Strategic Abstention based on Preference Extensions:
Positive Results and Computer-Generated Impossibilities

Florian Brandl  Felix Brandt  Christian Geist  Johannes Hofbauer
Technische Universität München
85748 Garching bei München, Germany
{brandlfl,brandtf,geist,hofbauej}@in.tum.de

Abstract

Voting rules are powerful tools that allow multiple agents to aggregate their preferences in order to reach joint decisions. A common flaw of some voting rules, known as the no-show paradox, is that agents may obtain a more preferred outcome by abstaining an election. We study strategic abstention for set-valued voting rules based on Kelly’s and Fishburn’s preference extensions. Our contribution is twofold. First, we show that, whenever there are at least five alternatives, every Pareto-optimal majoritarian voting rule suffers from the no-show paradox with respect to Fishburn’s extension. This is achieved by reducing the statement to a finite—yet very large—problem, which is encoded as a formula in propositional logic and then shown to be unsatisfiable by a SAT solver. We also provide a human-readable proof which we extracted from a minimal unsatisfiable core of the formula. Secondly, we prove that every voting rule that satisfies two natural conditions cannot be manipulated by strategic abstention with respect to Kelly’s extension. We conclude by giving examples of well-known Pareto-optimal majoritarian voting rules that meet these requirements.

1 Introduction

Whenever a group of multiple agents aims at reaching a joint decision in a fair and satisfactory way, they need to aggregate their (possibly conflicting) preferences. Voting rules are studied in detail in social choice theory and are coming under increasing scrutiny from computer scientists who are interested in their computational properties or want to utilize them in computational multiagent systems [Chevaleyre et al., 2007; Brandt et al., 2013].

A common flaw of many such rules, first observed by Fishburn and Brams [1983], who called it the no-show paradox, is that agents may obtain a more preferred outcome by abstaining an election. Following Moulin [1988], a voting rule is said to satisfy participation if it is immune to the no-show paradox. Moulin has shown that all resolute, i.e., single-valued, scoring rules (such as Borda’s rule) satisfy participation while all resolute Condorcet extensions suffer from the no-show paradox. Condorcet extensions comprise a large class of voting rules that satisfy otherwise rather desirable properties.

In this paper, we study participation for irresolute, i.e., set-valued, social choice functions (SCFs). A proper definition of participation for irresolute SCFs requires the specification of preferences over sets of alternatives. Rather than asking the agents to specify their preferences over all subsets (which would be bound to various rationality constraints and require exponential space), it is typically assumed that the preferences over single alternatives can be extended to preferences over sets. Of course, there are various ways how to extend preferences to sets [see, e.g., Gärdenfors, 1979; Barberá et al., 2004], each of which leads to a different version of participation. A function that yields a (possibly incomplete) preference relation over subsets of alternatives when given a preference relation over single alternatives is called a preference extension. In this paper, we focus on two common preference extensions due to Kelly [1977] and Fishburn [1972], both of which arise under natural assumptions about the agents’ knowledge of the tie-breaking mechanism that eventually picks a single alternative from the choice set [see, e.g., Gärdenfors, 1979; Ching and Zhou, 2002; Sanver and Zwicker, 2012; Brandt, 2011; Brandt and Brill, 2011; Brandt, 2015]. Kelly’s extension, for example, can be motivated by assuming that the agents possess no information whatsoever about the tie-breaking mechanism. A common interpretation of Fishburn’s extension, on the other hand, is that ties are broken according to the unknown preferences of a chairman. Since Fishburn’s extension is a refinement of Kelly’s extension it follows that Fishburn-participation is stronger than Kelly-participation. The idea pursued in this paper is to exploit the uncertainty of the agents about the tie-breaking mechanism in order to prevent strategic abstention. Our two main results are as follows.

• Whenever there are at least four alternatives, Pareto-optimality and Fishburn-participation are incompatible in the context of majoritarian SCFs. When there are at least five alternatives, this even holds for strict preferences.

• Every SCF that satisfies set-monotonicity and independence of indifferent voters satisfies Kelly-participation. Every set-monotonic majoritarian SCF satisfies Kelly-participation when preferences are strict.
The first result is obtained using computer-aided theorem proving techniques. In particular, we reduce the statement to a finite—yet very large—problem, which is encoded as a formula in propositional logic and then shown to be unsatisfiable by a SAT solver. We also provide a human-readable proof for this result, which we extracted from a minimal unsatisfiable core of the SAT formula.

The conditions for the second result are easy to check and satisfied by a number of well-studied SCFs, including Pareto-optimal majoritarian SCFs. In contrast to Moulin’s negative result for resolute SCFs, there are appealing Condorcet extensions that satisfy Kelly-participation.

Our negative result holds even for strict preferences while our positive result holds even for weak preferences. The latter is somewhat surprising and stands in sharp contrast to the related finding that no Condorcet extension satisfies Kelly-strategyproofness when preferences are weak (recall that an SCF is strategyproof if no agent can obtain a more preferred outcome by misrepresenting his preferences) [Brandt, 2011, 2015].

Participation is similar to, but logically independent from, strategyproofness. Manipulation by abstention is arguably a more severe problem than manipulation by misrepresentation for two reasons. First, agents might not be able to find a beneficial misrepresentation. It was shown in various papers that the corresponding computational problem can be intractable [see, e.g., Faliszewski et al., 2010]. Finding a successful manipulation by strategic abstention, on the other hand, is never harder than computing the outcome of the respective SCF. Secondly, one could argue that agents will not lie about their preferences because this is considered immoral (Borda famously exclaimed “my scheme is intended only for honest men”), while strategic abstention is deemed acceptable. \(^1\)

3 Preliminaries

Let \( A \) be a finite set of alternatives and \( \mathbb{N} = \{1, 2, \ldots \} \) a set of agents. \( \mathcal{T}(\mathbb{N}) \) denotes the set of all finite and non-empty subsets of \( \mathbb{N} \). A (weak) preference relation is a complete, reflexive, and transitive binary relation on \( A \). The preference relation of agent \( i \) is denoted by \( R_i \). The set of all preference relations is denoted by \( \mathcal{R} \). In accordance with conventional notation, we write \( P_i \) for the strict part of \( R_i \), i.e., \( x P_i y \) if \( x R_i y \) but not \( y R_i x \), and \( I_i \) for the indifference part of \( R_i \), i.e., \( x I_i y \) if \( x R_i y \) and \( y R_i x \). A preference relation \( R_i \) is called strict if it additionally is anti-symmetric, i.e., \( x P_i y \) or \( y P_i x \) for all distinct alternatives \( x, y \). We will compactly represent a preference relation as a comma-separated list with all alternatives among which an agent is indifferent placed in a set. For example \( x P_i y I_i z \) is represented by \( R_i: x, \{ y, z \} \).

A preference profile \( R \) is a function from a set of agents \( A \) to the set of preference relations \( \mathcal{R} \). The set of all preference profiles is denoted by \( \mathcal{R}^A(\mathbb{N}) \). For a preference profile \( R \) and a preference relation \( R_i \), we define

\[
R_{-i} = R \setminus \{(i, R_i)\} \quad \text{and} \quad R_{+i} = R \cup \{(i, R_i)\}.
\]

The majority relation of \( R \) is denoted by \( R_M \), where

\[
x R_M y \iff |\{ i \in N : x R_i y \}| \geq |\{ i \in N : y R_i x \}|.
\]

An alternative \( x \) is a Condorcet winner in \( R \) if \( x P_M y \) for all \( y \in A \setminus \{ x \} \).

Our central objects of study are social choice functions (SCFs), i.e., functions that map a preference profile to a set of alternatives. Formally, an SCF is a function

\[
f : \mathcal{R}^A(\mathbb{N}) \to 2^A \setminus \emptyset.
\]

\(^1\)Alternatively, one could also argue that manipulation by misrepresentation is more critical because agents are tempted to act immorally, which is a valid, but different, concern.
Two minimal fairness conditions for SCFs are anonymity and neutrality. An SCF is anonymous if the outcome does not depend on the identities of the agents and neutral if it is symmetric with respect to alternatives. An SCF \( f \) is majoritarian (or a neutral C1 function) if it is neutral and for all \( R, R' \in \mathcal{R}^N \), \( f(R) = f(R') \) whenever \( R_M = R'_M \). Even the seemingly narrow class of majoritarian SCFs contains a variety of interesting functions (sometimes called tournament solutions). Examples include Copeland’s rule, the top cycle, and the uncovered set [see, e.g., Brandt et al., 2015]. These functions are usually also Condorcet extensions, i.e., SCFs that uniquely return a Condorcet winner whenever one exists.

Next we introduce a very weak variable electorate condition which requires that a completely indifferent agent does not affect the outcome. An SCF \( f \) satisfies independence of indifferent voters (IIV) if

\[
f(R) = f(R_{i+1}) \text{ for all } R \in \mathcal{R}^N,
\]

where \( i \) is an agent who is indifferent between all alternatives, i.e., \( x I_i y \) for all \( x, y \in A \). It is easy to see that every majoritarian SCF satisfies anonymity, neutrality, and IIV.

We say that \( R' \) is an \( f \)-improvement over \( R \) if alternatives chosen in \( R \) are improved from \( R \) to \( R' \), i.e., \( x R'_i y \) implies \( x R_i y \) and \( y R'_i x \) implies \( y R_i x \) for all \( x \in f(R), y \in A \), and \( i \in N \). An SCF \( f \) satisfies set-monotonicity if

\[
f(R) = f(R') \text{ whenever } R' \text{ is an } f \text{-improvement over } R.
\]

The two preference extensions we consider in this paper are Kelly’s extension and Fishburn’s extension. For all \( X, Y \subseteq A \) and \( R, R' \in \mathcal{R} \),

\[
X R_i^K Y \iff x R_i y \text{ for all } x \in X, y \in Y, \quad \text{(Kelly)} \quad X R_i^F Y \iff X \setminus Y R_i^K Y \text{ and } X R_i^K Y \setminus X. \quad \text{(Fishburn)}
\]

It follows from the definitions that Fishburn’s extension is a refinement of Kelly’s extension, i.e., \( R_i^F \subseteq R_i^K \) for every \( R_i \in \mathcal{R} \).

The following example illustrates the definition of Kelly’s extension and Fishburn’s extension. Consider the preference relation \( R_i: a, b, c, d \) and the sets of alternatives \( X = \{a, b\} \), \( Y = \{a, b, c\} \), and \( Z = \{b, d\} \). Then, \( X P_i^F Y \) and \( X P_i^K Z \). The sets \( Y \) and \( Z \) are not comparable with respect to either one of the two extensions. In the interest of space, we refer to Section 1 (and the references therein) for justifications of these extensions.

With the preference extensions at hand, we can now formally define participation and strategyproofness. An SCF \( f \) is Kelly-manipulable by strategic abstention if there exists a preference profile \( R \in \mathcal{R}^N \) and an agent \( i \in N \) such that \( f(R_{i+1}) P_i^K f(R) \). An SCF \( f \) is Kelly-manipulable if there exist preference profiles \( R, R' \in \mathcal{R}^N \), and an agent \( i \in N \), such that \( R_j = R'_j \) for all agents \( j \neq i \) and \( f(R') P_i^K f(R) \). \( f \) is said to satisfy Kelly-participation or Kelly-strategyproofness if it is not Kelly-manipulable by strategic abstention or Kelly-manipulable, respectively. Fishburn-participation and Fishburn-strategyproofness are defined analogously.

| \( |A| \) | Brandt and Geist [2014] | This paper |
|---|---|---|
| 3 | 49 | 823,543 |
| 4 | 50,625 | \( \sim 2.5 \cdot 10^{40} \) |
| 5 | \( \sim 7.9 \cdot 10^{17} \) | \( \sim 9.4 \cdot 10^{807} \) |
| 6 | \( \sim 5.8 \cdot 10^{100} \) | \( \sim 6.8 \cdot 10^{39649} \) |

Table 1: Number of different majoritarian SCFs. While Brandt and Geist [2014] could assume an odd number of agents with strict preferences, participation requires us to deal with variable electorates, and thus weak tournaments.

## 4 Computer-aided Theorem Proving

For some of our results, we are going to make use of the computer-aided proving methodology described by Brandt and Geist [2014]. The main idea is to prove statements by encoding a finite instance as a satisfiability problem, which can be solved by a computer using a SAT solver, and providing a (simple) reduction argument, which extends this result to arbitrary domain sizes. We extend their framework to also cater for indifferences in the majority relations (corresponding to weak tournaments), which is an important requirement for being able to deal with the notion of participation: if an agent with at least one strict preference abstains the election, the corresponding majority relation might already contain indifferences.\(^2\)

Note that the introduction of weak tournaments significantly increases the size of the search space (see Table 1), which makes any type of exhaustive search even less feasible. Apart from being able to treat such large search spaces, another major advantage of the computer-aided approach is that many similar conjectures and hypotheses (here, e.g., statements about other preference extensions) can be checked quickly using the same framework.

In the coming subsections, we are going to explain our extension and some core features of the computer-aided method; for details of the original approach, however, the reader is referred to Brandt and Geist [2014].

### 4.1 Encoding Participation

At the core of the computer-aided approach lies an encoding of the problem to be solved as a SAT instance. For this, all axioms involved need to be stated in propositional logic. We take over the formalization of the optimized encoding by Brandt and Geist [2014], which contains the following relevant axioms: functionality of the choice function, the orbit condition, and Pareto-optimality.\(^3\) What remains is to encode the notion of participation. While this encoding turns out to be similar to the one of strategyproofness defined by Brandt and Geist [2014], it is more complex and not straightforward. In particular, it requires a novel condition that is equivalent to participation for majoritarian SCFs, which we are going to call tournament-participation.

\(^2\)Brandt and Geist [2014] avoided this problem in their work on strategyproofness by assuming an odd number of agents with strict preferences. Obviously, one cannot assume a constant odd number of agents when considering participation.

\(^3\)The latter is encoded as being a refinement of the uncovered set.
For convenience, we first define some additional notation. We are going to identify preference profiles with their corresponding majority relations, which in turn can be represented by weak tournaments \(T\) (i.e., we write \(f(T)\) instead of \(f(R)\) for any preference profile \(R\) that has \(T\) as its majority relation). We denote the edge difference of two weak tournaments \(T, T'\) by \(\Delta_{T,T'} = \{c \in T : c \notin T'\}\) and the inverse of an edge set \(E \subseteq A \times A\) by \(E^{-1} = \{(x, y) : (y, x) \in E\}\).

**Definition 1.** A majoritarian SCF \(f\) is Fishburn-tournament-manipulable by strategic abstention if there exist weak tournaments \(T, T'\) and a preference relation \(R_{\mu} \in \mathcal{R}\) such that \(f(T') \neq f(T)\), with

\[
\begin{align*}
(\Delta_{T,T'} \cap \Delta_{T',T}^{-1}) & = \emptyset, \\
(\Delta_{T,T'} \cup \Delta_{T',T}^{-1}) & \subseteq R_{\mu}, \text{ and} \\
(T \cap T^{-1} \cap T' \cap T'^{-1}) & \subseteq R_{\mu}.
\end{align*}
\]

If the agents’ preferences are required to be strict, it additionally has to hold that either \(T\) or \(T'\) is a proper, i.e., anti-symmetric, tournament. A majoritarian SCF \(f\) satisfies Fishburn-tournament-participation if it is not Fishburn-tournament-manipulable by strategic abstention.

Conditions (1) to (3) can intuitively be phrased as follows: (1) no strict edge may be reversed between \(T\) and \(T'\), (2) \(R_{\mu}\) has to be in line with the changes from \(T\) to \(T'\), and (3) majority ties that occur in both tournaments must be reflected by an indifference in \(R_{\mu}\).

In the following lemma, we show that, for majoritarian SCFs, the condition of Fishburn-tournament-manipulability corresponds to an abstaining agent with preferences \(R_{\mu}\) who thereby obtains a preferred outcome.\(^{4}\) The proof is omitted due to space restrictions.

**Lemma 1.** A majoritarian SCF satisfies Fishburn-participation iff it satisfies Fishburn-tournament-participation.

Fishburn-tournament-participation can then be encoded in propositional logic (with variables \(c_{T,X}\) representing \(f(T) = X\)) as the following simple transformation shows:

\[\neg(f(T') \neq f(T)) \equiv \bigwedge_{Y \neq X} (\neg c_{T,X} \vee \neg c_{T,Y})\]

for all weak tournaments \(T, T'\) and preference relations \(R_{\mu}\) satisfying conditions (1) to (3).

**4.2 Proof Extraction**

A very interesting feature of the approach by Brandt and Geist [2014] is the possibility to extract human-readable proofs from an unsatisfiability result by the SAT solver. This is done by computing a minimal unsatisfiable set (MUS), an inclusion-minimal set of clauses that is still unsatisfiable.\(^{5}\) This MUS can then, assisted by our encoder/decoder program, be read and transformed into a standard human-readable proof. Different proofs can be found by varying the MUS extractor or by encoding the problem for different subdomains, such as neighborhoods of a set of profiles or randomly sampled subdomains, respectively.

5. **Results and Discussion**

In general, participation and strategyproofness are not logically related. However, extending an observation by Brandt [2011, 2015], it can be shown that strategyproofness implies participation under certain conditions. The proof of this statement is omitted due to space restrictions.

**Lemma 2.** Consider an arbitrary preference extension. Every SCF that satisfies IV and strategyproofness satisfies participation. When preferences are strict, every majoritarian SCF that satisfies strategyproofness satisfies participation.

As a consequence of Lemma 2, some positive results for Kelly-strategyproofness and Fishburn-strategyproofness carry over to participation. We will complement these results by impossibility theorems for Fishburn-participation and a positive result for Kelly-participation, which specifically does not hold for Kelly-strategyproofness.

5.1 Fishburn-participation

It turns out that Pareto-optimality is incompatible with Fishburn-participation in majoritarian social choice. The corresponding Theorems 1 and 2 and their proofs were obtained using the computer-aided method laid out in Section 4. In order to simplify the original proof, which was found by the computer, we first state a lemma, which offers further insights into the possible choices of majoritarian SCFs that satisfy Fishburn-participation and Pareto-optimality.

To state Lemma 3 we introduce some additional notation: an alternative \(x\) (McKelvey) covers an alternative \(y\) if \(x\) is at least as good as \(y\) compared to every other alternative. Formally, \(x\) covers \(y\) if \(x \triangleright M y\) and, for all \(z \in A\), both \(y \triangleright M z\) implies \(x \triangleright M z\) and \(z \triangleright M y\). The uncovered set of \(R_{M}\), denoted \(UC(R_{M})\), is the set of all alternatives that are not covered by any other alternative. By definition, \(UC\) is a majoritarian SCF.

Brandt et al. [2014] have shown that every majoritarian and Pareto-optimal SCF selects a subset of the (McKelvey) uncovered set. We show that an SCF that additionally satisfies Fishburn-participation furthermore only depends on the majority relation between alternatives in the uncovered set.

**Lemma 3.** Let \(f\) be a majoritarian and Pareto-optimal SCF that satisfies Fishburn-participation. Let \(R, R'\) be preferences profiles such that \(R_{M} \cap UC(R_{M}) \subseteq R'_{M} \cap UC(R'_{M})\). Then \(f(R_{M}) = f(R'_{M}) \subseteq UC(R_{M})\).

The proof of Lemma 3 is omitted since a variant of this statement was shown by Brandt et al. [2015].\(^{6}\)

Now, let us turn to our impossibility theorems. The computer found these impossibilities even without using

\(^{4}\)Note that both the definition of tournament-participation and Lemma 1 are independent of a specific preference extension, and thus also applicable to, e.g., Kelly’s extension.

\(^{5}\)We used picosat, which is part of the Picosat distribution [Biere, 2008].

\(^{6}\)Lemma 3 can be strengthened in various respects. It also holds for the iterated uncovered set, all preference extensions satisfying some milder conditions, and probabilistic social choice functions.
Lemma 3. However, the formalization of the lemma allowed the SAT solver to find smaller proofs and makes the human-readable proofs more intuitive.

**Theorem 1.** There is no majoritarian and Pareto-optimal SCF that satisfies Fishburn-participation if \(|A| \geq 4\).

**Proof.** Let \(f\) be a majoritarian and Pareto-optimal SCF satisfying Fishburn-participation. We first prove the statement for \(A = \{a, b, c, d\}\). Throughout this proof, we are going to make extensive use of Lemma 1, which allows us to apply Fishburn-tournament-participation instead of regular Fishburn-participation. The conditions (1) to (3) are easy to verify and will not explicitly refer to them. Intuitively, the proof strategy is to alter the weak tournaments \(T_\alpha, T_\beta,\) and \(T_\gamma\) as depicted below by letting varying agents join some underlying electorate, which will exclude certain choices of \(f\) (by an application of Fishburn-tournament-participation), until we reach a contradiction. In the figures of weak tournaments we only show the strict part of the majority relation for the sake of clarity and highlight alternatives that have to be chosen by \(f\) with a thick border.

First consider \(T_\alpha\). Adding an agent with preferences \(R_{\alpha_1}: \{a, b, c, d\}\) yields \(T_{\alpha_1}\), where due to symmetry \(f(T_{\alpha_1}) = \{a, b, c, d\}\). As \(f\) satisfies Fishburn-participation, nothing that is Fishburn-preferred to \(\{a, b, c, d\}\) according to \(R_{\alpha_1}\) could have been chosen in \(T_\alpha\). Thus \(d \notin f(T_\alpha)\). Adding an agent with \(R_{\alpha_2}: \{a, c, d\}\) also leads to \(T_{\alpha_1}\). Hence \(f(T_\alpha) = \{b, d\}\). If \(T_\alpha\) is altered to \(T_{\alpha_2}\) by adding an agent with \(R_{\alpha_2}: \{b, d\}, c, a\), we get \(f(T_{\alpha_2}) = \{b, c, d\}\) by the fact that \(a\) is covered by \(d\). Therefore, \(f(T_\alpha) = \{d\}\). Adding an agent with \(R_{\alpha_3}: \{a, b, d\}\) leads to \(T_{\alpha_3}\). We have \(f(T_{\alpha_3}) = \{a, c, d\}\) by the fact that \(b\) is covered by \(a\) and Lemma 3. We correspondingly get \(f(T_\alpha) = \{a, d\}\). Hence, we deduce that \(f(T_\alpha) = \{a, c, d\}\).

Now consider \(T_\beta\). Neutrality implies that \(f(T_\beta)\) contains either neither or both of \(a\) and \(b\). Hence, \(f(T_\beta) = \{a\}\). Adding an agent with \(R_{\beta_1}: \{c, a, b, d\}\) changes \(T_\beta\) to \(T_{\beta_1}\) where \(\{a, b, c, d\}\) is selected. Thus, \(f(T_{\beta_1}) = \{c\}\). Additionally, an agent with \(R_{\beta_2}: \{c, d\}, a, b\) might alter \(T_{\beta_1}\) to \(T_{\beta_2}\). Hence, \(f(T_{\beta_2}) = \{d\}\). As \(f(T_\beta) = \{c\}\), an agent with \(R_{\beta_3}: \{b, c, a, d\}\) changes \(T_{\beta_2}\) to \(T_{\beta_3}\). \(T_{\beta_3}\) is isomorphic to \(T_\alpha\) which implies \(f(T_{\beta_3}) = \{a, b, d\}\). Thus, \(f(T_{\beta_3}) = \{a, b, c, d\}\). Hence, we deduce that \(f(T_{\beta_3}) = \{a, b, d\}\).

Finally consider \(T_\gamma\). Neutrality implies that \(f(T_\gamma)\) contains either neither or both of \(b\) and \(c\). Thus \(f(T_\gamma) = \{b\}\). Adding an agent with \(R_{\gamma_1}: \{a, c, d\}\) changes \(T_\gamma\) to \(T_{\gamma_1}\). Analogously to \(T_{\alpha_2}\), \(f(T_{\gamma_1}) = \{a, b, d\}\). Therefore, \(f(T_\gamma) = \{a, d\}\). An agent with \(R_{\gamma_2}: \{a, b, c, d\}\) changes \(T_\gamma\) to \(T_{\gamma_2}\). Neutrality implies that \(f(T_{\gamma_2}) = \{a, b, c, d\}\). This implies that \(f \neq \{a\}, \{b, c\}\). Adding an agent with preferences \(R_{\gamma_3}: \{c, b, d\}, a\) alters \(T_\gamma\) to \(T_{\gamma_3}\). Alternative \(d\) is a Condorcet winner in \(T_{\gamma_3}\), which implies that \(f(T_{\gamma_3}) = \{d\}\) because \(d\) covers \(a, b, c\). Hence \(f(T_{\gamma_3}) = \{a, b, c, d\}\). Both an agent with \(R_{\gamma_4}: \{a, b, c\}, d\) as well as with \(R_{\gamma_5}: \{a, b, c\}, d\) can alter \(T_{\gamma_3}\) to \(T_{\gamma_4}\) which is isomorphic to \(T_\beta\). Thus \(f(T_{\gamma_4}) = \{b, c, d\}\). This implies that \(f(T_{\gamma_4}) = \{d\}\), \{a, b, c, d\}, respectively, otherwise an agent with \(R_{\gamma_4}\) or \(R_{\gamma_5}\), respectively, can manipulate. This excludes all possible choices from \(T_\gamma\), a contradiction.

Now let \(|A| \geq 5\). It follows from Lemma 3 that the choice of \(f\) does not depend on covered alternatives. Hence the statement follows by extending the weak tournaments depicted above to \(A\) such that all alternatives but \(a, b, c, d\) are covered.

We could verify with our computer-aided approach that this impossibility still holds for strict preferences when there are at least 5 alternatives.

**Theorem 2.** There is no majoritarian and Pareto-optimal SCF that satisfies Fishburn-participation if \(|A| \geq 5\), even when preferences are strict.

The shortest proof of Theorem 2 that we were able to extract from our computer-aided approach still uses 124 different instances of manipulation by abstention. Even after grouping them, close to a hundred instances that need to be distinguished remain. Thus, the complete proof has to be omitted here; a computer-generated version, however, is available from the authors upon request. Theorems 1 and 2 are both tight in the sense that, whenever there are less than four or five alternatives, respectively, there exists an SCF that satisfies the desired properties.

An interesting question is whether these impossibilities also extend to other preference extensions. Given the computer-aided approach, this can be easily checked by simply replacing the preference extension in the SAT encoder. For instance, it turns out that the impossibility of Theorem 1 still holds if we consider a coarsening of Fishburn’s extension which can only compare sets that are contained in each other. Kelly’s extension on the other hand does not lead to an impossibility for \(|A| \leq 5\), which will be confirmed more generally in the next section.

5.2 Kelly-participation

Theorems 1 and 2 are sweeping impossibilities within the domain of majoritarian SCFs. For Kelly’s extension, we ob-
tain a much more positive result that covers attractive majoritarian and non-majoritarian SCFs. Brandt [2011, 2015] has shown that set-monotonicity implies Kelly-strategyproofness for strict preferences and that no Condorcet extension is Kelly-strategyproof when preferences are weak. We prove that set-monotonicity (and the very mild IIV axiom) imply Kelly-participation even for weak preferences. We have thus found natural examples of SCFs that violate Kelly-strategyproofness but satisfy Kelly-participation.\footnote{It is easily seen that the proof of Theorem 3 straightforwardly extends to group-participation, i.e., no group of agents can obtain a unanimously more preferred outcome by abstaining.}

**Theorem 3.** Let \( f \) be an SCF that satisfies IIV and set-monotonicity. Then \( f \) satisfies Kelly-participation.

**Proof.** Let \( f \) be an SCF that satisfies IIV and set-monotonicity. Assume for contradiction that \( f \) does not satisfy Kelly-participation. Hence there exist a preference profile \( R \) and an agent \( i \) such that \( f(R_{−i}) P^K_i f(R) \). Let \( X = f(R), Y = f(R_{−i}), \) and \( Z = A \setminus (X \cup Y) \). By definition of \( Y \) \( P^K_i X \) we have that \( x I_i^y \) for all \( x, y \in X \cap Y \).

We define a new preference relation \( R' \) in which all alternatives in \( Y \) are tied for the first place, followed by all alternatives in \( X \) as they are ordered in \( R \), and all remaining alternatives in one indifference class at the bottom of the ranking. Formally,

\[
R' = (Y \times A) \cup R_{|X} \cup (A \times Z).
\]

Let \( i' \) be an agent who is indifferent between all alternatives, i.e., \( x I_{i'} y \) for all \( x, y \in A \). Since \( f \) satisfies IIV we have that \( f(R_{−i'+i'}) = f(R_{−i}) \).

By definition, \( R_{−i'+i'} \) is an \( i \)-improvement over both \( R \) and \( R_{−i'+i'} \). Hence, set-monotonicity implies that \( f(R_{−i'+i'}) = f(R) \) and \( f(R_{−i'+i'}) = f(R_{−i'+i'}) \). In summary, we obtain

\[
f(R_{−i+i'}) = f(R_{−i+i'}) = f(R_{−i}) P^K_i f(R) = f(R_{−i+i'}),
\]

which is a contradiction. \( \square \)

Two rather undiscriminating SCFs that satisfy both IIV and set-monotonicity are the Pareto rule and the omninomination rule (which returns all alternatives that are ranked first by at least one agent). Majoritarian SCFs satisfy IIV by definition and there are several appealing majoritarian SCFs that satisfy set-monotonicity, among those for instance the top cycle, the minimal covering set, and the bipartisan set [see, e.g., Brandt, 2011, 2015; Brandt et al., 2015]. While these majoritarian SCFs are sometimes criticized for not being discriminating enough, the computer-aided approach could be used to generate a refinement of the bipartisan set that, for \( |A| = 5 \), selects only 1.43 alternatives on average, and yet satisfies Kelly-participation.\footnote{For comparison, the bipartisan set (the smallest previously known majoritarian SCF satisfying Kelly-participation) yields 2.68 alternatives on average.}

### Table 2: Overview of results for participation and strategyproofness with respect to Kelly’s and Fishburn’s extension and strict/weak preferences.

<table>
<thead>
<tr>
<th>Extension</th>
<th>Property</th>
<th>Strict preferences</th>
<th>Weak preferences</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kelly</td>
<td>Participation</td>
<td>( \oplus ) All set monotonic SCFs that satisfy IIV (Theorem 3)</td>
<td>( \ominus ) No Condorcet extension (^{\text{b}})</td>
</tr>
<tr>
<td></td>
<td>Strategy-proofness</td>
<td>( \oplus ) All set monotonic SCFs (^{\text{b}})</td>
<td>( \ominus ) No Condorcet extension (^{\text{b}})</td>
</tr>
<tr>
<td>Fishburn</td>
<td>Participation</td>
<td>( \oplus ) No majoritarian &amp; Pareto-optimal SCF ((</td>
<td>A</td>
</tr>
<tr>
<td></td>
<td>Strategy-proofness</td>
<td>( \ominus ) No majoritarian &amp; Pareto-optimal SCF ((</td>
<td>A</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \oplus ) Few undiscriminating SCFs, e.g., ( \text{COND}^e ), ( \text{TC}^g ), and ( \text{PO}^c ) (Lemma 2), and all scoring rules</td>
<td>( \oplus ) Few undiscriminating SCFs, e.g., ( \text{COND}^e ), ( \text{TC}^g ), and ( \text{PO}^c )</td>
</tr>
</tbody>
</table>

\(^{\text{a}}\): follows from a result by Aziz et al. [2014] about probabilistic social choice functions, \(^{\text{b}}\): Brandt [2011, 2015], \(^{\text{c}}\): Brandt and Brill [2011], \(^{\text{d}}\): Brandt and Geist [2014], \(^{\text{e}}\): Feldman [1979], \(^{\text{f}}\): Moulin [1988], \(^{\text{g}}\): Sanver and Zwicker [2012]

### 6 Conclusion

Previous results have indicated a conflict between strategic non-manipulability and Condorcet-consistency [Moulin, 1988; Pérez, 2001; Jimeno et al., 2009; Brandt, 2011, 2015]. For example, Moulin [1988] has shown that no resolute Condorcet extension satisfies participation and Brandt [2011, 2015] has shown that no irresolute Condorcet extension satisfies Kelly-strategyproofness. Theorem 3 addresses an intermediate question and finds that—perhaps surprisingly—there are attractive Condorcet extensions that satisfy Kelly-participation, even when preferences are weak. On the other hand, we have presented elaborate computer-generated impossibilities (Theorems 1 and 2), which show that these encouraging results break down once preferences are extended by the more refined Fishburn extension. This significantly improves our understanding of which behavioral assumptions allow for aggregation functions that are immune to strategic abstention.

An overview of the main findings of this paper and how they relate to other related results is given in Table 2.
References


