used in the proof of Theorem 4.22 have girth 6. This is in contrast to the polynomial-time algorithm by Aziz et al. [3] for computing the core on FHGs with girth at least 5.

5 CONCLUSION

We have investigated the computational complexity of finding and recognizing popular, strongly popular, and mixed popular partitions in various types of ordinal hedonic games (see Table 1) and cardinal hedonic games. Two important factors that govern the complexity of computing these partitions in ordinal hedonic games are whether preferences may contain ties and whether coalitions of size 3 are allowed. When preferences are weak, computing mixed popular and strongly popular partitions is only difficult for representations for which we cannot even compute Pareto optimal partitions efficiently. For strict preferences, however, Pareto optimal partitions can be found efficiently while computing mixed popular and strongly popular partitions remains intractable, even when preferences are globally ranked. Our positive results are obtained via a single linear programming approach that unifies a number of existing results and exploits the relationships between the different types of popularity. Finally, we complete the picture by providing a variety of results showing the intractability of popular, strongly popular, and mixed popular partitions in ASHGs and FHGs.

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A APPENDIX: PROOFS

This appendix contains all omitted proofs.

Throughout the appendix, we denote for any subset $M \subseteq N$ of agents and partitions σ, σ' of N, $\phi_M(\sigma, \sigma') = |N(\sigma, \sigma') \cap M| - |N(\sigma', \sigma) \cap M|$, that is, the popularity margin on a the subset M with respect to σ and σ'.

A.1 Ordinal Hedonic Games

Proof of Proposition 4.4. Assume that a vector $x \in \mathbb{R}^E$ is given. The separation problem for the matching polytope can be solved in polynomial time. For the popularity constraints, we assign weights w_x to the edges of the underlying graph such that for all matchings M on G, $w_x(M) = \phi(x_M, x)$. Therefore, their separation problem turns into finding a maximum weight matching, which can be done in polynomial time.

We define the weights by letting

$$w_x(i,j) = \sum_{a \in N_G(i)} x(i,a)\phi_i(j,a) + \sum_{a \in N_G(j)} x(j,a)\phi_j(i,a).$$

Given a matching M and an agent a, denote by $M(a)$ the agent, a is matched with. We compute

$$\begin{aligned}\phi(x_M, x) &= \sum_{a \in N} \sum_{i,j \in N_G(a)} x_M(a,i)x(a,j)\phi_a(i,j) \\ &= \sum_{a \in N} \sum_{i,j \in N_G(a)} \chi_M(a,i)x(a,j)\phi_a(i,j) \\ &= \sum_{a \in N} \sum_{j \in N_G(a), i = M(a)} x(a,j)\phi_a(i,j).\end{aligned}$$

On the other hand,

$$\begin{aligned}w_x(M) &= \sum_{\{i,j\} \in M} \left[\sum_{b \in N_G(i)} x(i,b)\phi_i(j,b) \right. \\ &\quad \left. + \sum_{b \in N_G(j)} x(j,b)\phi_j(i,b) \right] \\ &= \sum_{\{i,j\} \in M} \left[\sum_{b \in N_G(i), j = M(i)} x(i,b)\phi_i(j,b) \right. \\ &\quad \left. + \sum_{b \in N_G(j), i = M(j)} x(j,b)\phi_j(i,b) \right] \\ &= \sum_{a \in N, a \text{ matched}} \sum_{j \in N_G(a), i = M(a)} x(a,j)\phi_a(i,j) \\ &= \sum_{a \in N} \sum_{j \in N_G(a), i = M(a)} x(a,j)\phi_a(i,j)\end{aligned}$$

The last equation is due to the fact that the inner sum is empty for unmatched agents in M. Putting everything together, we conclude that $\phi(x_M, x) = w_x(M)$ which completes the proof. □

Proof of Theorem 4.12. We provide a Turing reduction from X3C to the problem of finding a partition in the support of a mixed popular partition together with its probability in this mixed partition.

Given an instance $X3C$, we compute a very similar game as in the proof of Theorem 4.11. We have $N' = N \cup \{z_1, z_2\}$ where the agents N are as in the above construction with the identical preferences and $\{c_0^1, z_1, z_2\} \succ_{z_i} \{z_i\}$ for $i = 1, 2$. By Lemma 4.10, the partition $\pi^* = \{\{a_j^i, b_j^i, c_j^i\} : j = 0, \dots, k, i = 1, \dots, 2^j\} \cup \{\{\alpha_j^i, \beta_j^i\}, \{\gamma_j^i, \delta_j^i\} : j = 0, \dots, k, i = 1, \dots, 2^j \text{ odd}\} \cup \{\{z_1\}, \{z_2\}\}$ is strongly popular if there exists no 3-partition of R through sets in S. Therefore the unique mixed popular partition assigns probability 1 to π^*.

On the other hand, assume that there exist a 3-partition $S' \subseteq S$ of R. Define $\pi = \{\{b_k^v, b_k^w, b_k^x\} : \{v, w, x\} \in S'\} \cup \{\{b_k^i\} : i = |R| + 1 \dots, 2^k\} \cup \{\{\delta_{k-1}^i, a_k^{2i-1}, a_k^{2i}\} : i = 1, \dots, 2^{k-1}\} \cup \{\{b_j^i, c_{j+1}^{2i-1}, c_{j+1}^{2i}\}, \{\delta_j^i, \alpha_{j+1}^{2i-1}, \alpha_{j+1}^{2i}\}, \{a_j^i, \beta_j^i, \gamma_j^i\} : j = 1, \dots, k-1, i = 1, \dots, 2^j\} \cup \{\{\alpha_0^1\}, \{z_1, z_2, c_0^1\}\}$. It is easily checked that $\phi(\pi, \pi^*) = 1$. Therefore, there exists no mixed popular partition that assigns probability 1 to π^*.

We can solve X3C by computing a partition π in the support of a mixed popular partition and checking its probability in case $\pi = \pi^*$. □

Proof of Theorem 4.13. The problem is in coNP, because a more popular partition serves as a polynomial-time certificate for a 'no'-instance.

Let therefore an instance (R, S) of X3C be given. We define a flatmate game $(N, \succsim)$ (in IRLC representation) with $N = R_1 \cup R_2 \cup B \cup \bigcup_{k=1}^6 S_k$. There, we define $R_j = \{r_j : r \in R\}$, $S_i = \{s_i : s \in S\}$, $j = 1, 2$, $i = 1, \dots, 6$, $V = \bigcup_{k=1}^6 S_k$, and $B = \cup_{j=1}^l B_j$ for $l = \lceil \log_2(|R|) \rceil$.

It remains to define the sets B_j, $j = 1, \dots, l$. To this extent, we define $m_0 = |R|$ and iteratively for $j = 1, \dots, l$, $m_j = \lceil m_{j-1}/2 \rceil$. Note that, by construction, $m_{l-1} = 2$ and $m_l = 1$. For $j = 1, \dots, l-1$, set $B_j = \{b_j^i, c_j^i : i = 1, \dots, m_j\}$ and $B_l = \{b\}$.

To define the preference lists, we order the set $R = \{r^1, \dots, r^{|R|}\}$ arbitrarily. In addition, the agents in $V \cup R_1$ are only allowed to be part of very specific coalitions of size 3 that rely on fixed orders of the sets $s \in S$. Therefore, take for $s \in S$ a fixed order $s = \{\alpha^1, \alpha^2, \alpha^3\}$ and have the possible sets as $\mathcal{S} = \{\{\alpha_1^1, \alpha_1^2, s_1\}, \{\alpha_1^3, s_3\} : s \in S\}$. Preference lists are given as follows:

- $\{b, c_{l-1}^1, c_{l-1}^2\} \succ_b \{b\}$
- $\{b_j^i, c_{j-1}^{2i-1}, c_{j-1}^{\max\{2i, m_{j-1}\}}\} \succ_{b_j^i} \{b_j^i, c_j^i\} \succ_{b_j^i} \{b_j^i\}$, $j = 1, \dots, l-1$, $i = 1, \dots, m_j$
- $\{c_{l-1}^i, b_{l-1}^i\} \succ_{c_{l-1}^i} \{b, c_{l-1}^1, c_{l-1}^2\} \succ_{c_{l-1}^i} \{c_{l-1}^i\}$, $i = 1, 2$
- $\{c_j^i, b_j^i\} \succ_{c_j^i} \{c_j^i, c_j^{\max\{i+1, m_j\}}, b_{j+1}^{(i+1)/2}\} \succ_{c_j^i} \{c_j^i\}$, $j = 1, \dots, l-2$, $i = 1, \dots, m_j$, i odd
- $\{c_j^i, b_j^i\} \succ_{c_j^i} \{c_j^i, c_j^{i-1}, b_{j+1}^{i/2}\} \succ_{c_j^i} \{c_j^i\}$, $j = 1, \dots, l-2$, $i = 1, \dots, m_j$, i even
- $\{r_2^i, r_1^i\} \succ_{r_2^i} \{r_2^i, r_2^{\max\{i+1, m_0\}}, b_1^{(i+1)/2}\} \succ_{r_2^i} \{r_2^i\}$, $i = 1, \dots, m_0$, i odd
- $\{r_2^i, r_1^i\} \succ_{r_2^i} \{r_2^i, r_2^{i-1}, b_1^{i/2}\} \succ_{r_2^i} \{r_2^i\}$, $i = 1, \dots, m_0$, i even
- $\{C \in \mathcal{S} : r_1 \in C\} \succ_{r_1} \{r_1, r_2\} \succ_{r_1} \{r_1\}$, $r \in R$
- $\{\alpha_1^1, \alpha_1^2, s_1\} \succ_{s_1} \{s_1, s_2, s_5\} \succ_{s_1} \{s_1\}$, $\{\alpha^1, \alpha^2, \alpha^3\} = s \in S$
- $\{s_1, s_2, s_5\} \succ_{s_2} \{s_2\}$, $s \in S$
- $\{\alpha_1^3, s_3\} \succ_{s_3} \{s_3, s_4, s_6\} \succ_{s_3} \{s_3\}$, $\{\alpha^1, \alpha^2, \alpha^3\} = s \in S$
- $\{s_3, s_4, s_6\} \succ_{s_4} \{s_4\}$, $s \in S$

- $\{s_1, s_2, s_5\} \succ_{s_5} \{s_5, s_6\} \succ_{s_5} \{s_5\}$, $s \in S$
- $\{s_5, s_6\} \succ_{s_6} \{s_3, s_4, s_6\} \succ_{s_6} \{s_6\}$, $s \in S$

Here, $\{\alpha^1, \alpha^2, \alpha^3\}$ is in the order that was used for the definition of $\mathcal{S}$. The sets in the preferences of the agents of type r_1, $r \in R$, can be ordered arbitrarily in a linear way. The reduction is visualized in Figure 2. The graph on vertex set N contains an edge whenever two agents are in a common individually rational coalition.

Note that the number of agents is polynomially bounded in $|R|$ and $|S|$ (it is even still linear) and all lists contain polynomially many sets of size at most 3. We want to verify whether the partition $\pi = \{\{s_1, s_2, s_5\}, \{s_3, s_4, s_6\} : s \in S\} \cup \{\{r_1, r_2\} : r \in R\} \cup \{\{b^i_j, c^i_j\} : j = 1, \dots, l-1, i = 1, \dots, m_j\} \cup \{b\}$ is popular. We claim that (R, S) is a 'yes'-instance of X3C if and only if π is not popular for the hedonic game given by $(N, \succsim)$.

Assume first that (R, S) is a 'yes'-instance. Then, there exists a 3-partition $S' \subseteq S$ of R. We label $c^i_0 = r^i_2$, $i = 1, \dots, |R|$.

The partition

$$\pi' = \{\{b, c^1_{l-1}, c^2_{l-1}\}\} \cup$$
$$\{\{b^i_j, c^{2i-1}_{j-1}, c^{\max\{2i, m_{j-1}\}}_{j-1}\} : 1 \le j \le l-1, 1 \le i \le m_j\} \cup$$
$$\{\{\alpha^1_1, \alpha^2_1, s_1\}, \{\alpha^3_1, s_3\}, \{s_2\}, \{s_4\}, \{s_5, s_6\} : \{\alpha^1, \alpha^2, \alpha^3\}$$
$$= s, s \in S'\} \cup \{\{s_1, s_2, s_5\}, \{s_3, s_4, s_6\} : s \in S \setminus S'\}$$

is more popular than π.

Conversely, assume that π' is more popular than π. We collect the following facts:

- For $j = 1, \dots, l-1$, $i = 1, \dots, m_j$, $c^i_j \notin N(\pi', \pi)$ and if $b^i_j \in N(\pi', \pi)$, then $c^i_j \in N(\pi, \pi')$.
- For $r \in R$, $r_2 \notin N(\pi', \pi)$ and if $r_1 \in N(\pi', \pi)$, then $r_2 \in N(\pi, \pi')$.
- For $s \in S$, $s_2, s_4, s_5 \notin N(\pi', \pi)$. Moreover, if $s_1 \in N(\pi', \pi)$, then $s_2 \in N(\pi, \pi')$, if $s_3 \in N(\pi', \pi)$, then $s_4 \in N(\pi, \pi')$, and if $s_6 \in N(\pi', \pi)$, then $s_5 \in N(\pi, \pi')$.
- For $s \in S$, $s_1 \in N(\pi', \pi)$ if and only if $s_3 \in N(\pi', \pi)$.

For the last fact observe, e.g., if $s_1 \in N(\pi', \pi)$ and $s_3 \notin N(\pi', \pi)$, then $s_2, s_5 \in N(\pi, \pi')$ and $(s_3, s_4, s_6 \notin N(\pi', \pi) \vee s_3, s_4 \in N(\pi, \pi'))$. Together with the first three facts (and even if $b \in N(\pi', \pi)$), this implies that $\phi(\pi', \pi) \le 0$, a contradiction.

The only possibility for $\phi(\pi', \pi) > 0$ is that $b \in N(\pi', \pi)$. It follows inductively for $j = l-1, \dots, 1$, $i = 1, \dots, m_j$, that $c^i_j \in N(\pi, \pi')$ and $b^i_j \in N(\pi', \pi)$. Therefore, $R_2 \subseteq N(\pi, \pi')$, hence $R_1 \subseteq N(\pi', \pi)$. From the last fact we can deduce that the coalitions of the agents in R_1 induce a 3-partition of R with sets in S. ◻

A.2 Additively separable hedonic games

Next, we consider the existence problem for ASHGs. The overall strategy is as follows:

For a reduction from X3C, given an instance (R, S), we have R-gadgets for every element of the ground set R and S-gadgets for every 3-elementary set in S. The gadgets for elements of R rely on an instance of a hedonic game that admits no popular partition. There are smaller examples with this property than the one used in the proof, but the instance used in the proof fulfills further properties required for the reduction to work. The gadgets for the sets in S consist of three agents that are very happy in a coalition of their own, but one of them is linked to the R-gadgets of the set and can simultaneously prevent the agents in the R-gadgets from voting down a partition. This is of course at the expense of the happiness of agents in the S-gadgets and can only happen if at least three R-gadgets are simultaneously dealt with. This is where we achieve the correspondence of the covering with 3-partitions.

We start by having a look at the instance of the hedonic game that defines the R-gadgets later on. In the whole proof there will only be one negative weight present that is large enough such that agents cannot form a coalition if they negatively influence each other and we want to end up with a popular partition. All instances in this section are symmetric and we therefore denote $v(i, j) = v_i(j) = v_j(i)$.

Proposition A.1. *Let $0 < \epsilon < 1$ and $K \ge 4$. Consider the following ASHG, depicted in Figure 3: $N = \{a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2\}$ such that $v(a_i, c_1) = 2, v(a_i, c_2) = 1, v(a_i, b_i) = \epsilon, v(b_i, c_2) = 0$ for all $i = 1, 2, 3$ and $v(x, y) = -K$ for all other valuations not defined. Then, there exists no popular partition.*

Proof. Assume for contradiction that π was a popular partition. Then the following facts hold:

- $a_i \notin \pi(a_j), i \ne j$
- $a_i \notin \pi(b_j), i \ne j$
- $b_i \notin \pi(b_j), i \ne j$
- $c_1 \notin \pi(c_2), c_1 \notin \pi(b_j)$

In all of these cases, dissolving the coalition in question would be more popular, because all but possibly one agent in the coalition have negative utility and an agent with non-negative utility can only be contained in the coalition if it contains at least 3 agents. Note that K is larger than the sum of positive weights incident to any agent and therefore its utility is negative once it is in a coalition with an agent that gives negative utility.

Now, for every j, exactly one of the following holds: $c_1 \in \pi(a_j)$ or $b_j \in \pi(a_j)$. In fact, both cannot hold as excluded above. If none holds, then $\pi(a_j) \subseteq \{a_j, c_2\}$ and we could delete b_2 from its coalition (making no agent worse) and add it to $\pi(a_j)$, resulting in a more popular partition.

Next, for $i = 1, 2$, there exists j with $c_i \in \pi(a_j)$. Otherwise, there existed k with $\pi(a_k) \subseteq \{a_k, b_k\}$ and removing b_k and adding c_i is more popular.

Up to symmetry, the only possible partition π is therefore $\{\{a_1, c_1\}, \{b_1\}, \{a_2, c_2, b_2\}, \{a_3, b_3\}\}$. But then $\{\{a_2, c_1\}, \{b_2\}, \{a_3, c_2, b_3\}, \{a_1, b_1\}\}$ is more popular. Hence, π was not popular. ◻

Proof of Theorem 4.15. The reduction is from X3C to deciding whether there exists a popular partition.

Let (R, S) be an instance of X3C. This can be reduced to an instance $(N, \succsim)$, where $(N, \succsim)$ is an ASHG defined in the following way.

Let $N = \{a^r_1, a^r_2, a^r_3, b^r_1, b^r_2, b^r_3, c^r_1, c^r_2 : r \in R\} \cup \{y^s, z^s_1, z^s_2 : s \in S\}$ and edge weights be as follows:

- $v(a^r_i, c^r_1) = 2$ and $v(a^r_i, c^r_2) = 1, v(a^r_i, b^r_i) = \epsilon, v(b^r_i, c^r_2) = 0$ for all $i = 1, 2, 3$ and $r \in R$;
- $v(a^r_3, a^{r'}_3) = 0, v(b^r_3, a^{r'}_3) = 0, v(b^r_3, b^{r'}_3) = 0$ for all $r, r' \in s \in S$;

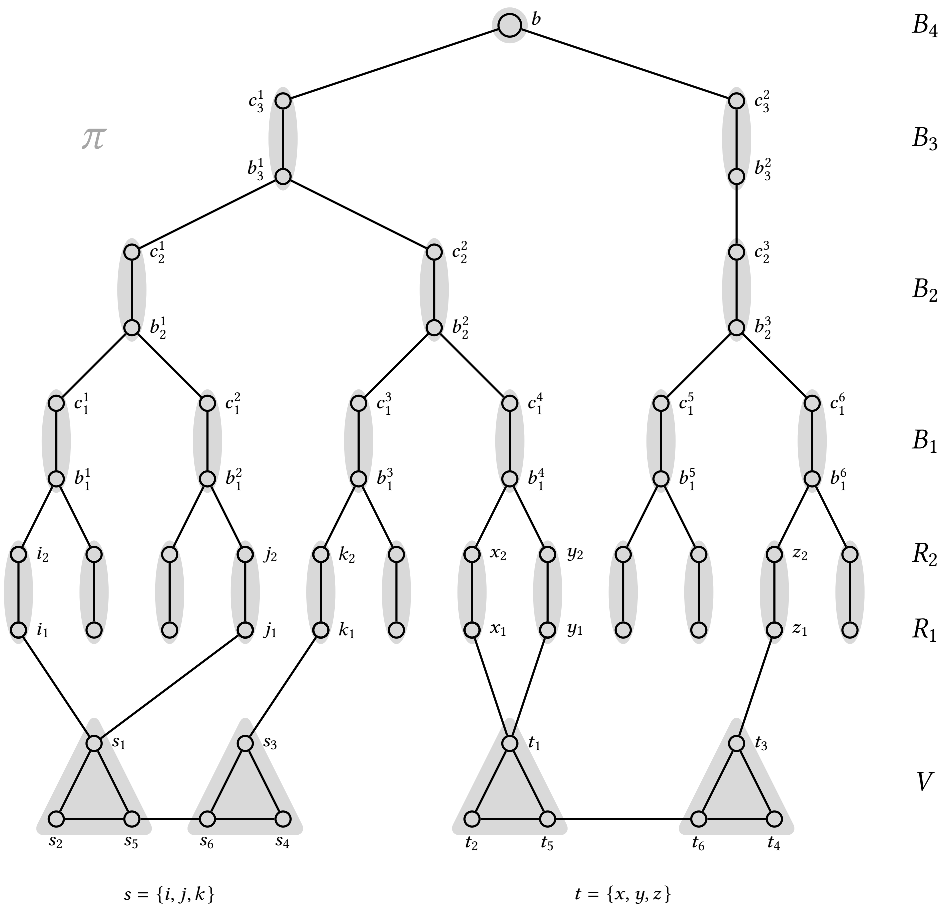

Figure 2: Schematic of the reduction for the verification problem for flatmate games. There is an edge between two agents if they are in a common individually rational coalition. The partition π in question for verification is marked in gray.

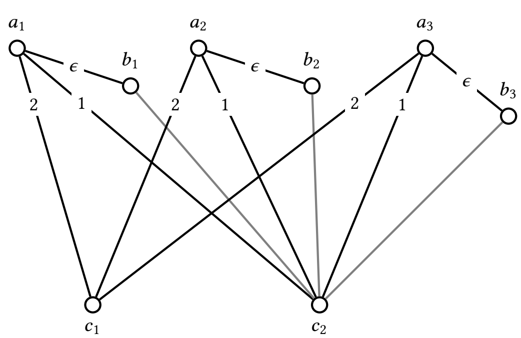

Figure 3: Instance of an ASHG with no popular partition. Grey edges have weight 0, omitted edges have weight $-K$.

- $v(a_3^r, y^s) = 5$ and $v(b_3^r, y^s) = 0$ for all $s \in S$ and $r \in R$ such that $r \in s$;
- $v(y^s, z_1^s) = v(y^s, z_2^s) = 10$ and $v(z_1^s, z_2^s) = 0$ for all $s \in S$
- $v(x, y) = -40$ for all other valuations not defined.

For the reduction to work, we can choose, e.g., $\epsilon = \frac{1}{2}$.

A schematic of the reduction for a certain set $s = \{i, j, k\} \in S$ is depicted in Figure 4. We abbreviate in the figure and the rest of the proof $V^r = \{a_1^r, a_2^r, a_3^r, b_1^r, b_2^r, b_3^r, c_1^r, c_2^r\}$, where $r \in R$, and $W^s = \{y^s, z_1^s, z_2^s\}$, where $s \in S$. Also denote $V^R = \cup_{r \in R} V^r$, $W^S = \cup_{s \in S} W^s$ and $A_3 = \{a_3^r : r \in R\}$.

We show that there exists a popular partition of $(N, \succsim)$ if and only if (R, S) is a 'yes' instance of X3C.

Assume (R, S) is a 'yes' instance of X3C. Then, there exists $S' \subseteq S$ such that S' is a partition of R. The following partition π is then popular: $\{\{a_1^r, c_1^r\} : r \in R\} \cup \{\{a_2^r, b_2^r, c_2^r\} : r \in R\} \cup \{\{y^s, a_3^i, a_3^j, a_3^k, b_3^i, b_3^j, b_3^k\} : s = \{i, j, k\} \in S'\} \cup \{W^s : s \in N \setminus S'\} \cup \{\{z_1^s, z_2^s\} : s \in S'\}$.

Assume for contradiction that π' is more popular than π.

We prove the following claims:

(1) Let $r \in R$ such that for all $s \in S$ with $r \in s$ holds that $y^s \notin \pi'(a_3^r)$. Then, $|N(\pi, \pi') \cap V^r| - |N(\pi', \pi) \cap V^r| \geq 1$.
(2) Let $r \in R$. If $|\{y^s : s \in S\} \cap \pi'(a_3^r)| \leq 1$ then, $|N(\pi, \pi') \cap V^r| - |N(\pi', \pi) \cap V^r| \geq 0$. If $|\{y^s : s \in S\} \cap \pi'(a_3^r)| \geq 2$ then, $|N(\pi, \pi') \cap V^r| - |N(\pi', \pi) \cap V^r| \geq -1$.

We start with the proof of the first claim.

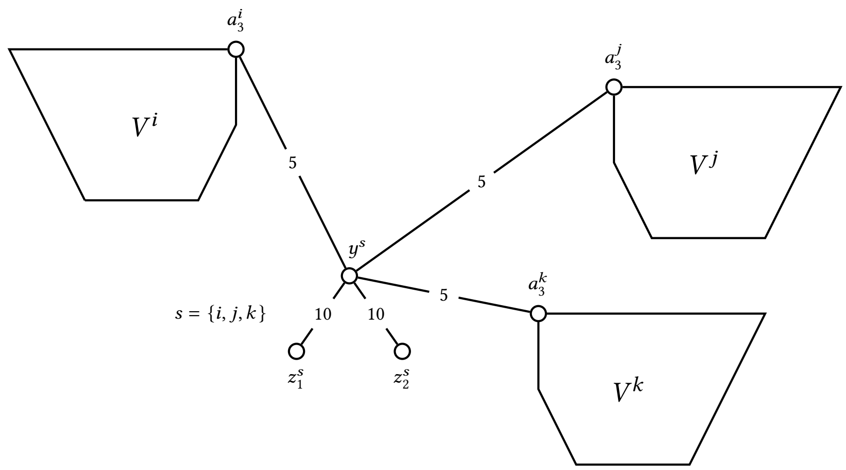

Figure 4: Schematic of the reduction of the existence problem for ASHGs. Edges of weight 0 and of negative weight are omitted.

Let therefore $r \in R$ such that for all $s \in S$ with $r \in s$ holds that $y^s \notin \pi'(a_3^r)$. Since $r \in R$ is fixed, we omit the superscript r for proving this claim. We know that $a_3 \in N(\pi, \pi')$ and $b_2, b_3 \notin N(\pi', \pi)$ We distinguish several cases:

- First, consider the case that $c_1 \in \pi'(a_1)$. Then, $b_1, a_2 \notin N(\pi', \pi)$. In addition, we may assume $a_1 \notin N(\pi', \pi)$, because otherwise $c_1, c_2 \in N(\pi, \pi')$ and the claim is true.
If $c_i \in N(\pi', \pi)$, then $c_{3-i} \notin N(\pi', \pi)$ and either $(a_1 \in N(\pi, \pi') \vee a_2 \in N(\pi, \pi')) \wedge b_3 \in N(\pi, \pi')$ or $a_1, a_2 \in N(\pi, \pi')$. In every case, $|N(\pi', \pi)| \leq 2$ and $|N(\pi, \pi')| \geq 3$ and the claim follows.
Hence, we may assume that $c_i \notin N(\pi', \pi)$ and no agent can be in $N(\pi', \pi)$. In this case, the claim follows.
- Second, assume $c_1 \in \pi'(a_2)$. Then, $a_1, b_2 \in N(\pi, \pi')$. If $a_2 \notin N(\pi', \pi)$, then it has a negative neighbor, i.e., $a_2 \in N(\pi, \pi')$. We have $|N(\pi, \pi')| \geq 4$, $|N(\pi', \pi)| \leq 3$.
Hence, $a_2 \in N(\pi', \pi)$. As a consequence, $c_1 \notin N(\pi', \pi)$ and $c_2 \notin N(\pi', \pi) \vee b_1 \notin N(\pi', \pi)$ and we conclude with $|N(\pi, \pi')| \geq 3$, $|N(\pi', \pi)| \leq 2$.
- Third, assume $c_1 \in \pi'(a_3)$. Then, $a_1, b_3 \in N(\pi, \pi')$. If $c_2 \in \pi'(a_3)$, then $c_1, c_2, a_2 \in N(\pi, \pi')$ and we conclude with $|N(\pi, \pi')| \geq 6$. If $c_2 \notin \pi'(a_3)$, then $\{a_1, a_3, b_3\} \subseteq N(\pi, \pi')$ and $a_2, b_2 \notin N(\pi', \pi)$ and either $b_2 \in N(\pi, \pi')$ or $c_2 \notin N(\pi', \pi)$.
- Finally, assume $c_1 \notin \pi'(a_1) \cup \pi'(a_2) \cup \pi'(a_3)$. Then $a_1, c_1 \in N(\pi, \pi')$ and $a_2 \notin N(\pi', \pi) \vee c_2 \notin N(\pi', \pi)$. Hence, $|N(\pi, \pi')| \geq 3$, $|N(\pi', \pi)| \leq 2$. This concludes the proof of the first claim.

Before we prove the second claim, we argue that we can assume without loss of generality that for all $r \in R$, $\pi'(a_3^r) \cap V^r \subseteq \{a_3^r, b_3^r\} \vee \{y^s : s \in S\} \cap \pi'(a_3^r) = \emptyset$. Indeed, if both conditions are not met, then leaving with $y^s \in \{y^s : s \in S\} \cap \pi'(a_3^r)$ and forming a coalition with W^s yields a partition π'' with the following properties:

- $|N(\pi'', \pi) \cap (N \backslash W^s)| \geq |N(\pi', \pi) \cap (N \backslash W^s)| - 1$ (Note that the only agent that is not still better off is possibly a_3^r since the other $a_3^{r'}$ are worse off since they would get negative utility in $\pi'(a_3^r)$.)
- $|N(\pi, \pi'') \cap (N \backslash W^s)| \geq |N(\pi, \pi') \cap (N \backslash W^s)| + 1$ (the only candidate is again a_3^r)
- $|N(\pi'', \pi) \cap W^s| \geq |N(\pi', \pi) \cap W^s| + 3$ if $\pi(y^s) \neq W^s$
- $|N(\pi, \pi'') \cap W^s| \geq |N(\pi, \pi') \cap W^s| - 3$ if $\pi(y^s) = W^s$

Other changes in W^s cannot occur at the same time and we conclude $\phi(\pi'', \pi) \geq \phi(\pi', \pi)$ (in fact the inequality is strict).

For the second claim, this means that if some $y^s \in \pi'(a_3^r)$ we can consider π' modified such that y^s leaves its coalition. This can only decrease the size of $N(\pi', \pi) \cap V^r$ if $|\{y^s : s \in S\} \cap \pi'(a_3^r)| \geq 2$ and cannot increase the size of $N(\pi, \pi') \cap V^r$ by more than 1. Hence, the claim follows from the first case.

We define the set of critical subsets $s \in S$ as $Y^c := \{s \in S : \exists r \in R \text{ with } y^s \in \pi'(a_3^r)\}$ and the set of happy R gadgets as $R^h = \{r \in R : |\{y^s : s \in S\} \cap \pi'(a_3^r)| \geq 2\}$.

We know that for every $y^s \in Y^c$ at most 3 of the a_3^r do not satisfy the condition of the first claim. Hence, a total of $\max\{|R| - 3|Y^c| + |R^h|, 0\}$ of the agents a_3^r does so. Putting together the claims yields

$$\begin{aligned}
&|N(\pi, \pi') \cap V^R| - |N(\pi', \pi) \cap V^R| \\
&\geq \max\{|R| - 3|Y^c| + |R^h|, 0\} - |R^h| \geq |R| - 3|Y^c|.
\end{aligned} \tag{1}$$

We claim that in addition

$$|N(\pi', \pi) \cap W^S| - |N(\pi, \pi') \cap W^S| \leq |R| - 3|Y^c|. \tag{2}$$

The idea to prove this inequality is that every agent y^s has to decide whether the agents in W^s or the a_3^r with $r \in s$ should be happy. Without loss of generality, we can assume that for all $s \in S$, $\pi(y^s) \cap A_3 = \emptyset$ or $\pi(y^s) \cap W^s = \{y^s\}$. Indeed, if both conditions are not met, then leaving with y^s and forming a coalition with W^s yields a partition π'' with $\phi(\pi'', \pi) \geq \phi(\pi', \pi)$.

To prove Equation (2) note that $W^s \subseteq N(\pi, \pi') \cap W^S$ for every $s \in Y^c$ such that $\pi(y^s) = W^s$. In other words, $|N(\pi, \pi') \cap W^S| \geq 3|\{s \in Y^c : \pi(y^s) = W^s\}|$.

In addition, the only agents that get better in W^S can be in a W^s such that $\pi(y^s) \neq W^s$ and $y^s \notin Y^c$. This is, $|N(\pi', \pi) \cap W^S| \leq 3|\{s \notin Y^c : \pi(y^s) \neq W^s\}|$.

Combining the inequalities yields

$$\begin{aligned}&|N(\pi',\pi)\cap W^S| - |N(\pi,\pi')\cap W^S|\\ &\le 3(|\{s\notin Y^c\colon \pi(y^s)\neq W^s\}| - |\{s\in Y^c\colon \pi(y^s)=W^s\}|)\\ &= 3(|\{s\notin Y^c\colon \pi(y^s)\neq W^s\}| + |\{s\in Y^c\colon \pi(y^s)\neq W^s\}|\\ &- |\{s\in Y^c\colon \pi(y^s)\neq W^s\}| - |\{s\in Y^c\colon \pi(y^s)=W^s\}|)\\ &= 3|S'| - 3|Y^c| = |R| - 3|Y^c|.\end{aligned}$$

Combining Equation (1) and Equation (2) yields $|N(\pi,\pi')| - |N(\pi',\pi)| \ge 0$, contradicting the assumption that π' was more popular than π.

It remains to prove that every popular partition yields a 3-partition of R with sets in S. Therefore, assume that π is a popular partition in $(N,\succsim)$. The partition will be found by checking intersections of $\pi(y^s)\cap A_3$ as captured in the following claims:

(1) For all $r\in R$ there exists a unique $s\in S$ with $y^s\in\pi(a_3^r)$. For this s holds that $r\in s$.
(2) For all $s\in S$ holds: $(\exists i\in s: a_3^i\in\pi(y^s)) \Rightarrow (\forall j\in s, a_3^j\in\pi(y^s))$

If the claim is true, $S' := \{s\in S: A_3\cap\pi(y^s)\neq\emptyset\}$ covers R due to existence and is a partition due to uniqueness and the second claim that ensures that either all three or none of the agents in A_3 corresponding to elements in s are present in a coalition $\pi(y^s)$.

We start to show the existence part of the first claim which will follow directly from the property that $N|_{V^r}$ contains no popular partition (Proposition A.1).

Assume for contradiction that there exists a $r\in R$ such that for all $s\in S$ holds $y^s\notin\pi(a_3^r)$. We obtain a more popular partition in two steps. First, we modify π such that for all agents in $v\in V^r$ we split their coalition into $\pi(v)\cap V^r$ and $V^r\backslash\pi(v)$. This cannot decrease the utility of any agent. Application of Proposition A.1 yields a more popular partition locally on V^r that can be extended to the whole N via the remaining (modified) coalitions in π.

For the uniqueness part assume for contradiction that there is $r\in R$ and $s\neq s'\in S$ with $\{y^s, y^{s'}\}\subseteq\pi(a_3^r)$. We distinguish two cases.

First, assume that $|\pi(a_3^r)\cap A_3|\le 3$. Then, there exists (without loss of generality using symmetry amongst s and s') an agent $r'\in R$ with $r'\in s$ and $a_3^{r'}\notin\pi(a_3^r)$. Then, the partition π' obtained from π by removing the agents in W^s from their partitions in π and letting them form a coalition is more popular. Indeed, $|N(\pi,\pi')|\le 2$ (the two remaining agents a_3^t with $t\neq r'$ and $t\in s$ are the only ones to possibly loose utility) and $W^s\subseteq N(\pi',\pi)$.

Second, assume that $|\pi(a_3^r)\cap A_3|\ge 4$. Then, there exists an agent $u\in A_3\cap\pi(a_3^r)$ with $u\notin s$. The same partition π' as in the first case yields $|N(\pi,\pi')|\le 3$ and $|N(\pi',\pi)|\ge|W^s\cup\{u\}| = 4$.

In both cases, we have found a more popular partition, a contradiction.

Finally, for the second claim, in the case that there exists a $s\in S$ with $1\le|\{j\in s: a_3^j\in\pi(y^s)\}|\le 2$, the same rearrangement of coalitions (i.e., forming the coalition W^s) is more popular. □

Proof of Theorem 4.16. The problem is in coNP, because a more popular partition serves as a polynomial-time certificate for a 'no'-instance.

For hardness, we reduce again from X3C. Given an instance (R,S) of X3C, we assume without loss of generality that $|R|\ge 6$. We define an ASHG $(N,\succsim)$ given by $N = R\cup\{s_1,s_2,s_3\colon s\in S\}\cup\{b_1,b_2,b_3\}$ and weights as follows:

- $v(i,s_3) = 1$ for $i\in s, s\in S$
- $v(s_1,s_3) = v(s_2,s_3) = 4$ for $s\in S$
- $v(s_j,b_j) = 1$ for $s\in S, j=1,2$
- $v(b_1,b_3) = v(b_2,b_3) = \alpha$ for $\frac{|R|}{3}-1<\alpha<\frac{|R|}{3}$
- $v(i,j) = 0$ for $i,j\in R$, $v(s_1,s_2)=0$ for $s\in S$, and $v(b_1,b_2)=0$
- $v(x,y) = -\max\{12, |S|+|R|/3\}$ for all agents $x,y\in N$ such that no utility is defined, yet.

One can choose, e.g., $\alpha = (|R|-1)/3$, but for the reduction, only the above bounds matter. We introduce some useful notation for the proof. Denote $V^s = \{s_1,s_2,s_3\}$ for $s\in S$, $B=\{b_1,b_2,b_3\}$, and $V = \cup_{s\in S}V^s$.

The partition in question is $\pi = \{V^s\colon s\in S\}\cup\{\{r\}\colon r\in R\}\cup\{B\}$. We claim that (R,S) is a 'yes'-instance of X3C if and only if π is not popular for the ASHG given by G.

If (R,S) is a 'yes'-instance, there exists a subset $S'\subseteq S$ that partitions R. In particular $|R| = 3|S'|$.

Consider the partition given by $\pi' = \{V^s\colon s\in S\backslash S'\}\cup\{\{s_3,i,j,k\}\colon\{i,j,k\} = s\in S'\}\cup\{\{b_j,s_j\colon s\in S'\}\colon j=1,2\}\cup\{\{b_3\}\}$.

Then, $N(\pi',\pi) = R\cup\{b_1,b_2\}$ and $N(\pi,\pi') = \cup_{s\in S'}V^s\cup\{b_3\}$. Hence, π' is more popular than π.

Conversely, assume that there exists a more popular partition π' and fix one that maximizes $\phi(\pi',\pi)$. We have to prove that there exists a subset $S'\subseteq S$ that yields a partition of R. Note that the negative weight is chosen so large that agents in a coalition linked by negative utility are always worse off.

First, we claim that for all $s\in S$, $N(\pi',\pi)\cap V^s=\emptyset$. Assume for contradiction that for $j=1,2$, $s_j\in N(\pi',\pi)$. Then, $\{s_j,s_3,b_j\}\subseteq\pi'(s_j)\subseteq V^s\cup\{b_j\}$. Thus, $s_{3-j},s_3,b_j,b_3\in N(\pi,\pi')$.

We form a new coalition π'' from π' by having the coalitions V^s and B (these agents leave their coalitions in π') and all other coalitions remain the same. We consider two cases:

- If $|\pi'(b_{3-j})\cap V|\le 1$, then $b_{3-j}\in N(\pi,\pi')$. (We used that $|R|\ge 6$.) We have that $s_3,s_{3-j},b_1,b_2,b_3\in N(\pi,\pi')\backslash N(\pi,\pi'')$, $s_2\in N(\pi',\pi)\backslash N(\pi,\pi'')$ and possibly the agent $t\in\pi'(b_{3-j})\cap V$ yields $t\in N(\pi',\pi)\cap N(\pi,\pi'')$. Hence, $\phi(\pi'',\pi)>\phi(\pi',\pi)$.
- Otherwise, $\pi'(b_{3-j})\cap V\subseteq N(\pi,\pi')$, but possibly $b_{3-j}\in N(\pi',\pi)\backslash N(\pi,\pi'')$ in addition. However, $\phi(\pi'',\pi)>\phi(\pi',\pi)$ remains valid.

In any case, we derived a contradiction to the maximality condition on π'.

If $s_3\in N(\pi',\pi)$, then $\{s_1,s_2\}\subseteq\pi'(s_3)$, $s\cap\pi'(s_3)\neq\emptyset$, and $\pi'(s_3)\subseteq V^s\cup s$ (here $s\subseteq R$ is the set of R-agents corresponding to elements of the set s). Hence, forming a coalition π'' by leaving with the agents in s moves these agents and s_1,s_2 out of $N(\pi,\pi')$, while only removing s_3 from $N(\pi',\pi)$. Hence, we again contradict the maximality of $\phi(\pi',\pi)$.

For the rest of the analysis, we narrow down the possible more popular partitions to a very specific situation that corresponds to 3-partitions. The idea is basically that whenever we 'sacrifice' a set V^s of agents, we can improve only 3 agents in R. Due to the

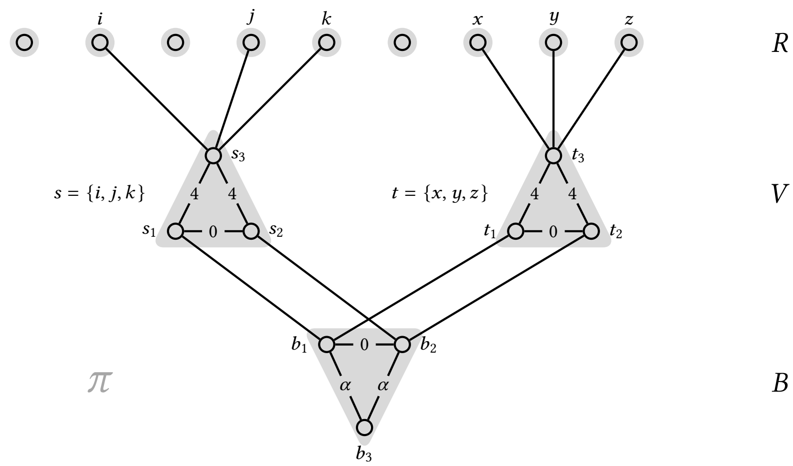

Figure 5: Schematic of the reduction for the verification problem of popular partitions on symmetric ASHGs. Edges without explicit weight have weight 1. Omitted edges for agents in R have weight 0. All other omitted edges have weight -12. The partition π marked in gray is the one under consideration for verification.

boundaries on α, we will cross the threshold, where we can have a popularity margin of precisely 1 exactly at the moment when we gathered $\frac{|R|}{3}$ neighbors for b_1 and b_2 in order to improve these.

We introduce the sets $R_I = R \cap N(\pi', \pi)$ and $S_C = \{s \in S : \pi'(s_3) \cap R \neq \emptyset\}$. Our goal is to prove $|R| = |R_I| = 3|S_C|$.

For $s \in S_C$ holds $V^s \subseteq N(\pi, \pi')$ (which follows for s_3 since $s_3 \notin N(\pi', \pi)$). Consequently, $|N(\pi, \pi') \cap V| \geq 3|S_C|$. In addition, $|N(\pi', \pi) \cap R| = |R_I| \leq 3|S_C|$ and $\phi_B(\pi', \pi) \leq 1$.

If $|R_I| < 3|S_C|$, then $\phi(\pi, \pi') = \phi_B(\pi, \pi') + \phi_V(\pi, \pi') + \phi_R(\pi, \pi') \geq -1 + 3|S_C| - (|R_I|) = 3|S_C| - |R_I| - 1 \geq 0$ and π' is not more popular. We conclude that $|R_I| = 3|S_C|$.

Before we conclude the proof, we show two auxiliary claims:

(1) If $B \subseteq \pi'(b_3)$ then $b_1 \notin N(\pi', \pi) \vee b_2 \notin N(\pi', \pi)$.
(2) For $j = 1, 2$, if $b_j \in N(\pi', \pi)$, then $b_j \in \pi'(b_3) \vee |\{s \in S : s_j \in \pi'(b_j)\} \cap \pi'(b_j)| \geq \frac{|R|}{3}$.

The first claim follows from the fact that if b_j forms a coalition with an agent outside B that gives her positive utility, then b_{3-j} cannot be both in this coalition and improve her utility. The second claim follows from $u_\pi(b_j) = \alpha > \frac{|R|}{3} - 1$.

We are ready to prove $|R| = 3|S_C|$. We consider the agents in B. The only possibility for $\phi(\pi', \pi) > 0$ is that $\phi_B(\pi', \pi) \geq 1$ which can only happen if $\{b_1, b_2\} \subseteq N(\pi', \pi)$. Due to the auxiliary claims, there exists $j \in \{1, 2\}$ with $|\{s \in S : s_j \in \pi'(b_j)\} \cap \pi'(b_j)| \geq \frac{|R|}{3}$.

If $s^* \in \{s \in S : s_j \in \pi'(b_j)\} \setminus S_C$, then $s^*_j \in N(\pi, \pi')$ (using $|R| \geq 6$, i.e., $|\pi'(b_j) \cap \{s \in S : s_j \in \pi'(b_j)\}| \geq 2$).[6]

Consequently, $\phi(\pi, \pi') = \phi_B(\pi, \pi') + \phi_V(\pi, \pi') + \phi_R(\pi, \pi') \geq -1 + (3|S_C| + 1) - 3|S_C| \geq 0$, a contradiction. Therefore, $\{s \in S : s_j \in \pi'(b_j)\} \subseteq S_C$ and $\frac{|R|}{3} \leq |\{s \in S : s_j \in \pi'(b_j)\}| \leq |S_C| = \frac{|R_I|}{3} \leq \frac{|R|}{3}$.

Consider the set $S' = S_C$. Then, S_C covers R since $R_I = R$. In addition, since $|R| = 3|S_C|$, every agent $r \in R$ is present in exactly one $s \in S_C$. Hence, S' is a partition of R with sets in S. In total, (R, S) is a 'yes'-instance of X3C. ◻

We first prove the existence of the graph that underlies the reductions for mixed and strong popularity on ASHGs. It satifies similar properties as the flatmate game considered in Lemma 4.10.

Lemma A.2. *Let a instance (R, S) of X3C be given. Then, there exists a symmetric ASHG $(N, \succeq)$ with a number of agents and weights polynomial in $|R|$, an agent $x \in N$ and an individually rational partition π^* of N such that for all partitions $\pi \neq \pi^*$,*

(1) $\phi(\pi^*, \pi) \geq 1$
(2) *If $x \in N(\pi^*, \pi)$, then $\phi(\pi^*, \pi) \geq 3$ or (R, S) is a 'yes'-instance.*
(3) $v_{\pi^*}(x) > 0$

In addition, if (R, S) is a 'yes'-instance, there exists a partition π' with

(1) $\phi(\pi^*, \pi') = 1$
(2) *$\pi'(x) = \{x\}$ and $x \in N(\pi^*, \pi')$*

Proof. Let (R, S) be an instance of X3C. We construct the following game. Let $k = \min\{k \in \mathbb{N} : 2^k \geq |R|\}$ define the smallest power of 2 that is larger than the cardinality of R. We define a flatmate game on vertex set $N = \{y_1, y_2\} \cup \bigcup_{j=0}^{k} N_j$, where $N_j = \bigcup_{i=1}^{2^j} A^i_j$ consists of 2^j sets of agents A^i_j.

We define the sets of agents as

- $A^i_k = \{a^i_k, b^i_k, c^i_k\}$ for $i = 1, \dots, 2^k$
- $A^i_j = \{a^i_j, b^i_j, c^i_j, \alpha^i_j, \beta^i_j, \gamma^i_j, \delta^i_j\}$ for $j = 0, \dots, k-1$, $i = 1, \dots, 2^j$

We order the set R in an arbitrary but fixed way, say $R = \{r^1, \dots, r^{|R|}\}$ and for a better understanding of the proof and the preferences, we label the agents $b^i_k = r^i$ for $i = 1, \dots, |R|$. If we view the set of agents N as $k + 1$ levels of agents, then the ground set R of the instance of X3C is identified with some specific agents in the top level k. We are ready to define the preferences.

- $v(y_1, y_2) = 1$
- $v(y_2, b^i_k) = 2k + 3$, $i = |R| + 1, \dots, 2^k$

[6] This argument is stronger than what is needed for ASHGs, but is needed for the case of FHGs.

- $v(b_k^i, b_k^{i'}) = 0$, $i, i' \in \{|R| + 1, \dots, 2^k\}$
- $v(b_k^i, b_k^{i'}) = k + 2$ if there exists $s \in S$ with $r^i, r^{i'} \in s$
- $v(a_k^i, b_k^i) = v(a_k^i, c_k^i) = v(b_k^i, c_k^i) = k + 1$, $i = 1, \dots, 2^k$
- For $j = 0, \dots, k-1$, $i = 1, \dots, 2^k$
  - $v(a_j^i, b_j^i) = v(a_j^i, c_j^i) = j + 1$, $v(b_j^i, c_j^i) = j + 1.5$
  - $v(b_j^i, c_{j+1}^{2i-1}) = v(b_j^i, c_{j+1}^{2i}) = j + 1.5$
  - $v(\alpha_j^i, \beta_j^i) = j + 1$, $v(\beta_j^i, \gamma_j^i) = 0$
  - $v(\beta_j^i, a_j^i) = j + 1.75$, $v(\gamma_j^i, a_j^i) = j + 1.25$
  - $v(\gamma_j^i, \delta_j^i) = j + 2$, $v(\delta_j^i, \alpha_{j+1}^{2i-1}) = v(\delta_j^i, \alpha_{j+1}^{2i}) = j + 1.5$
- $v(g, h) = -2^k(2k + 3)$ for all $g, h \in N$ such that the utility is not yet defined.

Let $\pi^* = \{\{a_j^i, b_j^i, c_j^i\} : j = 0, \dots, k, i = 1, \dots, 2^j\} \cup \{\{\alpha_j^i, \beta_j^i\}, \{\gamma_j^i, \delta_j^i\} : j = 0, \dots, k-1, i = 1, \dots, 2^j\} \cup \{\{y_1, y_2\}\}$ and $x = c_0^1$.

Now consider a partition $\pi \neq \pi^*$.

We will prove the following claim by induction over $j = k, \dots, 0$. For every $i = 1, \dots, 2^j$ holds:

(1) If $\{b_j^i, a_j^i\} \cap \pi(c_j^i) = \emptyset$, then $\phi_{T_j^i}(\pi^*, \pi) \geq 1$ and $\phi_{T_j^i}(\pi^*, \pi) \geq 3 \vee \{b_k^i : i = 1, \dots, 2^k\} \cap T_j^i \subseteq N(\pi, \pi^*)$.

(2) If $\alpha_j^i \notin N(\pi, \pi^*)$ and there exists an agent $z \in T_j^i$ with $\pi(z) \neq \pi^*(z)$. Then $\phi_{T_j^i}(\pi^*, \pi) \geq 1$.

Note that $y_1 \notin N(\pi, \pi^*)$ and if $y_2 \in N(\pi, \pi^*)$, then $y_1 \in N(\pi^*, \pi)$. In addition, if for $i \in \{1, \dots, |R|\}$, $b_k^i \in N(\pi, \pi^*)$, then there exists $i' \in \{1, \dots, |R|\} \backslash \{i\}$ with $b_k^{i'} \in \pi(b_k^i)$. Hence, $\{b_k^{i'}, b_k^i\} \subseteq N(\pi, \pi^*)$ can only happen if there exists a $s \in S$ and $v \in \{1, \dots, |R|\} \backslash \{i, i'\}$ with $s = \{r^i, r^{i'}, r^v\}$ and $\pi(b_k^i) = \{b_k^i, b_k^{i'}, b_k^v\}$. Thus, if $\{b_k^1, \dots, b_k^{|R|}\} \subseteq N(\pi, \pi^*)$, the instance (R, S) is a 'yes'-instance. Finally, $\alpha_0^i \notin N(\pi, \pi^*)$ is true. Hence, the assertion on π^* follows from the case $j = 0$ of the claim.

We will now proceed with the proof of the claim.

For the base case $j = k$, we observe that if $A_k^i \cap N(\pi, \pi^*) \neq \emptyset$, then clearly $\phi_{A_k^i}(\pi^*, \pi) \geq 1$. In addition, if $\{b_k^i, a_k^i\} \cap \pi(c_k^i) = \emptyset$, then $\{a_k^i, c_k^i\} \subseteq N(\pi^*, \pi)$ and $b_k^i \in N(\pi^*, \pi) \cup N(\pi, \pi^*)$.

For the induction step, let $j \in \{k-1, \dots, 0\}$ and fix $i \in \{1, \dots, 2^j\}$. Assume first that there exists an agent $z \in T_j^i$ with $\pi(z) \neq \pi^*(z)$ but no such agent in A_j^i. Then, $z \in T_{j+1}^{2i-1} \vee z \in T_{j+1}^{2i}$ and the claim follows by induction. Assume therefore that there exists an agent $z \in A_j^i$ with $\pi(z) \neq \pi^*(z)$.

We make the following observations.

- If $\alpha_j^i \in N(\pi, \pi^*)$, then $\beta_j^i \in N(\pi^*, \pi)$
- If $\beta_j^i \in N(\pi, \pi^*)$, then $\alpha_j^i \in N(\pi^*, \pi)$
- If $\gamma_j^i \in N(\pi, \pi^*)$, then $\delta_j^i \in N(\pi^*, \pi)$
- If $\delta_j^i \in N(\pi, \pi^*)$, then $\gamma_j^i \in N(\pi^*, \pi)$

Now, we consider the case that $\pi(a_j^i) \neq \pi^*(a_j^i)$.

- We consider first the subcase that $b_j^i \in N(\pi, \pi^*)$. Then $c_j^i \in N(\pi^*, \pi)$.
  - If $\pi(b_j^i) \supseteq \{c_{j+1}^{2i-1}, c_{j+1}^{2i}\}$, then $\phi_{A_j^i}(\pi, \pi^*) \leq 1$ (with the above observations), while by induction $\phi_{T_{j+1}^{2i-1} \cup T_{j+1}^{2i}}(\pi^*, \pi) \geq 2$ and $\phi_{T_{j+1}^{2i-1} \cup T_{j+1}^{2i}}(\pi^*, \pi) \geq 4 \vee \{b_k^i : i = 1, \dots, 2^k\} \cap (T_{j+1}^{2i-1} \cup T_{j+1}^{2i}) \subseteq N(\pi, \pi^*)$ and we are done.
  - Otherwise, $c_j^i \in \pi(b_j^i)$. Then $\phi_{A_j^i}(\pi^*, \pi) \geq 1$ or $a_j^i \in N(\pi, \pi^*)$. The second case can only occur for $\pi(a_j^i) = \{a_j^i, \beta_j^i, \gamma_j^i\}$. Hence, $\phi_{A_j^i}(\pi^*, \pi) \geq 1$ or $\pi(\delta_j^i) = \{\delta_j^i, \alpha_{j+1}^{2i-1}, \alpha_{j+1}^{2i}\}$. But then $\phi_{A_j^i}(\pi^*, \pi) \geq -1$ and $\phi_{T_{j+1}^{2i-1} \cup T_{j+1}^{2i}}(\pi^*, \pi) \geq 2$ and we are done.
- We can even assume that $b_j^i \in N(\pi^*, \pi)$, since otherwise $a_j^i \in \pi(b_j^i)$ and $a_j^i, c_j^i \in N(\pi^*, \pi)$ and it follows $\phi_{A_j^i}(\pi^*, \pi) \geq 1$.
- If $c_j^i \in N(\pi, \pi^*)$, then $a_j^i, b_j^i \in N(\pi^*, \pi)$ and therefore $\phi_{A_j^i}(\pi^*, \pi) \geq 1$ and we are done.
- Since $\pi(c_j^i) \neq \pi^*(c_j^i)$, we can assume $c_j^i \in N(\pi^*, \pi)$
- Next, consider the case that $a_j^i \in N(\pi, \pi^*)$ and, by the previous cases, $c_j^i, b_j^i \in N(\pi^*, \pi)$.
  - If $\pi(a_j^i) = \{a_j^i, \beta_j^i, \gamma_j^i\}$, then $\phi_{A_j^i}(\pi^*, \pi) \geq 3$ or $\pi(\delta_j^i) = \{\delta_j^i, \alpha_{j+1}^{2i-1}, \alpha_{j+1}^{2i}\}$. In the latter case, $\phi_{A_j^i}(\pi^*, \pi) \geq 1$ and $\phi_{T_{j+1}^{2i-1} \cup T_{j+1}^{2i}}(\pi^*, \pi) \geq 2$ by induction and we are done.
  - Otherwise, $\beta_j^i \in \pi(a_j^i) \cap N(\pi^*, \pi)$ or $\gamma_j^i \in \pi(a_j^i) \cap N(\pi^*, \pi)$. In the former case, $\alpha_j^i \in N(\pi^*, \pi)$ and in total $\phi_{A_j^i}(\pi^*, \pi) \geq 3$. In the latter case, again, $\phi_{A_j^i}(\pi^*, \pi) \geq 3$ or $\pi(\delta_j^i) = \{\delta_j^i, \alpha_{j+1}^{2i-1}, \alpha_{j+1}^{2i}\}$ and the case is similar as before.
- It remains that $a_j^i, b_j^i, c_j^i \in N(\pi^*, \pi)$ in which case $\phi_{A_j^i}(\pi^*, \pi) \geq 3$.

We may therefore assume that $\pi(a_j^i) = \pi^*(a_j^i)$. Only for the remaining cases, we need that $\alpha_j^i \notin N(\pi, \pi^*)$. If $\pi(\alpha_j^i) \neq \pi^*(\alpha_j^i)$, then $\alpha_j^i, \beta_j^i \in N(\pi^*, \pi)$ and consequently $\phi_{A_j^i}(\pi^*, \pi) \geq 2$. If $\pi(\gamma_j^i) \neq \pi^*(\gamma_j^i)$, then $\phi_{A_j^i}(\pi^*, \pi) \geq 2$ or $\phi_{A_j^i}(\pi, \pi^*) \geq 0 \wedge \pi(\delta_j^i) \cap \{\alpha_{j+1}^{2i-1}, \alpha_{j+1}^{2i}\} \neq \emptyset$ and the claim follows by induction.

For the second part of the lemma, assume that S' is a 3-partition of R through sets in S.

Define $\pi' = \{\{b_k^v, b_k^w, b_k^x\} : \{v, w, x\} \in S'\} \cup \{\{b_k^{|R|+1}, \dots, b_k^{2^k}, y_2\}, \{y_1\}\} \cup \{\{\delta_{k-1}^i, a_k^{2i-1}, a_k^{2i}\} : i = 1, \dots, 2^{k-1}\} \cup \{\{b_j^i, c_{j+1}^{2i-1}, c_{j+1}^{2i}\}, \{\delta_j^i, \alpha_{j+1}^{2i-1}, \alpha_{j+1}^{2i}\}, \{a_j^i, \beta_j^i, \gamma_j^i\} : j = 1, \dots, k-1, i = 1, \dots, 2^j\} \cup \{\{\alpha_0^1\}, \{c_0^1\}\}$. It is easily checked that $\phi(\pi', \pi^*) = 1$ and $c_0^1 \in N(\pi^*, \pi')$. □

Proof of Theorem 4.17. The reduction is from X3C. Given an instance (R, S) of X3C, we consider the symmetric ASHG of Lemma A.2 on agent set N with utility function v together with the partition π^* and the special agent $x \in N$. Set $M = \max\{\sum_{w \in N : v(y,w) > 0} v(y, w) : y \in N\}$. We define a symmetric ASHG on agent set $N' = N \cup \{z\}$ where the utilities are given by $v'(y, w) = v(y, w)$ if $y, w \in N$, $v'(z, x) = v_{\pi^*}(x)/2$, and $v'(z, y) = -M - 1$ for $y \in N \backslash \{x\}$. Note that by Lemma A.2, this reduction is in polynomial time.

Consider the partition $\sigma^* = \pi^* \cup \{\{z\}\}$ and let $\sigma \neq \sigma^*$ be given and define $\pi = (\sigma \backslash \sigma(z)) \cup \{\sigma(z) \backslash \{z\}\}$, that is, the partition of agent set N where z leaves her coalition. If $\sigma(z) = \{z\}$, then $\phi(\sigma^*, \sigma) > 0$ by Lemma A.2. If $\sigma(z) = \{z, x\}$, then $\phi(\sigma^*, \sigma) = -1 + \phi(\pi^*, \pi) \geq 0$ and $\phi(\sigma^*, \sigma) > 0$ if (R, S) is a 'no'-instance. Otherwise, $z \in N(\sigma^*, \sigma)$ and the only agent in N that can be worse off in π compared to σ is x. If $\pi \neq \pi^*$, then $\phi(\sigma^*, \sigma) \geq -1 + \phi(\pi^*, \pi) \geq 0$ and $\phi(\sigma^*, \sigma) > 0$ if (R, S) is a 'no'-instance. If $\pi = \pi^*$, then $\phi(\sigma^*, \sigma) = \phi_{\sigma^*(x)}(\sigma^*, \sigma) > 0$. Note that $|\pi(x)| \geq 2$ because $v_{\pi^*}(x) > 0$. It follows that σ^* is popular and it is a strongly popular partition if (R, S) is a 'no'-instance.

If (R, S) is a 'yes'-instance, then σ^* is the only candidate that might be strongly popular. Consider the partition π' from Lemma A.2 and define $\sigma' = (\pi' \backslash \{\{x\}\}) \cup \{\{x, z\}\}$. Then, $x \in N(\pi^*, \pi') \cap N(\sigma^*, \sigma')$, whereas $z \in N(\sigma', \sigma^*)$. Therefore, $\phi(\sigma', \sigma) = 1 + \phi(\pi', \pi^*) = 0$. Hence, π^* is not strongly popular and there exists no strongly popular partition. □

Proof of Theorem 4.18. In the proof of Theorem 4.17, the partition σ^* is strongly popular if, and only if, (R, S) is a 'no'-instance of X3C. □

Proof of Theorem 4.19. We give a Turing reduction from X3C. Given an instance (R, S) of X3C, we consider the symmetric ASHG of Lemma A.2 on agent set N with utility function v together with the partition π^* and the special agent $x \in N$. Set $M = \max\{\sum_{w \in N\colon v(y,w)>0} v(y, w)\colon y \in N\}$. We define a symmetric ASHG on agent set $N' = N \cup \{z_1, z_2\}$ where the utilities are given by $v'(y, w) = v(y, w)$ if $y, w \in N$, $v'(z_1, z_2) = v'(z_1, x) = v'(z_2, x) = v_{\pi^*}(x)/3 > 0$, and $v'(z_i, y) = -M - 1$ for $i = 1, 2, y \in N \backslash \{x\}$. Note that by Lemma A.2, this reduction is in polynomial time.

Consider the partition $\sigma^* = \pi^* \cup \{\{z_1, z_2\}\}$ and let $\sigma \neq \sigma^*$ be given and define $\pi = (\sigma \backslash (\sigma(z_1) \cup \sigma(z_2))) \cup \{\sigma(z_1) \backslash \{z_1, z_2\}, \sigma(z_2) \backslash \{z_1, z_2\}\}$, that is, the partition of agent set N where z_1 and z_2 leave their coalitions. Assume that (R, S) is a 'no'-instance. We will prove that $\phi(\sigma^*, \sigma) > 0$, and therefore that σ^* is strongly popular. We may assume that $\sigma(z_1) = \{z_1, z_2\}$ or $x \in \sigma(z_i)$ for some i, because otherwise it is a Pareto improvement if z_1 and z_2 leave their coalitions and form a coalition of their own.

If $\sigma(z_1) = \{z_1, z_2\}$, then by Lemma A.2, $\phi(\sigma^*, \sigma) = \phi(\pi^*, \pi) > 0$, because $\pi \neq \pi^*$. Otherwise assume without loss of generality that $x \in \sigma(z_1)$. If $\pi(z_1) \subseteq \{x, z_1, z_2\}$, then $\phi(\sigma^*, \sigma) \geq -2 + \phi(\pi^*, \pi) = 1$. If there exists $y \in N \backslash \{x\}$ with $y \in \sigma(z_1)$, then $z_1, z_2 \in N(\sigma^*, \sigma)$ and with $\phi_x(\sigma^*, \sigma) - \phi_x(\pi^*, \pi) \geq -2$, it follows, $\phi(\sigma^*, \sigma) \geq 2 - 2 + \phi_x(\pi^*, \pi) > 0$ if $\pi \neq \pi^*$ and $\phi(\sigma^*, \sigma) \geq 2 - 1 + \phi_x(\pi^*, \pi) = 1$ if $\pi = \pi^*$. In particular, the unique mixed popular partition consists of σ^* with probability 1.

Now assume that (R, S) is a 'yes'-instance. Consider the partition π' from Lemma A.2 and define $\sigma' = (\pi' \backslash \{\{x\}\}) \cup \{\{x, z_1, z_2\}\}$. Then, $x \in N(\pi^*, \pi') \cap N(\sigma^*, \sigma')$, whereas $z_1, z_2 \in N(\sigma', \sigma^*)$. Therefore, $\phi(\sigma', \sigma) = 2 + \phi(\pi', \pi^*) = 1$. Hence, the pure mixed partition $\{\sigma^*\}$ is not mixed popular.

We can solve X3C by computing a partition σ in the support of a mixed popular partition and checking its probability in case $\sigma = \sigma^*$. □

Proof of Theorem 4.20. We provide a reduction from X3C. Given an instance (R, S) of X3C, we consider the symmetric ASHG of Lemma A.2 on agent set N with utility function v together with the partition π^* and the special agent $x \in N$. Set $M = \max\{\sum_{w \in N\colon v(y,w)>0} v(y, w)\colon y \in N\}$. For $i = 1, 2$, let $N_i = \{y_i\colon y \in N\}$ be two copies of N. Accordingly, let π_i^* be their respective copies of π^*.

We define a symmetric ASHG on agent set $N' = N_1 \cup N_2 \cup Z$ where $Z = \{z_k^j\colon k = 1, 2, j = 1, 2, 3\}$. Define $Z^j = \{z_1^j, z_2^j\}$. Utilities are as follows.

- $v'(y_i, w_i) = v(y, w)$ if $y, w \in N_i$ for $i = 1, 2$
- $v'(z_k^j, x_1) = v_{\pi^*}(x)/7$, $v'(z_k^j, x_2) = v_{\pi^*}(x)/8$ for $k = 1, 2, j = 1, 2, 3$
- $v'(z_1^j, z_2^j) = v_{\pi^*}(x)$ for $j = 1, 2, 3$
- $v'(u, y) = -M - 1$ for every pair of agents $u, y \in N'$ such that their utility is not yet defined

Note that by Lemma A.2, this reduction is in polynomial time.

First assume that (R, S) is a 'no'-instance. Then, $\sigma^* = \pi_1^* \cup \pi_2^* \cup \{Z^j\colon j = 1, 2, 3\}$ is popular. To prove this, let σ be an arbitrary partition and define $\pi_i = \{\sigma(y) \cap N_i\colon y \in N_i\}$ be the coalitions restricted to N_i. For each $j \in \{1, 2, 3\}$, we can assume that $\sigma(z_k^j) = Z^j$ or there exists a $i \in \{1, 2\}$ with $Z^j \cap \sigma(x_i) \neq \emptyset$. Otherwise, one can obtain a Pareto-improvement σ' over σ and it suffices to prove that $\phi(\sigma^*, \sigma') \geq 0$. Indeed, if $\sigma(z_k^j) = \{z_k^j\}$ for $k = 1, 2$, then creating Z^j is a Pareto-improvement. On the other hand, if $\{z_{3-k}, x_1, x_2\} \cap \sigma(z_k^j) = \emptyset$ and $|\sigma(z_k^j)| \geq 2$, then leaving her coalition with z_k^j yields a Pareto-improvement over σ. In addition, we may assume that it does not happen that $\sigma(z_k^j) = \{z_k^j\}$, because adding z_k^j to $\sigma(z_{3-k}^j)$ yields a Partition σ' with

- $y \in N(\sigma^*, \sigma)$ if and only if $y \in N(\sigma^*, \sigma')$ for $y \in N'$, and
- $y \in N(\sigma, \sigma^*)$ if and only if $y \in N(\sigma', \sigma^*)$ for $y \in N'$.

By a similar argument, we can assume that $\sigma(x_i) \subseteq Z \cup N_i$.

We can therefore partition the agent set N' into sets of the type Z^j such that $\sigma(z_1^j) = Z^j$, of the type N_i such that $Z \cap \sigma(x_i) = \emptyset$, and of the type$N_i \cup \sigma\{x_i\}$ such that $Z \cap \sigma(x_i) \neq \emptyset$. For the first type, $\phi_{Z^j}(\sigma^*, \sigma) = 0$ and by Lemma A.2, $\phi_{N_i}(\sigma^*, \sigma) \geq 0$ for the second type of sets. We prove that $\phi_{N_i \cup \sigma\{x_i\}}(\sigma^*, \sigma) \geq 0$ if $Z \cap \sigma(x_i) \neq \emptyset$.

If $\sigma(x_i) \subseteq Z \cup \{x_i\}$, then $x_i \in N(\sigma^*, \sigma)$ and $\phi_{\sigma(x_i) \backslash \{x_i\}}(\sigma^*, \sigma) \geq -2$. Hence, $\phi_{N_i \cup \sigma(x_i)}(\sigma^*, \sigma) \geq -2 + \phi(\pi_i^*, \pi_i) \geq 0$ by Lemma A.2.

Otherwise, $Z \cap \sigma(x_i) \subseteq N(\sigma^*, \sigma)$ and the only agent in N_i that can be worse off in π_i compared to σ is x. It follows $\phi_{N_i \cup \sigma(x_i)}(\sigma^*, \sigma) = \phi_{N_i}(\sigma^*, \sigma) + \phi_{\sigma(x_i) \cap Z}(\sigma^*, \sigma) \geq \phi_{N_i}(\sigma^*, \sigma) + 2 \geq -2 + \phi(\pi_i^*, \pi_i) + 2 \geq 0$.

Together, it is shown that σ^* is popular.

Conversely, assume that (R, S) is a 'yes'-instance and assume for contradiction that σ is popular and define $\pi_i = \{\sigma(y) \cap N_i\colon y \in N_i\}$ as above. The Pareto-improvements of the first implication show that for all j, $Z^j \in \sigma$ or $\sigma(x_i) \cap Z^j \neq \emptyset$. Define $I = \{i \in \{1, 2\}\colon Z \cap \sigma(x_i) \neq \emptyset$. The first crucial step is to prove that for all $i \in I$, it holds that there exists a $j \in \{1, 2, 3\}$ with $\sigma(x_i) = \{x_i\} \cup Z^j$.

Let therefore $i \in I$. First, $\sigma(x_i) \cap N_i = \{x_i\}$ since otherwise splitting $\sigma(x_i)$ into singleton coalitions is more popular. In addition, $x_{3-i} \notin \sigma(x_i)$. If this happens and $|\sigma(x_i) \cap Z| \neq 2$, then splitting into singleton coalitions is more popular. On the other hand, if $|\sigma(x_i) \cap Z| = 2$, there exists $j^* \in \{1, 2, 3\}$ with $S^{j^*} \in \sigma$. We form the partition σ' by leaving her coalition with x_1 and forming $\{x_1, z_1^{j^*}, z_2^{j^*}\}$. Then,

$\{x_1, x_2, z_1^{j^*}, z_2^{j^*}\} \subseteq N(\sigma', \sigma)$ while $N(\sigma, \sigma') \subseteq \sigma(x_i) \cap Z$. Hence, σ' is more popular.

Hence, $\sigma(x_i) \subseteq Z \cup \{x_i\}$. If for $j \neq j'$, $Z^j \cap \sigma(x_i) \neq \emptyset$ and $Z^{j'} \cap \sigma(x_i) \neq \emptyset$, then dissolving $\sigma(x_i)$ is again more popular. Finally, if $|\sigma(x_i) \cap Z| = 1$, we find again a $j^* \in \{1,2,3\}$ with $S^{j^*} \in \sigma$. We form the partition σ' by forming $\pi(x_i) \cap Z$ and $\{x_i, z_1^{j^*}, z_2^{j^*}\}$ which is more popular.

The next step is to show that $I = \{1,2\}$. Assume for contradiction that $Z \cap \sigma(x_i) = \emptyset$. Then we can assume that for all $y \in N_i$, $\sigma(y) \subseteq N_i$. If $\pi_i \neq \pi_i^*$, then replacing π_i by π_i^* is more popular (by Lemma A.2). Otherwise $\pi_i = \pi_i^*$ and we consider the partition π_i' of the last part of Lemma A.2 for N_i. By the pigeon hole principle, here exists a $j^* \in \{1,2,3\}$ with $S^{j^*} \in \sigma$. We obtain $\sigma' = (\sigma \backslash (\pi_i \cup \{S^{j^*}\})) \cup ((\pi_i' \backslash \{\{x_i\}\}) \cup \{\{x_i, z_1^{j^*}, z_2^{j^*}\}\})$. Then, $\phi(\sigma', \sigma) = \phi_{N_i \cup Z^{j^*}}(\sigma', \sigma) = -1 + 2 = 1$ and σ' is more popular.

Together, we can assume that there exist $j_1, j_2 \in \{1,2,3\}$ with $\sigma(x_i) = \{x_i, z_1^{j_i}, z_2^{j_i}\}$, for $i = 1,2$. Let $j_3 \in \{1,2,3\} \backslash \{j_1, j_2\}$ be the third index. Note that $Z^{j_3} \in \sigma$. Define $\sigma' = (\sigma \backslash \{\sigma(z_1^j) : j = 1,2,3\}) \cup \{\{x_1, z_1^{j_2}, z_2^{j_2}\}, \{x_2, z_1^{j_3}, z_2^{j_3}\}, Z^{j_1}\}$. Then, $N(\sigma', \sigma) = Z^{j_2} \cup Z^{j_3}$ while $N(\sigma, \sigma') = Z^{j_1}$. Hence, σ' is more popular.

All in all, it is shown that there exists no popular partition if (R, S) is a 'yes'-instance. This concludes the proof of the theorem. □

A.3 Fractional Hedonic Games

Now, we consider the existence and verification problem for popular partitions in fractional hedonic games. The strategy is similar to the case of ASHGs. Again, there exist gadgets for every element of R and the sets in S. The R-gadgets rely on rather simple graphs, namely stars.

We define by S_k the star graph with k leaves, i.e. $S_k \cong G$, where $G = (V, E)$ with $V = \{c, l_1, \dots, l_k\}$, $E = \{\{c, l_j\} : j = 1, \dots, k\}$.

Proposition A.3. *Let S_k be a star with center c and leaves $l_1, \dots, l_k$. Assume that all utilities are 1 (or constant). For $k \leq 5$, the (sub-)partition (of) $\pi = \{\{c, l_1, l_2, l_3\}, \{l_4\}, \{l_5\}\}$ is popular. For $k \geq 6$, S_k admits no popular partition.*

Proof. The first part is easily seen.

For the second assertion, let $k \geq 6$ and assume that π was a popular partition. Then, $|\pi(c)| \leq 4$, since otherwise we obtain a more popular partition if one leaf leaves $\pi(c)$. But in this case, the grand coalition is more popular (having c and at least $k - 3$ leaves better off).

□

Proof of Theorem 4.21. The reduction is from X3C to deciding whether there exists a popular partition.

Let (R, S) be an instance of X3C. We transform it into an FHG $(N, \succsim)$ defined by the graph $G = (N, E)$ that is given as follows:

$N = \{c^r, l_j^r : r \in R, j = 1, \dots, 6\} \cup \{y^s, z_j^s : s \in S, j = 1,2\}$ and $E = E^R \cup E^C \cup E_6 \cup E^S$ where $E^R = \{\{c^r, l_j^r\} : r \in R, j = 1, \dots, 6\}$, $E^C = \{\{l_6^r, y^s\} : r \in s \in S\}$, $E_6 = \{\{l_6^r, l_6^t\} : r \neq t, r, t \in s \text{ for } s \in S\}$, $E^S = \{\{y^s, z_j^s\}, \{z_1^s, z_2^s\} : s \in S, j = 1,2\}$. The edge set E^C connects the gadgets for the ground set and the subsets for the X3C instance.

The weights are 1, except $v(e) = \frac{1}{2}$ for $e \in E^C$ and $v(e) = \frac{1}{4}$ for $e \in E_6$. A schematic of the reduction for a certain set $s = \{i, j, k\} \in S$ is depicted in Figure 6.

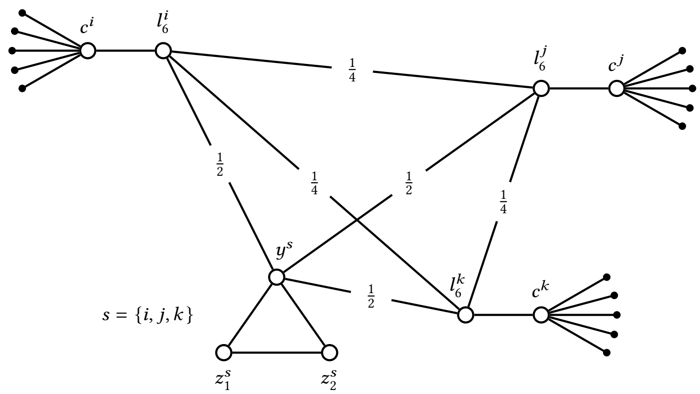

Figure 6: Reduction for existence problem of popular partitions in FHGs. The schematic displays the part of the network corresponding to one specific set $s = \{i, j, k\}$.

We show that there exists a popular partition of $(N, \succsim)$ if and only if (R, S) is a 'yes' instance of X3C.

Assume (R, S) is a 'yes' instance of X3C. Then, there exists $S' \subseteq S$ such that S' is a partition of R. The following partition π is then popular: $\pi = \{\{c^r, l_1^r, l_2^r, l_3^r\} : r \in R\} \cup \{\{l_j^r\} : r \in R, j = 4,5\} \cup \{\{y^s, l_6^i, l_6^j, l_6^k\} : s = \{i,j,k\} \in S'\} \cup \{\{z_1^s, z_2^s\} : s \in S'\} \cup \{\{y^s, z_1^s, z_2^s\} : s \in S \backslash S'\}$.

Assume for contradiction that π' is more popular than π and let π be with $\phi(\pi'.\pi)$ maximal. We will prove that $\phi(\pi, \pi') \geq 0$, a contradiction.

We introduce some notation for the proof. $V^r = \{c^r, l_j^r : j = 1, \dots, 6\}$, where $r \in R$, and $W^s = \{y^s, z_1^s, z_2^s\}$, where $s \in S$. Also denote $V^R = \cup_{r \in R} V^r$, $W^S = \cup_{s \in S} W^s$ and $A_6 = \{a_6^r : r \in R\}$ and $Y^c = \{s \in S : \exists a \in A_6 \text{ with } a \in \pi'(y^s)\}$.

Recall that for any subset $M \subseteq N$ of agents and partitions σ, σ' of N, $\phi_M(\sigma, \sigma') = |N(\sigma, \sigma') \cap M| - |N(\sigma', \sigma) \cap M|$.To derive a contradiction, we prove several claims.

(1) Let $r \in R$ such that for all $s \in S$, $y^s \notin \pi'(a_6^r)$. Then $\phi_{V^r} \geq 1$.
(2) $\nexists r \in R, s, s' \in S$ with $s \neq s'$ and $\{y^s, y^{s'}\} \subseteq \pi'(a_6^r)$.
(3) $\forall s \in S$ holds: $\pi'(y^s) \cap W^s = \{y^s\} \vee \pi'(y^s) \subseteq W^s$.
(4) For all $r \in R$, $\phi_{V^r}(\pi, \pi') \geq 0$.
(5) $\phi_{W^S}(\pi', \pi) \leq |R| - 3|Y^c|$.

The first claim says that we need sufficient external influence for V^r to be locally popular. The second and third claim give insight on the structure of possible more popular partitions. The forth claim shows that we locally do best for every V^r. The final claim calculates the tradeoff between forming a coalition W^s and joining the agents in V^r.

In order to complete the proof from the claims, we apply claims 1 and 4 to obtain $\phi_{V^R}(\pi, \pi') \geq \max\{0, |R| - 3|Y^c|\} \geq |R| - 3|Y^c|$. Combining this inequality with the one of Claim 5 yields $\phi(\pi, \pi') \geq 0$.

The first claim is a straightforward case distinction considering $\pi'(c^r)$. Observe that by construction of its neighboring agents, $a_6^r \in N(\pi,\pi') \vee a_6^r \in \pi'(c^r)$. This property makes it equivalent to the agents a_5^r and a_4^r in the analysis.

We proceed with the second claim. Therefore, assume for contradiction that $r \in R, s, s' \in S$ with $s \neq s'$ and $\{y^s, y^{s'}\} \subseteq \pi'(a_6^r)$. We denote $C = \pi'(a_6^r)$ for this part. We claim that we can change π' while strictly increasing $\phi(\pi',\pi)$. This is done by forming a partition π'' that consists of coalitions W^t whenever $y^t \in C$. The agents outside W^S in C form a coalition of their own. Other coalitions are not changed.

- Let $t \in S$ with $y^t \in C$. If $\pi(y^t) = W^t$, then $W^t \subseteq N(\pi,\pi')$. This is immediate for the z_j^t. In addition, by assumption on C, at least 3 agents are present, and the utility is estimated as $u_{\pi'}(y^t) \leq \max\{\frac{\frac{1}{2}}{3}, \frac{\frac{3}{2}}{4}, \frac{\frac{5}{2}}{5}, \frac{\frac{6}{2}}{6}, \frac{\frac{7}{2}}{7}\} = \frac{1}{2} < \frac{2}{3} = u_\pi(y^t)$
- If $\pi(y^t) \neq W^t$, then $z_1^t, z_2^t \notin N(\pi',\pi)$ and $y^t \notin N(\pi',\pi) \vee (\exists i : z_i^t \in N(\pi,\pi'))$.
  Define $Y := |\{t \in S : y^t \in C\}|$. These first two insights yield, that $\phi_{\{W^s : s \in Y\}}(\pi'',\pi) \geq 3|Y| + \phi_{\{W^s : s \in Y\}}(\pi',\pi)$. There is an increase of at least 6 by the assumption that $\{y^s, y^{s'}\} \subseteq C$.
- The only agents that can decrease (π'',π) compared to (π',π) are in A_6. Note that if $a \in A_6 \cap C$ has at most one neighbor in Y, then for some p (the number of neighbors in A_6), $u_{\pi'}(a) = \frac{\frac{1}{2}+\frac{p}{4}}{3+p} < \frac{1}{4} = u_\pi(a)$. Define the improving agents in A_6 via $\mathcal{I} = C \cap A_6 \cap N(\pi',\pi)$ and the non-worsened agents as $\mathcal{I}' = C \cap A_6 \setminus (\mathcal{I} \cup N(\pi,\pi'))$.
  – If $|\mathcal{I}| \leq 2$, then $\phi_{C\cap A_6}(\pi'',\pi) \geq \phi_{C\cap A_6}(\pi',\pi) - 4$ (the agents in $\mathcal{I}$ each counted twice for being worse instead of better off).
  – If $|\mathcal{I}| \geq 3$, we know that $|Y| \geq 3$ (otherwise, three agents in $\mathcal{I}$ are incident to the same two y^t, but then in the instance of X3C, we had two identical 3-elementary sets). This means for any $a \in A_6 \cap C$ that has exactly two neighbors in Y that for some p, $u_{\pi'}(a) \leq \frac{1+\frac{p}{4}}{4+p} = \frac{1}{4}$. Hence, $a \notin N(\pi',\pi)$.
  Agents in $\mathcal{I}$ need therefore three neighbors in Y and agents in $\mathcal{I}'$ two. Since every agent in Y has at most three neighbors, this accumulates to $|Y| \geq |\mathcal{I}| + \frac{2}{3}|\mathcal{I}'|$.
  Consequently, for $M = C \cap (A_6 \cup W^S)$

$$\begin{aligned}\phi(\pi'',\pi) =&\phi_{N\setminus M}(\pi'',\pi) + \phi_{C\cap A_6}(\pi'',\pi)\\ &+ \phi_{W^S\cap C}(\pi'',\pi)\\ \geq&\phi_{N\setminus M}(\pi',\pi) + \phi_{C\cap A_6}(\pi',\pi) - 2|\mathcal{I}|\\ &- |\mathcal{I}'| + \phi_{W^S\cap C}(\pi',\pi) + 3|Y|\\ >&\phi(\pi',\pi)\end{aligned}$$

In both cases, we contradict the maximality of $\phi(\pi',\pi)$.

The third claim is proven similarly, but we have to refine some calculation of the previous claim, since we do not get the same lower bounds for the denominators of the utilities.

Assume for contradiction that $s \in S$ with $\pi'(y^s) \cap A_6 \neq \emptyset$ and $\pi'(y^s) \cap W^s \neq \emptyset$. We set $C = \pi'(y^s)$.

- First, we argue that we may assume that $A_6 \cap C \cap N(\pi',\pi) = \emptyset$. Otherwise, by the previous claim, if $a_6^r \in A_6 \cap C \cap N(\pi',\pi)$, then $c^r \in C$. Consequently, $a_j^r \in N(\pi,\pi')$ for $j = 1,2,3$ and $c^r \in N(\pi,\pi')$. The latter is due to $u_{\pi'}(c^r) \leq \frac{6}{9} < \frac{3}{4} = u_\pi(c^r)$. Also, $(\exists j \in \{4,5\} : a_j^r \notin C) \vee a_6^r \notin N(\pi',\pi)$. Indeed, if the first is wrong, then for some p, $u_{\pi'}(a_6^r) \leq \frac{1+\frac{1}{2}+\frac{p}{4}}{6+p} = \frac{1}{4} = u_\pi(a_6^r)$. Hence resetting the coalition within V^r to π yields a coalition contradicting the maximality of $\phi(\pi',\pi)$.
- We consider two cases. First assume that $\pi(y^s) \neq W^s$. We claim that rearranging π' by means of removing agents of W^s from $\pi'(y^s)$ improves $\phi(\pi',\pi)$. Indeed, $z_j^s \notin N(\pi',\pi)$, but they will be after the rearrangement, and $y^s \in N(\pi',\pi)$ afterwards. Also, for all $a \in A_6 \cap C$, $u_{\pi'}(a) \leq \frac{\frac{1}{2}+\frac{p}{4}}{p+3} < \frac{1}{4}$ and these agents are already worse off in the original π'.
- If $\pi(y^s) = W^s$, the same holds for agents in $A_6 \cap C$. Since $W^s \subseteq N(\pi,\pi')$, the same rearrangement improves $\phi(\pi',\pi)$.

We proceed with the next claim and fix $r \in R$. We may assume that for some s, $y^s \in \pi'(a_6^r)$ (since the other case is already covered in the first claim). In addition, if $c^r \notin \pi'(a_6^r)$, then $a_6^r \notin N(\pi',\pi)$ (by the previous claims). In this case, the coalition π restricted to $V^r \setminus \{a_6^r\}$ is popular and the claim is true.

Denote $C = \pi'(a_6^r)$ and assume therefore $c^r \in C$. We also know that $\{a_1,a_2,a_3\} \cap N(\pi',\pi) = \emptyset$ and $|\{a_1,a_2,a_3\} \cap N(\pi,\pi')| \geq 2$. Consequently, if $\{a_4,a_5\} \cap C = \emptyset$, we are done. If $\{a_4,a_5\} \cap C \neq \emptyset$, $\{a_1,a_2,a_3\} \subseteq N(\pi,\pi')$. Putting the final case together, $|N(\pi',\pi)| \leq 3$ while $|N(\pi,\pi')| \geq 3$ and the claim is true.

For the fifth claim, we consider the coalitions in π for different y^s:

- If $W^s = \pi(y^s)$, then $W^s \cap N(\pi',\pi) = \emptyset$ (by Claim 3) and if $s \in Y^c$, then $W^s \subseteq N(\pi,\pi')$. This gives $|N(\pi,\pi') \cap W^S| \geq 3|\{s \in Y^c : \pi(y^s) = W^s\}|$.
- If $W^s \neq \pi'(y^s)$ and $s \in Y^c$, then $W^s \cap N(\pi',\pi) = \emptyset$ (again using Claim 3). Consequently, $|N(\pi',\pi) \cap W^S| \leq 3|\{s \notin Y^c : \pi(y^s) \neq W^s\}|$.
  Combining the inequalities yields

$$\begin{aligned}&|N(\pi',\pi) \cap W^S| - |N(\pi,\pi') \cap W^S|\\ &\leq 3(|\{s \notin Y^c : \pi(y^s) \neq W^s\}| - |\{s \in Y^c : \pi(y^s) = W^s\}|)\\ &= 3(|\{s \notin Y^c : \pi(y^s) \neq W^s\}| + |\{s \in Y^c : \pi(y^s) \neq W^s\}|\\ &- |\{s \in Y^c : \pi(y^s) \neq W^s\}| - |\{s \in Y^c : \pi(y^s) = W^s\}|)\\ &= 3|S'| - 3|Y^c| = |R| - 3|Y^c|.\end{aligned}$$

This proves the final claim and we have proved that 'yes'-instances of X3C map to popular partitions of the FHG.

For the reverse implication, assume that π is a popular partition. We exhibit the coalitions of the agents in A_6.

(1) For all $r \in R$, there exists a unique $s \in S$ with $y^s \in \pi(a_6^r)$. For this s holds that $r \in s$.
(2) For all $r \in R$, $|A_6 \cap \pi(a_6^r)| = 3$.

If the claims are true, $S' := \{s \in S : A_6 \cap \pi(y^s) \neq \emptyset\}$ covers R due to existence and is a partition due to uniqueness and the fact, that uniqueness and the second claim imply that the coalition of the unique y^s must contain precisely a_6^i for $i \in s$.

We start with the first claim. Existence is clear because otherwise the subpartition of π on V^r (possibly restricted to V^r) is popular on V^r, contradicting Proposition A.3.

For uniqueness, assume for contradiction that there is $r \in R$ and $s \neq s' \in S$ with $\{y^s, y^{s'}\} \subseteq \pi(a_3^r)$. We obtain a more popular coalition π' as follows: remove the agents in W^s from their partitions in π and let them form a coalition. Then $W^s \cup \{y^{s'}\} \subseteq N(\pi', \pi)$ and $N(\pi, \pi') \subseteq \{a_6^r : r \in s\}$. Hence, π' is more popular.

For the second claim, we know due to uniqueness in the first claim that $|A_6 \cap \pi(a_6^r)| \leq 3$. Assume for contradiction that $|A_6 \cap \pi(a_6^r)| < 3$ and let $y^s \in \pi(a_6^r)$. Then, the same coalition π' as in the proof of the previous claim is more popular. This time, $W^s \subseteq N(\pi', \pi)$ and $N(\pi, \pi') \subseteq \{a_6^r : r \in s\}$, hence by assumption $|N(\pi, \pi')| \leq 2$.

□

Proof of Theorem 4.22. First of all, the verification problem is in coNP, because a more popular partition serves as a polynomial-time certificate for a 'no'-instance.

For hardness, we reduce again from X3C. Given an instance (R, S) of X3C, we assume without loss of generality that $|R| \geq 6$. We define an FHG $(N, \succsim)$ given by the underlying graph $G = (N, E)$ depicted in Figure 7 and defined as:

$N = R \cup \{s_1, s_2, s_3 : s \in S\} \cup \{b_1, b_2, b_3\}$, $E = \{\{s_3, r\} : r \in R \cap s\} \cup \{\{s_1, s_3\}, \{s_2, s_3\} : s \in S\} \cup \{\{s_j, b_j\} : s \in S, j = 1, 2\} \cup \{\{b_1, b_3\}, \{b_2, b_3\}\}$.

The symmetric weights v are given as follows:

- $v(i, s_3) = \frac{1}{2}$ if $i \in s$
- $v(s_1, s_3) = v(s_2, s_3) = 1$ for $s \in S$
- $v(s_j, b_j) = \frac{1}{4}$ for $s \in S, j = 1, 2$
- $v(b_1, b_3) = v(b_2, b_3) = \alpha$ for $\frac{3(|R|-3)}{4|R|} < \alpha < \frac{3|R|}{4(|R|+3)}$

One can choose α with a size bounded polynomially in the input size. For the reduction, only the above bounds matter. We introduce the same notation as in the proof for ASHGs. Denote $V^s = \{s_1, s_2, s_3\}$ for $s \in S$, $B = \{b_1, b_2, b_3\}$, and $V = \cup_{s \in S} V^s$.

G is bipartite with bipartition $(R \cup \{s_1, s_2 : s \in S\} \cup \{b_3\}, \{s_3 : s \in S\} \cup \{b_1, b_2\})$ and all weights on present edges are positive.

The verification problem is asked for the partition $\pi = \{V^s : s \in S\} \cup \{\{r\} : r \in R\} \cup \{B\}$. We claim that (R, S) is a 'yes'-instance of X3C if and only if π is not popular for the FHG given by G.

If (R, S) is a 'yes'-instance, there exists a subset $S' \subseteq S$ that partitions R. In particular $|R| = 3|S'|$.

Consider the partition given by $\pi' = \{V^s : s \in S \backslash S'\} \cup \{\{s_3, i, j, k\} : \{i, j, k\} = s \in S'\} \cup \{\{b_j, s_j : s \in S'\} : j = 1, 2\} \cup \{\{b_3\}\}$.

Then, for $j = 1, 2$, $u_{\pi'}(b_j) = \frac{\frac{1}{4}|S'|}{|S'|+1} = \frac{|R|}{4(|R|+3)} > \frac{\alpha}{3} = u_\pi(b_j)$. Since all agents in R have clearly improved their utility, $R \cup \{b_1, b_2\} \subseteq N(\pi', \pi)$ (and in fact equality holds here). Moreover, the utilities of agents in V^s for $s \in S \backslash S'$ have not changed. Consequently, $N(\pi, \pi') \subseteq \cup_{s \in S'} V^s \cup \{b_3\}$. Hence, π' is more popular than π.

Conversely, assume that there exists a more popular partition π' and fix one that maximizes $\phi(\pi', \pi)$. We have to prove that there exists a subset $S' \subseteq S$ that yields a partition of R.

First, we make the observation that if $b_j \in N(\pi', \pi)$ for $j = 1, 2$, then $b_3 \in N(\pi, \pi')$. Hence, $\phi_B(\pi', \pi) \leq 1$.

Second, we claim that for all $s \in S$, $N(\pi', \pi) \cap V^s = \emptyset$. Clearly, $s_3 \notin N(\pi', \pi)$ (by construction, since she receives a top coalition with respect to the given utilities). Assume for $j = 1, 2$, $s_j \in N(\pi', \pi)$. Then, $\pi'(s_j) = \{s_j, s_3, b_j\}$. Note that both neighbors of s_j are needed to improve utility, but no other agent may be present since for $|\pi'(s_j)| \geq 4$ follows $u_{\pi'}(s_j) \leq \frac{\frac{5}{4}}{4} < \frac{1}{3} = u_\pi(s_j)$. In addition, $s_{3-j}, b_3 \in N(\pi, \pi')$.

We form a new coalition π'' from π' by having the coalitions V^s and B and all other coalitions remain the same. The exact same case distinction for b_{3-j} as in the case of ASHGs yields a contradiction to the maximality condition on π'.

The remainder of the proof follows a similar strategy as for ASHGs, but some arguments are more involved.

To make this more formal, we introduce the sets $R_I = R \cap N(\pi', \pi)$ of agents in R that form a coalition with a neighbor in π' and $S_C = \{s \in S : \pi'(s_3) \cap R \neq \emptyset\}$. The latter is the set of critical sets in S whose corresponding agents s_3 form a coalition with agents in R. We split it into $S_{C,1} = \{s \in S : |\pi'(s_3) \cap R| = 1\}$ and $S_{C,2} = S_C \backslash S_{C,1}$.

We have the following facts:

- For $s \in S_C$, $s_3 \in N(\pi, \pi')$.
- For $s \in S_{C,1}$, $s_1 \in N(\pi, \pi') \vee s_2 \in N(\pi, \pi')$.
- For $s \in S_{C,2}$, $s_1 \in N(\pi, \pi') \wedge s_2 \in N(\pi, \pi')$.

Consequently, $|N(\pi, \pi') \cap V| \geq 2|S_{C,1}| + 3|S_{C,2}|$. In addition, $|N(\pi', \pi) \cap R| = |R_I| \leq |S_{C,1}| + 3|S_{C,2}|$.

If $S_{C,1} \neq \emptyset$, then $\phi(\pi, \pi') = \phi_B(\pi, \pi') + \phi_V(\pi, \pi') + \phi_R(\pi, \pi') \geq -1 + 2|S_{C,1}| + 3|S_{C,2}| - (|S_{C,1}| + 3|S_{C,2}|) = |S_{C,1}| - 1 \geq 0$ and π' is not more popular. We conclude that $S_{C,1} = \emptyset$ or equivalently $S_C = S_{C,2}$.

A similar calculation excludes the case $|R_I| < 3|S_{C,2}|$ which means $|R_I| = 3|S_{C,2}|$.

We claim that in fact $|R| = 3|S_C| = 3|S_{C,2}|$. Before we prove this, we show the same two auxiliary claims as for ASHGs.

(1) If $B \subseteq \pi'(b_3)$ then $b_1 \notin N(\pi', \pi) \vee b_2 \notin N(\pi', \pi)$.
(2) For $j = 1, 2$, if $b_j \in N(\pi', \pi)$, then $b_j \in \pi'(b_3) \vee |\{s \in S : s_j \in \pi'(b_j)\} \cap \pi'(b_j)| \geq \frac{|R|}{3}$.

For the first claim, assume that $B \subseteq \pi'(b_3)$ and $b_1, b_2 \in N(\pi', \pi)$. Denote $p_j = |\{s \in S : s_j \in \pi'(b_3)\}|$. We know that $p_j \geq 1$, since otherwise $b_j \notin N(\pi', \pi)$.

The function $x \mapsto \frac{3(x-3)}{4x}$ is monotonically increasing for $x > 0$. Thus, by the lower bound on α, we know that $\alpha > \frac{3}{8}$ (using $|R| \geq 6$).

Let $j \in \{1, 2\}$ with $p_j = \min\{p_j, p_{3-j}\}$. Then $|\pi'(b_3)| \geq 3 + 2p_j$. We compute $u_\pi(b_j) - u_{\pi'}(b_j) = \frac{\alpha}{3} - \frac{\alpha + \frac{p_j}{4}}{3+2p_j} = \frac{p_j}{3(3+2p_j)}(2\alpha - \frac{3}{4}) > 0$. Hence, $b_j \notin N(\pi', \pi)$, a contradiction.

For the second claim, let $j \in \{1, 2\}$ with $b_j \in N(\pi', \pi)$ and assume $b_j \notin \pi'(b_3)$. Similarly as before, let $p = |\{s \in S : s_j \in \pi'(b_j)\}|$. Note that $u_\pi(b_j) = \frac{\alpha}{3} > \frac{|R|-3}{4|R|} = \frac{1}{4}\frac{\frac{|R|}{3}-1}{\left(\frac{|R|}{3}-1\right)+1}$. Therefore, $u_\pi(b_j) < u_{\pi'}(b_j) \leq \frac{1}{4}\frac{p}{p+1}$ only if $p > \frac{|R|}{3} - 1$ and since p is an integer, this implies $p \geq \frac{|R|}{3}$.

The remainder of the proof is identical to the one for ASHGs (Theorem 4.16). □

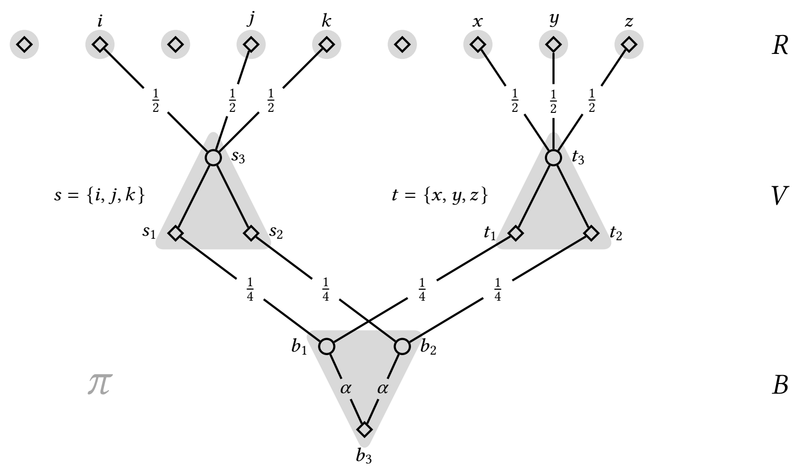

Figure 7: Schematic of the reduction for the verification problem of popular partitions on bipartite FHGs. The bipartition is indicated by the shapes of the agents. The partition π under consideration is marked in gray.