Minimal Stable Sets in Tournaments

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We propose a systematic methodology for defining tournament solutions as extensions of maximality. The central concepts of this methodology are maximal qualified subsets and minimal stable sets. We thus obtain an infinite hierarchy of tournament solutions, encompassing the top cycle, the uncovered set, the Banks set, the minimal covering set, and the tournament equilibrium set. Moreover, the hierarchy includes a new tournament solution, the *minimal extending set*, which is conjectured to refine both the minimal covering set and the Banks set.

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1 Introduction

Given a finite set of alternatives and choices between all pairs of alternatives, how to choose from the entire set in a way that is faithful to the pairwise comparisons? This simple, yet captivating, problem is studied in the literature on tournament solutions. A tournament solution thus seeks to identify the "best" elements according to some binary dominance relation, which is usually assumed to be asymmetric and complete. Since the ordinary notion of maximality may return no elements due to cyclical dominations, numerous alternative solution concepts have been devised and axiomatized [see, e.g., 22, 20]. Tournament solutions have numerous applications throughout economics, most prominently in social choice theory where the dominance relation is typically defined via majority rule [e.g., 12, 5]. Other application areas include multi-criteria decision analysis [e.g., 1, 6], non-cooperative game theory [e.g., 13, 19, 10], and cooperative game theory [15, 8].

In this paper, we approach the tournament choice problem using a methodology consisting of two layers: *qualified subsets* and *stable sets*. Our framework captures most known tournament solutions and allows us to provide unified proofs of axiomatic properties and inclusion relationships between tournament solutions.

In general, we consider six standard properties of tournament solutions: monotonicity (MON), independence of unchosen alternatives (IUA), the weak superset property (WSP), the strong superset property (SSP), composition-consistency (COM), and irregularity (IRR). The point of departure for our methodology is to collect the maximal elements of so-called qualified subsets, i.e., distinguished subsets that admit a maximal element. In general, families of qualified subsets are characterized by three properties (closure, independence, and fusion). Examples of families of qualified subsets are all subsets with at most two elements, all subsets that admit a maximal element, or all transitive subsets. Each family yields a corresponding tournament solution and we thus obtain an infinite hierarchy of tournament solutions. The tournament solutions corresponding to the three examples given above are the set of *all alternatives except the minimum*, the *uncovered set* [12, 21], and the *Banks set* [2]. Our methodology allows us to easily establish a number of inclusion relationships between tournament solutions defined via qualified subsets (Proposition 2) and to prove that all such tournament solutions satisfy WSP and MON (Proposition 1). Based on an axiomatic characterization using minimality and a new property called strong retentiveness, we show that the Banks set is the finest tournament solution definable via qualified subsets (Theorem 1).

Generalizing an idea by Dutta [11], we then propose a method for refining any suitable solution concept S by defining minimal sets that satisfy a natural stability criterion with respect to S. A crucial property in this context is whether S always admits a *unique* minimal stable set. For tournament solutions defined via qualified subsets, we show that this is the case if and only if no tournament contains two disjoint stable sets (Lemma 2). As a consequence of this characterization and a theorem by Dutta [11], we show that an infinite number of tournament solutions (defined via qualified subsets) always admit a unique minimal stable set (Theorem 3). Moreover, we show that all tournament solutions defined as unique minimal stable sets satisfy WSP and IUA (Proposition 4), SSP and various other desirable properties if the original tournament solution is defined via qualified subsets (Theorem 4), and MON and COM if the original tournament solution satisfies these properties (Proposition 5 and Proposition 6). The minimal stable sets with respect to the three tournament solutions mentioned in the paragraph above are the *minimal dominant set*, better known as the *top cycle* [16, 28], the *minimal covering set* [11], and a new tournament solution that we call the *minimal extending set (ME)*. Whether ME satisfies uniqueness turns out to be a highly non-trivial combinatorial problem and remains open. If true, ME would be contained in both the minimal covering set and the Banks set while satisfying all of the desirable properties listed above. We conclude the paper by axiomatically characterizing all tournament solutions definable via unique minimal stable sets (Proposition 7) and investigating the relationship between ME and the *tournament equilibrium set* [26].

When considering qualified subsets that are maximal in terms of cardinality rather than set inclusion and using a slightly modified definition of stability, our framework also captures quantitative tournament solutions such as the *Copeland set* and the *bipartisan set* [see 7].

2 Preliminaries

The core of the problem studied in the literature on tournament solutions is how to extend choices from sets consisting of only two elements to larger sets. Thus, our primary objects of study will be functions that select one alternative from any pair of alternatives. Any such function can be conveniently represented by a tournament, i.e., a binary relation on the entire set of alternatives. Tournament solutions then advocate different views on how to choose from arbitrary subsets of alternatives based on these pairwise comparisons [see, e.g., 20, for an excellent overview of tournament solutions].

2.1 Tournaments

Let X be a universe of alternatives. The set of all *finite* subsets of set X will be denoted by $\mathcal{F}_0(X)$ whereas the set of all *non-empty* finite subsets of X will be denoted by $\mathcal{F}(X)$. A *tournament* T is a pair $(A, \succ)$, where $A \in \mathcal{F}(X)$ and $\succ$ is an asymmetric and complete (and thus irreflexive) binary relation on X, usually referred to as the *dominance relation*.[1] Intuitively, $a \succ b$ signifies that alternative a is preferable to b. The dominance relation can be extended to sets of alternatives by writing $A \succ B$ when $a \succ b$ for all $a \in A$ and $b \in B$. When A or B are singletons, we omit curly braces to improve readability. We further write $\mathcal{T}(X)$ for the set of all tournaments on X. For a given tournament $T = (A, \succ)$ and an alternative a, the *dominion* of a is denoted by $D(a) = \{\, b \in A \mid a \succ b\}$ and the *dominators* of a by $\overline{D}(a) = \{\, b \in A \mid b \succ a\}$. The *order* of T refers to the cardinality of A and T is called *regular* if $|D(a)| = |D(b)|$ for all $a, b \in A$. A *tournament isomorphism* of two tournaments $T = (A, \succ)$ and $T' = (A', \succ')$ is a bijective mapping $\pi : A \to A'$ such that $a \succ b$ if and only if $\pi(a) \succ' \pi(b)$.

2.2 Components and Decompositions

An important structural concept in the context of tournaments is that of a *component*. A component is a subset of alternatives that bear the same relationship to all alternatives not in the set.

Let $T = (A, \succ)$ be a tournament. A non-empty subset B of A is a *component* of T if for all $a \in A \setminus B$ either $B \succ a$ or $a \succ B$. A *decomposition* of T is a set of pairwise disjoint components $\{B_1, \ldots, B_k\}$ of T such that $A = \bigcup_{i=1}^{k} B_i$. Given a particular decomposition $\tilde{B} = \{B_1, \ldots, B_k\}$ of T, the *summary* of T is defined as the tournament on the individual components rather than the alternatives. Formally, the summary $\tilde{T} = (\tilde{B}, \tilde{\succ})$ of T is the tournament such that for all $i, j \in \{1, \ldots, k\}$ with $i \neq j$,

$$B_i \mathrel{\tilde{\succ}} B_j \quad \text{if and only if} \quad B_i \succ B_j.$$

Conversely, a new tournament can be constructed by replacing each alternative with a component. For notational convenience, we tacitly assume that $\mathbb{N} \subseteq X$. For pairwise

[1] This definition slightly diverges from the common graph-theoretic definition where $\succ$ is defined on A rather than on X. However, it facilitates the sound definition of tournament functions (such as tournament solutions or concepts of qualified subsets).

disjoint sets $B_1, \ldots, B_k \subseteq X$ and tournaments $\tilde{T} = (\{1, \ldots, k\}, \tilde{\succ})$, $T_1 = (B_1, \succ_1)$, $\ldots$, $T_k = (B_k, \succ_k)$, the *product* of $T_1, \ldots, T_k$ with respect to $\tilde{T}$, denoted by $\Pi(\tilde{T}, T_1, \ldots, T_k)$, is a tournament $(A, \succ)$ such that $A = \bigcup_{i=1}^{k} B_i$ and for all $b_1 \in B_i, b_2 \in B_j$,

$$b_1 \succ b_2 \quad \text{if and only if} \quad i = j \text{ and } b_1 \succ_i b_2, \text{ or } i \neq j \text{ and } i \mathrel{\tilde{\succ}} j.$$

2.3 Tournament Functions

A central aspect of this paper are functions that, for a given tournament, yield one or more subsets of alternatives. We will therefore define the notion of a *tournament function*. A function on tournaments is a tournament function if it is independent of outside alternatives and stable with respect to tournament isomorphisms. A tournament function may yield a (non-empty) subset of alternatives—as in the case of tournament solutions—or a set of subsets of alternatives—as in the case of qualified or stable sets.

Definition 1. *Let $Z \in \{\mathcal{F}_0(X), \mathcal{F}(X), \mathcal{F}(\mathcal{F}(X))\}$. A function $f : \mathcal{T}(X) \to Z$ is a* tournament function *if*

(i) $f(T) = f(T')$ for all tournaments $T = (A, \succ)$ and $T' = (A, \succ')$ such that $\succ|_A = \succ'|_A$, and

(ii) $f((\pi(A), \succ')) = \pi(f((A, \succ)))$ for all tournaments $(A, \succ)$, $(A', \succ')$, and tournament isomorphisms[2] $\pi : A \to A'$ of $(A, \succ)$ and $(A', \succ')$.

For a given set $A \in \mathcal{F}(X)$ and tournament function f, we overload f by also writing $f(A)$, provided the dominance relation is known from the context. For two tournament functions f and f', we write $f' \subseteq f$ if $f'(T) \subseteq f(T)$ for all tournaments T.

2.4 Tournament Solutions

The first tournament function we consider is $\max_{\prec} : \mathcal{T}(X) \to \mathcal{F}_0(X)$, which returns the undominated alternatives of a tournament. Formally,

$$\max_{\prec}((A, \succ)) = \{a \in A \mid \overline{D}(a) = \emptyset\}.$$

Due to the completeness of the dominance relation, this function returns at most one alternative in any tournament. Moreover, maximal—i.e., undominated—and maximum—i.e., dominant—elements coincide. In social choice theory, the maximum of a majority tournament is commonly referred to as the *Condorcet winner*. Obviously, the dominance relation may contain cycles and thus fail to have a maximal element. For this reason, a variety of alternative concepts to single out the "best" alternatives of a tournament have been suggested. Formally, a *tournament solution S* is defined as a function that associates with each tournament $T = (A, \succ)$ a non-empty subset $S(T)$ of A.

[2] $\pi(A)$ is a shorthand for the set $\{\pi(a) \mid a \in A\}$.

Definition 2. *A* tournament solution S *is a tournament function* $S : \mathcal{T}(X) \to \mathcal{F}(X)$ *such that* $\max_{\prec}(T) \subseteq S(T) \subseteq A$ *for all tournaments* $T = (A, \succ)$.[3]

The set $S(T)$ returned by a tournament solution for a given tournament T is called the *choice set* of T whereas $A \setminus S(T)$ consists of the *unchosen alternatives*. If $S' \subseteq S$ for two tournament solutions S and S', we say that S' is a *refinement* of S or that S' is *finer* than S.

2.5 Properties of Tournament Solutions

The literature on tournament solutions has identified a number of desirable properties for tournament solutions. In this section, we will define six of the most common properties.[4] In a more general context, Moulin [23] distinguishes between *monotonicity* and *independence* conditions, where a monotonicity condition describes the positive association of the solution with some parameter and an independence condition characterizes the invariance of the solution under the modification of some parameter.

In the context of tournament solutions, we will further distinguish between properties that are defined in terms of the dominance relation and properties defined in terms of the set inclusion relation. With respect to the former, we consider *monotonicity* and *independence of unchosen alternatives*. A tournament solution is monotonic if a chosen alternative remains in the choice set when extending its dominion and leaving everything else unchanged.

Definition 3. *A tournament solution S satisfies* monotonicity *(MON) if $a \in S(T)$ implies $a \in S(T')$ for all tournaments $T = (A, \succ)$, $T' = (A, \succ')$, and $a \in A$ such that $\succ|_{A\setminus\{a\}} = \succ'|_{A\setminus\{a\}}$ and $a \succ b$ implies $a \succ' b$ for all $b \in A$.*

A solution is independent of unchosen alternatives if the choice set is invariant under any modification of the dominance relation between unchosen alternatives.

Definition 4. *A tournament solution S is* independent of unchosen alternatives *(IUA) if $S(T) = S(T')$ for all tournaments $T = (A, \succ)$ and $T' = (A, \succ')$ such that $\succ|_{S(T)\cup\{a\}} = \succ'|_{S(T)\cup\{a\}}$ for all $a \in A$.*

With respect to set inclusion, we consider a monotonicity property to be called the *weak superset property* and an independence property known as the *strong superset property*. A tournament solution satisfies the weak superset property if an unchosen alternative remains unchosen when other unchosen alternatives are removed.

Definition 5. *A tournament solution S satisfies the* weak superset property *(WSP) if $S(B) \subseteq S(A)$ for all tournaments $T = (A, \succ)$ and $B \subseteq A$ such that $S(A) \subseteq B$.*

[3] Laslier [20] is slightly more stringent here as he requires the maximum be the only element in $S(T)$ whenever it exists.

[4] Our terminology slightly differs from the one by Laslier [20] and others. *Independence of unchosen alternatives* is also called *independence of the losers* or *independence of non-winners*. The *weak superset property* has been referred to as ϵ^+ or the *Aïzerman property*.

The strong superset property states that a choice set is invariant under the removal of unchosen alternatives.

Definition 6. *A tournament solution S satisfies the* strong superset property *(SSP) if $S(B) = S(A)$ for all tournaments $T = (A, \succ)$ and $B \subseteq A$ such that $S(A) \subseteq B$.*

The difference between WSP and SSP is precisely another independence condition called *idempotency*. A solution is *idempotent* if the choice set is invariant under repeated application of the solution concept, i.e., $S(S(T)) = S(T)$ for all tournaments T. When S is not idempotent, we define $S^k(T) = S(S^{k-1}(T))$ inductively by letting $S^1(T) = S(T)$ and $S^\infty(T) = \bigcap_{k\in\mathbb{N}} S^k(T)$.

The four properties defined above (MON, IUA, WSP, and SSP) will be called *basic* properties of tournament solutions. The conjunction of MON and SSP implies IUA. It is therefore sufficient to show MON and SSP in order to prove that a tournament solution satisfies all four basic properties.

Two further properties considered in this paper are *composition-consistency* and *irregularity*. A tournament solution is composition-consistent if it chooses the "best" alternatives from the "best" components.

Definition 7. *A tournament solution S is* composition-consistent *(COM) if for all tournaments T, $T_1, \dots, T_k$, and $\tilde{T}$ such that $T = \Pi(\tilde{T}, T_1, \dots, T_k)$, $S(T) = \bigcup_{i\in S(\tilde{T})} S(T_i)$.*

Finally, a tournament solution is irregular if it is capable of excluding alternatives in regular tournaments.

Definition 8. *A tournament solution S satisfies* irregularity *(IRR) if there exists a regular tournament $T = (A, \succ)$ such that $S(T) \neq A$.*

3 Qualified Subsets

In this section, we will define a class of tournament solutions that is based on identifying significant subtournaments of the original tournament, such as subtournaments that admit a maximal alternative.

3.1 Concepts of Qualified Subsets

A *concept of qualified subsets* is a tournament function that, for a given tournament $T = (A, \succ)$, returns subsets of A that satisfy certain properties. Each such set of sets will be referred to as a *family of qualified subsets*. Two natural examples of concepts of qualified subsets are $\mathcal{M}$, which yields all subsets that admit a maximal element, and $\mathcal{M}^*$, which yields all non-empty transitive subsets. Formally,

$$\begin{aligned}\mathcal{M}((A,\succ)) &= \{B \subseteq A \mid \max_{\prec}(B) \neq \emptyset\} \text{ and}\\ \mathcal{M}^*((A,\succ)) &= \{B \subseteq A \mid \max_{\prec}(C) \neq \emptyset \text{ for all non-empty } C \subseteq B\}.\end{aligned}$$

$\mathcal{M}$ and $\mathcal{M}^*$ are examples of concepts of qualified subsets, which are formally defined as follows.

Definition 9. *Let* $\mathcal{Q} : \mathcal{T}(X) \to \mathcal{F}(\mathcal{F}(X))$ *be a tournament function such that* $\mathcal{M}_1(T) \subseteq \mathcal{Q}(T) \subseteq \mathcal{M}(T)$. $\mathcal{Q}$ *is a* concept of qualified subsets *if it meets the following three conditions for every tournament* $T = (A, \succ)$.

(Closure) $\mathcal{Q}(T)$ *is downward closed with respect to* $\mathcal{M}$*: Let* $Q \in \mathcal{Q}(T)$*. Then,* $Q' \in \mathcal{Q}(T)$ *for all* $Q' \in \mathcal{M}(T)$ *with* $Q' \subseteq Q$.

(Independence) *Qualified sets are independent of outside alternatives: Let* $A' \in \mathcal{F}(X)$ *and* $Q \subseteq A \cap A'$*. Then,* $Q \in \mathcal{Q}(A)$ *if and only if* $Q \in \mathcal{Q}(A')$.

(Fusion) *Qualified sets may be merged under certain conditions: Let* $Q_1, Q_2 \in \mathcal{Q}(T)$ *and* $Q_1 \setminus Q_2 \succ Q_2$*. Then* $Q_1 \cup Q_2 \in \mathcal{Q}(T)$ *if there is a tournament* $T' \in \mathcal{T}(X)$ *and* $Q \in \mathcal{Q}(T')$ *such that* $|Q_1 \cup Q_2| \leq |Q|$.

Whether a set is qualified only depends on its internal structure (due to independence and the isomorphism condition of Definition 1). While closure, independence, and the fact that all singletons are qualified are fairly natural, the fusion condition is slightly more technical. Essentially, it states that if a qualified subset dominates another qualified subset, then the union of these subsets is also qualified. The additional cardinality restriction is only required to enable *bounded* qualified subsets. For every concept of qualified subsets $\mathcal{Q}$ and every given $k \in \mathbb{N}$, $\mathcal{Q}_k : \mathcal{T}(X) \to \mathcal{F}(\mathcal{F}(X))$ is a tournament function such that

$$\mathcal{Q}_k(T) = \{B \in \mathcal{Q}(T) \mid |B| \leq k\}.$$

It is easily verified that $\mathcal{Q}_k$ is a concept of qualified subsets. Furthermore, $\mathcal{M}$ and $\mathcal{M}^*$ (and thus also $\mathcal{M}_k$ and $\mathcal{M}^*_k$) are concepts of qualified subsets. Since only tournaments of order 4 or more may be intransitive and admit a maximal element at the same time, $\mathcal{M}_k = \mathcal{M}^*_k$ for $k \in \{1, 2, 3\}$.

3.2 Maximal Elements of Maximal Qualified Subsets

For every concept of qualified subsets, we can now define a tournament solution that yields the maximal elements of all inclusion-maximal qualified subsets, i.e., all qualified subsets that are not contained in another qualified subset.

Definition 10. *Let* $\mathcal{Q}$ *be a concept of qualified subsets. Then, the tournament solution* $S_{\mathcal{Q}}$ *is defined as*

$$S_{\mathcal{Q}}(T) = \{\max_{\prec}(B) \mid B \in \max_{\subseteq}(\mathcal{Q}(T))\}.$$

Since any family of qualified subsets contains all singletons, $S_{\mathcal{Q}}(T)$ is guaranteed to be non-empty and contains the Condorcet winner whenever one exists. As a consequence, $S_{\mathcal{Q}}$ is well-defined as a tournament solution.

The following tournament solutions can be restated via appropriate concepts of qualified subsets.

Condorcet non-losers. $S_{\mathcal{M}_2}$ is arguably the largest non-trivial tournament solution. In tournaments of order two or more, it chooses every alternative that dominates at least one other alternative. We will refer to this concept as *Condorcet non-losers* (*CNL*) as it selects everything except the minimum (or *Condorcet loser*) in such tournaments.

Uncovered set. $S_{\mathcal{M}}(T)$ returns the *uncovered set* $UC(T)$ of a tournament T, i.e., the set consisting of the maximal elements of inclusion-maximal subsets that admit a maximal element. The uncovered set is usually defined in terms of a subrelation of the dominance relation called the *covering relation* [12, 21].

Banks set. $S_{\mathcal{M}^*}(T)$ yields the *Banks set* $BA(T)$ of a tournament T [2]. $\mathcal{M}^*(T)$ contains subsets that not only admit a maximum, but can be completely ordered from maximum to minimum such that all of their non-empty subsets admit a maximum. $S_{\mathcal{M}^*}(T)$ thus returns the maximal elements of inclusion-maximal *transitive* subsets.

In the remainder of this section, we will prove various statements about tournaments solutions defined via qualified subsets. For a set B and an alternative $a \notin B$, the short notation $[B, a]$ will be used to denote the set $B \cup \{a\}$ and the fact that $\max_{\prec}(B \cup \{a\}) = \{a\}$. In several proofs, we will make use of the fact that whenever $a \notin S_{\mathcal{Q}}(T)$, there is some $b \in S_{\mathcal{Q}}(T)$ for every qualified subset $[Q, a]$ such that $[Q \cup \{a\}, b] \in \mathcal{Q}(T)$. We start by showing that every tournament solution defined via qualified subsets satisfies the weak superset property and monotonicity.

Proposition 1. *Let $\mathcal{Q}$ be a concept of qualified subsets. Then, $S_{\mathcal{Q}}$ satisfies* WSP *and* MON*.*

Proof. Let $T = (A, \succ)$ be a tournament, $a \notin S_{\mathcal{Q}}(A)$, and $A' \subseteq A$ such that $S_{\mathcal{Q}}(T) \cup \{a\} \subseteq A'$. For WSP, we need to show that $a \notin S_{\mathcal{Q}}(A')$. Let $[Q, a] \in \mathcal{Q}(A')$. Due to independence, $[Q, a] \in \mathcal{Q}(A)$. Since $a \notin S_{\mathcal{Q}}(A)$, there has to be some $b \in S_{\mathcal{Q}}(A)$ such that $[Q \cup \{a\}, b] \in \mathcal{Q}(A)$. Again, independence implies that $[Q \cup \{a\}, b] \in \mathcal{Q}(A')$. Hence, $a \notin S_{\mathcal{Q}}(A')$.

For MON, observe that $a \in S_{\mathcal{Q}}$ implies that there exists $[Q, a] \in \max_{\subseteq}(\mathcal{Q}(T))$. Define $T' = (A, \succ')$ by letting $T'|_{A \setminus \{a\}} = T|_{A \setminus \{a\}}$ and $a \succ' b$ for some $b \in A$ with $b \succ a$. Clearly, $[Q, a]$ is contained in $\mathcal{Q}(T')$ due to independence and the fact that $b \notin Q$. Now, assume for contradiction that there is some $c \in A$ such that $[Q \cup \{a\}, c] \in \mathcal{Q}(T')$. Since $a \succ' b$, $c \neq b$. Independence then implies that $[Q \cup \{a\}, c] \in \mathcal{Q}(T)$, a contradiction. □

Proposition 1 implies several known statements, namely that *CNL*, *UC*, and *BA* satisfy MON and WSP. All three concepts are known to fail idempotency (and thus SSP). *CNL* trivially satisfies IUA whereas this is not the case for *UC* and *BA* [see 20]. We also obtain some straightforward inclusion relationships, which define an infinite hierarchy of tournament solutions ranging from *CNL* to *BA*.

Proposition 2. *$S_{\mathcal{M}^*} \subseteq S_{\mathcal{M}}$, $S_{\mathcal{M}^*_k} \subseteq S_{\mathcal{M}_k}$, $S_{\mathcal{Q}_{k+1}} \subseteq S_{\mathcal{Q}_k}$, and $S_{\mathcal{Q}} \subseteq S_{\mathcal{Q}_k}$ for every concept of qualified subsets $\mathcal{Q}$ and $k \in \mathbb{N}$.*

Proof. All inclusion relationships follow from the following observation. Let T be a tournament and $\mathcal{Q}$ and $\mathcal{Q}'$ concepts of qualified subsets such that for every $[Q, a] \in \max_{\subseteq}(\mathcal{Q}(T))$, there is $[Q', a] \in \max_{\subseteq}(\mathcal{Q}'(T))$. Then, $S_{\mathcal{Q}} \subseteq S_{\mathcal{Q}'}$. □

It turns out that the Banks set is the finest tournament solution definable via qualified subsets. In order to show this, we introduce a new property called *strong retentiveness*, which prescribes that the choice set of every dominator set is contained in the original choice set. Alternatively, it can be seen as a variant of WSP because it states that a choice set may not grow when an alternative and its entire dominion are removed from the tournament.

Definition 11. *A tournament solution S satisfies* strong retentiveness *if $S(\overline{D}(a)) \subseteq S(A)$ for all tournaments $T = (A, \succ)$ and $a \in A$.*

Lemma 1. *Let $\mathcal{Q}$ be a concept of qualified subsets. Then, $S_{\mathcal{Q}}$ satisfies strong retentiveness.*

Proof. Let $(A, \succ)$ be a tournament, $a \in A$ an alternative, and $B = \overline{D}(a)$. We show that $b \in S_{\mathcal{Q}}(B)$ implies that $b \in S_{\mathcal{Q}}(A)$. Let $[Q, b]$ be a maximal qualified subset in B, i.e., $[Q, b] \in \max_{\subseteq}(\mathcal{Q}(B))$. If $[Q, b] \in \max_{\subseteq}(\mathcal{Q}(A))$, we are done. Otherwise, there has to be some $c \in A$ such that $[Q \cup \{b\}, c] \in \mathcal{Q}(A)$. Furthermore, $[Q, b] \succ a$ and $a \succ c$ because otherwise $[Q \cup \{b\}, c]$ would be qualified in B as well. We can now merge the qualified subsets $[Q, b]$ and $[\{a\}, b]$ according to the fusion condition. We claim that $[Q \cup \{a\}, b] \in \max_{\subseteq}(\mathcal{Q}(A))$. Assume for contradiction that there is some $d \in A$ such that $[Q \cup \{a, b\}, d] \in \mathcal{Q}(A)$. Since $d \in \overline{D}(a)$, independence implies that $[Q \cup \{a, b\}, d] \in \mathcal{Q}(B)$. This is a contradiction because $[Q, b]$ was assumed to be a *maximal* qualified subset of B. □

Theorem 1. *The Banks set is the finest tournament solution satisfying strong retentiveness and thus the finest tournament solution definable via qualified subsets.*

Proof. Let S be a tournament solution that satisfies strong retentiveness and $T = (A, \succ)$ a tournament. We first show that $BA(A) \subseteq S(A)$. For every $a \in BA(A)$, there has to be maximal transitive set $[Q, a] \subseteq A$. Let $Q = \{q_1, \ldots, q_n\}$ with $q_i \succ q_j$ for all $i < j$. We show that $B = \bigcap_{i=1}^{n} \overline{D}(q_i) = \{a\}$. Since $a \succ Q$, $a \in B$. Assume for contradiction that $b \in B$ with $b \neq a$. Then $b \succ Q$ and either $[Q \cup \{b\}, a]$ or $[Q \cup \{a\}, b]$ is a transitive set, which contradicts the maximality of $[Q, a]$. The repeated application of strong retentiveness implies that

$$S(A) \supseteq S(\overline{D}(q_n)) \supseteq S(\overline{D}(q_n) \cap \overline{D}(q_{n-1})) \supseteq \cdots \supseteq S(B) = S(\{a\}) = \{a\}$$

and hence that $a \in S(A)$. The statement now follows from Lemma 1. □

Some appeal of the above characterization comes from the fact that several other tournament solutions have been characterized using related conditions and minimality. For example, the top cycle is the smallest tournament solution satisfying the strongest expansion consistency condition β^+ [4] and the uncovered set the smallest one satisfying a weakening of β^+ called γ [22]. Strong retentiveness, which characterizes the Banks set, is a weakening of γ and can be further weakened to *retentiveness* by restricting the inclusion of Definition 11 to all $a \in S(A)$. Retentiveness then characterizes the tournament equilibrium set [26].

4 Stable Sets

In this section, we propose a general method for refining any suitable solution concept S by formalizing the stability of sets of alternatives with respect to S. This method is based on the notion of stable sets [29] and generalizes covering sets as introduced by Dutta [11].

4.1 Stability and Directedness

The reason why we are interested in maximal—i.e., undominated—alternatives is that dominated alternatives can be upset by other alternatives; they are unstable. The rationale behind stable sets is that this instability is only meaningful if an alternative is upset by something which itself is stable. Hence, a set of alternatives B is said to be stable if it consists precisely of those alternatives not upset by B. In von Neumann and Morgenstern's original definition, a is upset by B if some element of B dominates a. In our generalization, a is upset by B if $a \notin S(B \cup \{a\})$ for some underlying solution concept S.[5]

As an alternative to this fixed-point definition, which will be formalized in Corollary 1, stable sets can be seen as sets that comply with internal and external stability in some well-defined way. First, there should be no reason to restrict the selection by excluding some alternative from it and, secondly, there should be an argument against each proposal to include an outside alternative into the selection.[6] In our context, external stability with respect to some tournament solution S is defined as follows.

Definition 12. *Let S be a tournament solution and $T = (A, \succ)$ a tournament. Then, $B \subseteq A$ is* externally stable *in T with respect to tournament solution S (or S-*stable*) if $a \notin S(B \cup \{a\})$ for all $a \in A \setminus B$. The set of S-stable sets for a given tournament $T = (A, \succ)$ will be denoted by $\mathcal{S}_S(T) = \{B \subseteq A \mid B \text{ is } S\text{-stable in } T\}$.*

Externally stable sets are guaranteed to exist since the set of all alternatives A is trivially S-stable in $(A, \succ)$ for every S. We say that a set $B \subseteq A$ is *internally stable* with respect to S if $S(B) = B$. For now, we will focus on external stability because it will be seen later that certain conditions imply the existence of a unique minimal externally stable set, which also satisfies internal stability. We define $\widehat{S}(T)$ to be the

[5]Von Neumann and Morgenstern's definition can be seen as the special case where a is upset by B if $a \notin \max_{\prec}(B \cup \{a\})$.

[6]A large number of solution concepts in the social sciences spring from similar notions of internal and/or external stability [see, e.g., 29, 24, 27, 25, 11, 3, 10]. Wilson [30] refers to stability as the *solution property*.

tournament solution that returns the union of all *inclusion-minimal* S-stable sets in T, i.e., the union of all S-stable sets that do not contain an S-stable set as a proper subset.

Definition 13. *Let S be a tournament solution. Then, the tournament solution $\widehat{S}$ is defined as*

$$\widehat{S}(T) = \bigcup \min_{\subseteq}(\mathcal{S}_S(T)).$$

It is easily verified that $\widehat{S}$ is well-defined as a tournament solution as there are no S-stable sets that do not contain the Condorcet winner whenever one exists. We will only be concerned with tournament solutions S that (presumably) admit a *unique* minimal S-stable set in any tournament. It turns out it is precisely this property that is most difficult to prove for all but the simplest tournament solutions. A tournament T contains a *unique* minimal S-stable set if and only if $\mathcal{S}_S(T)$ is a *directed* set with respect to set inclusion, i.e., for all sets $B, C \in \mathcal{S}_S(T)$ there is a set $D \in \mathcal{S}_S(T)$ contained in both B and C. We say that $\mathcal{S}_S$ is directed when $\mathcal{S}_S(T)$ is a directed set for all tournaments T. Throughout this paper, directedness of a set of sets $\mathcal{S}$ is shown by proving the stronger property of closure under intersection, i.e., $B \cap C \in \mathcal{S}$ for all $B, C \in \mathcal{S}$. A set of sets $\mathcal{S}$ *pairwise intersects* if $B \cap C \neq \emptyset$ for all $B, C \in \mathcal{S}$. We will prove that, for every concept of qualified subsets $\mathcal{Q}$, $\mathcal{S}_{S_\mathcal{Q}}$ is closed under intersection if and only if $\mathcal{S}_{S_\mathcal{Q}}$ pairwise intersects. In order to improve readability, we will use the short notation $\mathcal{S}_\mathcal{Q}$ for $\mathcal{S}_{S_\mathcal{Q}}$.

Lemma 2. *Let $\mathcal{Q}$ be a concept of qualified subsets. Then, $\mathcal{S}_\mathcal{Q}$ is closed under intersection if and only if $\mathcal{S}_\mathcal{Q}$ pairwise intersects.*

Proof. The direction from left to right is straightforward since the empty set is not stable. The opposite direction is shown by contraposition, i.e., we prove that $\mathcal{S}_\mathcal{Q}$ does not pairwise intersect if $\mathcal{S}_\mathcal{Q}$ is not closed under intersection. Let $T = (A, \succ)$ be a tournament and $B_1, B_2 \in \mathcal{S}_\mathcal{Q}(T)$ be two sets such that $C = B_1 \cap B_2 \notin \mathcal{S}_\mathcal{Q}(T)$. Since C is not $S_\mathcal{Q}$-stable, there has to be $a \in A \setminus C$ such that $a \in S_\mathcal{Q}(C \cup \{a\})$. In other words, there has to be a set $Q \subseteq C$ such that $[Q, a] \in \max_\subseteq(\mathcal{Q}(C \cup \{a\}))$. Define

$$B'_1 = \{b \in B_1 \mid b \succ Q\} \text{ and } B'_2 = \{b \in B_2 \mid b \succ Q\}.$$

Clearly, $(B'_1 \setminus B'_2) \cap C = \emptyset$ and $(B'_2 \setminus B'_1) \cap C = \emptyset$. Assume without loss of generality that $a \notin B_1$. It follows from the stability of B_1, that B_1 has to contain an alternative b_1 such that $b_1 \succ [Q, a]$. Hence, B'_1 is not empty. Next, we show that $B'_1 \cap B'_2 = \emptyset$. Assume for contradiction that there is some $b \in B'_1 \cap B'_2$. If $b \succ a$, independence implies that $[Q \cup \{a\}, b] \in \mathcal{Q}(C \cup \{a\})$, which contradicts the fact that $[Q, a]$ is a maximal qualified subset in $C \cup \{a\}$. If, on the other hand, $a \succ b$, the set $[Q \cup \{b\}, a]$ is isomorphic to $[Q \cup \{a\}, b_1]$, which is a qualified subset of $B_1 \cup \{a\}$. Thus, $[Q \cup \{b\}, a] \in \mathcal{Q}(C \cup \{a\})$, again contradicting the maximality of $[Q, a]$. Independence, the isomorphism of $[Q, a]$ and $[Q, b_1]$, and the stability of B_2 further require that there has to be an alternative $b_2 \in B_2$ such that $b_2 \succ [Q, b_1]$. Hence, B'_1 and B'_2 are disjoint and non-empty.

Let $a' \in B'_2$ and R be a maximal subset of $B'_1 \cup Q$ such that $[R, a'] \in \mathcal{Q}(B'_1 \cup Q \cup \{a'\})$. We claim that Q has to be contained in R. Assume for contradiction that there exists some $b \in Q \setminus R$. Clearly, $[Q, a]$ and $[Q, a']$ are isomorphic. It therefore follows from independence that $[Q, a'] \in \mathcal{Q}(B'_1 \cup Q \cup \{a'\})$ and from closure that $[(Q \cap R) \cup \{b\}, a'] \in \mathcal{Q}(B'_1 \cup Q \cup \{a'\})$. Due to the stability of B_1, $[R, a']$ is not a maximal qualified subset in $B_1 \cup \{a'\}$, i.e., there exists a qualified subset that contains more elements. We may thus merge the qualified subsets $[R, a']$ and $[(Q \cap R) \cup \{b\}, a']$ according to the fusion condition because $R \setminus Q \succ Q$ and consequently $R \setminus Q \succ (Q \cap R) \cup \{b\}$. We then have that $[R \cup \{b\}, a'] \in \mathcal{Q}(B'_1 \cup Q \cup \{a'\})$, which yields a contradiction because R was assumed to be a maximal set such that $[R, a'] \in \mathcal{Q}(B'_1 \cup Q \cup \{a'\})$. Hence, $Q \subseteq R$. Due to the stability of B_1 in T, there has to be a $c \in B_1$ such that $c \succ [R, a']$. Since B'_1 contains all alternatives in B_1 that dominate $Q \subseteq R$, it also contains c. Independence then implies that $[R, a'] \notin \max_{\subseteq}(\mathcal{Q}_k(B'_1 \cup Q \cup \{a'\}))$.

Thus, $B'_1 \cup Q$ is stable in $B'_1 \cup B'_2 \cup Q$. Since Q is contained in every maximal set $R \subseteq B'_1 \cup Q$ such that $[R, a'] \in \mathcal{Q}(B'_1 \cup Q \cup \{a'\})$ for some $a' \in B'_2$, B'_1 (and by an analogous argument B'_2) remains stable when removing Q. This completes the proof because B'_1 and B'_2 are two disjoint $S_{\mathcal{Q}}$-stable sets in $B'_1 \cup B'_2$. $\square$

Dutta has shown by induction on the tournament order that tournaments admit no disjoint $S_{\mathcal{M}}$-stable sets (so-called *covering sets*).

Theorem 2 (Dutta). *$\mathcal{S}_{\mathcal{M}}$ pairwise intersects.*

Dutta [11] also showed that covering sets are closed under intersection, which now also follows from Lemma 2.[7]

Naturally, finer solution concepts also yield smaller minimal stable sets (if their uniqueness is guaranteed).

Proposition 3. *Let S and S' be two tournament solutions such that $\mathcal{S}_{S'}$ is directed and $S' \subseteq S$. Then, $\widehat{S'} \subseteq \widehat{S}$ and $\mathcal{S}_S$ pairwise intersects.*

Proof. The statements follow from the simple fact that every S-stable set is also S'-stable. Let $B \subseteq A$ be a minimal S-stable set in tournament $(A, \succ)$. Then, $a \notin S(B \cup \{a\})$ for every $a \in A \setminus B$ and, due to the inclusion relationship, $a \notin S'(B \cup \{a\}) \subseteq S(B \cup \{a\})$. As a consequence, B is S'-stable and has to contain the unique minimal S'-stable set since $\mathcal{S}_{S'}$ is directed. $\mathcal{S}_S$ pairwise intersects because two disjoint S-stable sets would also be S'-stable, which contradicts the directedness of $\mathcal{S}_{S'}$. $\square$

As a corollary of the previous statements, the set of $S_{\mathcal{M}_k}$-stable sets for every k is closed under intersection.

Theorem 3. *$\mathcal{S}_{\mathcal{M}_k}$ is closed under intersection for all $k \in \mathbb{N}$.*

Proof. Let $k \in \mathbb{N}$. We know from Proposition 2 that $S_{\mathcal{M}} \subseteq S_{\mathcal{M}_k}$ and from Theorem 2 and Lemma 2 that $\mathcal{S}_{\mathcal{M}}$ is directed. Proposition 3 implies that $\mathcal{S}_{\mathcal{M}_k}$ pairwise intersects. The statement then straightforwardly follows from Lemma 2. $\square$

[7] As Dutta's definition requires a stable set to be internally and externally stable, he actually proves that the intersection of any pair of coverings sets *contains* a covering set. A simpler proof, which shows that *externally* $S_{\mathcal{M}}$-stable sets are closed under intersection, is given by Laslier [20].

Interestingly, $\mathcal{S}_{\mathcal{M}_2}$, the set of all dominant sets, is not only closed under intersection, but in fact totally ordered with respect to set inclusion.

We conjecture that the set of all $S_{\mathcal{M}^*}$-stable sets also pairwise intersects and thus admits a unique minimal element. However, the combinatorial structure of transitive subtournaments within tournaments is extraordinarily rich [see, e.g., 31, 14] and it seems that a proof of the conjecture would be significantly more difficult than Dutta's.

Conjecture 1. *$\mathcal{S}_{\mathcal{M}^*}$ is closed under intersection.*

Using Lemma 2, the conjecture entails that $\mathcal{S}_{\mathcal{M}^*_k}$ for all $k \in \mathbb{N}$ is also closed under intersection. This trivially holds for $k \leq 3$ since $\mathcal{S}_{\mathcal{M}^*_k} = \mathcal{S}_{\mathcal{M}_k}$ in this case. The weakest version of Conjecture 1 that is not directly implied by Theorem 2 is that $\mathcal{S}_{\mathcal{M}^*_4}$ is closed under intersection. We were able to show this by reducing it to a large, but finite, number of cases that were checked using a computer. Unfortunately, this exercise did not yield enough insight to prove Conjecture 1. We will see in Section 4.3 that Conjecture 1 is a weakened version of a conjecture by Schwartz [26], which was verified for small tournaments using a computer.

Two well-known examples of minimal stable sets are the top cycle of a tournament, which is the minimal stable set with respect to $S_{\mathcal{M}_2}$, and the minimal covering set, which is the minimal stable set with respect to $S_{\mathcal{M}}$.

Minimal dominant set. The *minimal dominant set* (or top cycle) of a tournament $T = (A, \succ)$ is given by $TC(T) = \widehat{S}_{\mathcal{M}_2}(T) = \widehat{CNL}$, i.e., it is the smallest set B such that $B \succ A \setminus B$ [16, 28].

Minimal covering set. The *minimal covering set* of a tournament T is given by $MC(T) = \widehat{S}_{\mathcal{M}}(T) = \widehat{UC}$, i.e., it is the smallest set B such that for all $a \in A \setminus B$, there exists $b \in B$ so that every alternative in B that is dominated by a is also dominated by b [11].

The proposed methodology also suggests the definition of a new tournament solution that has not been considered in the literature before.

Minimal extending set. The *minimal extending set* of a tournament T is given by $ME(T) = \widehat{S}_{\mathcal{M}^*}(T) = \widehat{BA}$, i.e., it is the smallest set B such that no $a \in A \setminus B$ is the maximal element of a maximal transitive subset in $B \cup \{a\}$.

The minimal extending set will be further analyzed in Section 4.3.

4.2 Properties of Minimal Stable Sets

If $\mathcal{S}_S$ is directed—and we will only be concerned with tournament solutions S for which this is (presumably) the case—$\widehat{S}$ satisfies a number of desirable properties.

Proposition 4. *Let S be a tournament solution such that $\mathcal{S}_S$ is directed. Then, $\widehat{S}$ satisfies WSP and IUA.*

Proof. Clearly, any minimal S-stable set B remains S-stable when losing alternatives are removed or when edges between losing alternatives are modified. In the latter case, B also remains minimal. In the former case, the minimal S-stable set is contained in B. $\square$

It can be shown that sets that are stable within a stable set are also stable in the original tournament when the underlying tournament solution is defined via a concept of qualified subsets $\mathcal{Q}$. This lemma will prove very useful when analyzing $\widehat{S}_{\mathcal{Q}}$.

Lemma 3. *Let $T = (A, \succ)$ be a tournament and $\mathcal{Q}$ a concept of qualified subsets. Then, $\mathcal{S}_{\mathcal{Q}}(B) \subseteq \mathcal{S}_{\mathcal{Q}}(A)$ for all $B \in \mathcal{S}_{\mathcal{Q}}(A)$.*

Proof. We prove the statement by showing that the following implication holds for all $B \subseteq A$, $C \in \mathcal{S}_{\mathcal{Q}}(B)$, and $a \in A$:

$$\text{if } a \notin S_{\mathcal{Q}}(B \cup \{a\}) \text{ then } a \notin S_{\mathcal{Q}}(C \cup \{a\}).$$

To see this, let $a \notin S_{\mathcal{Q}}(B \cup \{a\})$ and assume for contradiction that there exist $[Q, a] \in \max_{\subseteq} \mathcal{Q}(C \cup \{a\})$. Then there has to be $b \in B$ such that $[Q \cup \{a\}, b] \in \mathcal{Q}(B \cup \{a\})$ because $[Q, a] \notin \max_{\subseteq} \mathcal{Q}(B \cup \{a\})$. Now, if $b \in C$, closure and independence imply that $[Q \cup \{a\}, b] \in \mathcal{Q}(C \cup \{a\})$, contradicting the maximality of $[Q, a]$. If, on the other hand, $b \in B \setminus C$, then there has to be $c \in C$ such that $[Q \cup \{b\}, c] \in \mathcal{Q}(C \cup \{b\})$. No matter whether $c \succ a$ or $a \succ c$, $Q \cup \{a, c\}$ is isomorphic to $[Q \cup \{b\}, c]$ and thus also a qualified subset, which again contradicts the assumption that $[Q, a]$ was maximal. $\square$

We are now ready to show a number of appealing properties of *unique* minimal stable sets when the underlying solution concept is defined via qualified subsets.

Theorem 4. *Let $\mathcal{Q}$ be a concept of qualified subsets such that $\mathcal{S}_{\mathcal{Q}}$ is directed. Then,*

(i) $\widehat{S}_{\mathcal{Q}} \subseteq S_{\mathcal{Q}}^{\infty}$,

(ii) $S_{\mathcal{Q}}(\widehat{S}_{\mathcal{Q}}(T) \cup \{a\}) = \widehat{S}_{\mathcal{Q}}(T)$ *for all tournaments $T = (A, \succ)$ and $a \in A$ (in particular, $\widehat{S}_{\mathcal{Q}}(T)$ is internally stable),*

(iii) $\widehat{S}_{\mathcal{Q}}$ *satisfies* SSP, *and*

(iv) $\widehat{\widehat{S}}_{\mathcal{Q}} = \widehat{S}_{\mathcal{Q}}$.

Proof. Let $T = (A, \succ)$ be a tournament. The first statement of the theorem is shown by proving by induction on k that $S_{\mathcal{Q}}^{k}(T)$ is an $S_{\mathcal{Q}}$-stable set. For the basis, let $B = S_{\mathcal{Q}}(T)$. Then, $S_{\mathcal{Q}}(B \cup \{a\}) \subseteq B$ for every $a \in A \setminus B$ due to WSP of $S_{\mathcal{Q}}$ (Proposition 1) and thus B is $S_{\mathcal{Q}}$-stable. Now, assume that $B = S_{\mathcal{Q}}^{k}(T)$ is $S_{\mathcal{Q}}$-stable and let $C = S_{\mathcal{Q}}(B)$. Again, WSP implies that $a \notin S_{\mathcal{Q}}(C \cup \{a\})$ for every $a \in B \setminus C$, i.e., $C \in \mathcal{S}_{\mathcal{Q}}(B)$. We can thus directly apply Lemma 3 to obtain that $C = S_{\mathcal{Q}}^{k+1}(T) \in \mathcal{S}_{\mathcal{Q}}(T)$. As the minimal $S_{\mathcal{Q}}$-stable set is contained in every $S_{\mathcal{Q}}$-stable set, the statement follows.

Regarding internal stability, assume for contradiction that $S_{\mathcal{Q}}(\widehat{S}_{\mathcal{Q}}(T)) \subset \widehat{S}_{\mathcal{Q}}(T)$. However, Lemma 3 implies that $S_{\mathcal{Q}}(\widehat{S}_{\mathcal{Q}}(T))$ is $S_{\mathcal{Q}}$-stable, contradicting the minimality of

$\widehat{S}_{\mathcal{Q}}(T)$. The remainder of the second statement follows straightforwardly from internal stability. If $S_{\mathcal{Q}}(\widehat{S}_{\mathcal{Q}}(T) \cup \{a\}) = C \subset \widehat{S}_{\mathcal{Q}}(T)$ for some $a \in A \setminus \widehat{S}_{\mathcal{Q}}(T)$, WSP implies that $S_{\mathcal{Q}}(\widehat{S}_{\mathcal{Q}}(T)) \subseteq C$, contradicting internal stability.

Regarding SSP, let $B = \widehat{S}_{\mathcal{Q}}(T)$ and assume for contradiction that $C = \widehat{S}_{\mathcal{Q}}(A') \subset B$ for some A' with $B \subseteq A' \subset A$. Clearly, C is $S_{\mathcal{Q}}$-stable not only in A' but also in B, which implies that $C \in \mathcal{S}_{\mathcal{Q}}(B)$. According to Lemma 3, C is also contained in $\mathcal{S}_{\mathcal{Q}}(A)$, contradicting the minimality of $\widehat{S}_{\mathcal{Q}}(T)$.

Finally, for $\widehat{\widehat{S}}_{\mathcal{Q}}(T) = \widehat{S}_{\mathcal{Q}}(T)$, we show that every $S_{\mathcal{Q}}$-stable set is $\widehat{S}_{\mathcal{Q}}$-stable and that every *minimal* $\widehat{S}_{\mathcal{Q}}$-stable set is $S_{\mathcal{Q}}$-stable set. The former follows from $\widehat{S}_{\mathcal{Q}}(T) \subseteq S_{\mathcal{Q}}(T)$, which is a consequence of the first statement of this theorem. For the latter statement, let $B \in \min_{\subseteq}(\mathcal{S}_{\widehat{S}_{\mathcal{Q}}}(T))$. We first show that $\widehat{S}_{\mathcal{Q}}(B \cup \{a\}) = B$ for all $a \in A \setminus B$. Assume for contradiction that $\widehat{S}_{\mathcal{Q}}(B \cup \{a\}) = C \subset B$ for some $a \in A \setminus B$. Since $\widehat{S}_{\mathcal{Q}}$ satisfies SSP, $\widehat{S}_{\mathcal{Q}}(B \cup \{a\}) = C$ for all $a \in A \setminus B$. As a consequence, C is $\widehat{S}_{\mathcal{Q}}$-stable in $B \cup \{a\}$ for all $a \in A \setminus B$ and, due to the definition of stability, also in A. This contradicts the assumption that B was the *minimal* $\widehat{S}$-stable set. Hence, $\widehat{S}_{\mathcal{Q}}(B \cup \{a\}) = B$ for all $a \in A \setminus B$. By definition of $\widehat{S}_{\mathcal{Q}}$, this implies that $a \notin S_{\mathcal{Q}}(B \cup \{a\})$ and thus that B is $S_{\mathcal{Q}}$-stable. □

The second statement of Theorem 4 allows us to characterize stable sets using the fixed-point formulation mentioned at the beginning of this section, which unifies internal and external stability.

Corollary 1. *Let $\mathcal{Q}$ be a concept of qualified subsets such that $\mathcal{S}_{\mathcal{Q}}$ is directed and $T = (A, \succ)$ a tournament. Then,*

$$\widehat{S}_{\mathcal{Q}}(T) = \min_{\subseteq}\{B \subseteq A \mid B = \bigcup_{a \in A} S_{\mathcal{Q}}(B \cup \{a\})\}.$$

There may very well be more than one internally and externally $S_{\mathcal{Q}}$-stable set in a tournament. For example, the proof of Theorem 4 implies that $S_{\mathcal{Q}}^{\infty}(T)$ is internally and externally $S_{\mathcal{Q}}$-stable.

We have already seen that $\widehat{S}_{\mathcal{Q}}$ satisfies some of the basic properties defined in Section 2.5. It further turns out that $\widehat{S}$ inherits monotonicity and composition-consistency from S.

Proposition 5. *Let S be a tournament solution such that $\mathcal{S}_S$ is directed and S satisfies MON. Then, $\widehat{S}$ satisfies MON as well.*

Proof. Let $T = (A, \succ)$ be a tournament with $a, b \in A$, $a \in \widehat{S}(T)$, and $b \succ a$, and let the relation $\succ'$ be identical to $\succ$ except that $a \succ' b$. Denote $T' = (A, \succ')$ and assume for contradiction that $a \notin \widehat{S}(T')$. Then, there has to be a minimal S-stable set $B \subseteq A \setminus \{a\}$ in T'. We show that B is also S-stable in T, a contradiction. If $b \notin B$, this would clearly be the case because $\widehat{S}$ satisfies IUA. If, on the other hand, $b \in B$, the only reason for B not to be S-stable in T is that $a \in S((B \cup \{a\}), \succ)$. However, monotonicity of S then implies that $a \in S((B \cup \{a\}), \succ')$, which is a contradiction because B is S-stable in T'. □

Proposition 6. *Let S be a tournament solution that satisfies* COM*. Then, $\widehat{S}$ satisfies* COM *as well.*

Proof. Let S be a composition-consistent tournament solution and $T = (A, \succ) = \Pi(\tilde{T}, T_1, \dots, T_k)$ a product tournament with $T = (\{1, \dots, k\}, \tilde{\succ})$, $T_1 = (B_1, \succ_1)$, ..., $T_k = (B_k, \succ_k)$. For a subset C of A, let $C_i = C \cap B_i$ for all $i \in \{1, \dots, k\}$ and $\tilde{C} = \bigcup_{i:\, C_i \neq \emptyset} \{i\}$. We will prove that $C \subseteq A$ is S-stable if and only if

(i) $\tilde{C}$ is S-stable in $\tilde{T}$, and

(ii) C_i is S-stable in T_i for all $i \in \{1, \dots, k\}$.

Consider an arbitrary alternative $a \in A \setminus C$. For C to be S-stable, a should not be contained in $S(C \cup \{a\})$. Since S is composition-consistent, a may be excluded for two reasons. First, a may be contained in an unchosen component, i.e., $a \in B_i$ such that $i \notin S(\tilde{C} \cup \{i\})$. Secondly, a may not be selected despite being in a chosen component, i.e., $a \in B_i$ such that $i \in S(\tilde{C} \cup \{i\})$ and $a \notin S(C_i \cup \{a\})$. This directly establishes the claim above and consequently that $\widehat{S}$ is composition-consistent. □

The previous propositions and theorems allow us to deduce several known statements about TC and MC, in particular that both concepts satisfy all basic properties and that MC is a refinement of UC^∞ and satisfies COM.

We conclude this section by generalizing the axiomatization of the minimal covering set [11] to abstract minimal stable sets. One of the cornerstones of the axiomatization is S-exclusivity, which prescribes under which circumstances a single element may be dismissed from the choice set.[8]

Definition 14. *A tournament solution S' satisfies* S-exclusivity *if, for every tournament $T = (A, \succ)$, $S'(T) = A \setminus \{a\}$ implies that $a \notin S(A)$.*

If S always admits a *unique* minimal S-stable set and $\widehat{S}$ satisfies SSP, which is always the case if S is defined via qualified subsets, then $\widehat{S}$ can be characterized by SSP, S-exclusivity, and inclusion-minimality.

Proposition 7. *Let S be a tournament solution such that $\mathcal{S}_S$ is directed and $\widehat{S}$ satisfies* SSP*. Then, $\widehat{S}$ is the finest tournament solution satisfying* SSP *and S-exclusivity.*

Proof. Let S be a tournament solution as desired and S' a tournament solution that satisfies SSP and S-exclusivity. We first prove that $\widehat{S} \subseteq S'$ by showing that $S'(T)$ is S-stable for every tournament $T = (A, \succ)$. Let $B = S'(T)$ and $a \in A \setminus B$. It follows from SSP that $S'(B \cup \{a\}) = B$ and from S-exclusivity that $a \notin S(B \cup \{a\})$, which implies that B is S-stable. Since $\widehat{S}(T)$ is the unique inclusion-minimal S-stable set, it has to be contained in all S-stable sets. The statement now follows from the fact that $\widehat{S}$ satisfies SSP and S-exclusivity. □

Hence, TC is the finest tournament solution satisfying SSP and CNL-exclusivity, MC is the finest tournament solution satisfying SSP and UC-exclusivity, and ME is the finest tournament solution satisfying SSP and BA-exclusivity if Conjecture 1 holds.

[8] UC-exclusivity is the property γ^{**} used in the axiomatization of MC [20].

4.3 The Minimal Extending Set

As mentioned in Section 4.1, the minimal extending set is a new tournament solution that has not been considered before. In analogy to UC-stable sets, which are known as covering sets, we will call BA-stable sets *extending sets.* B is an extending set of tournament $T = (A, \succ)$ if, for all $a \notin B$, every transitive path (or so-called Banks trajectory) in $B \cup \{a\}$ with maximal element a can be extended, i.e., there is $b \in B$ such that b dominates every element on the path. In other words, $B \subseteq A$ is an *extending set* if for all $a \in A \setminus B$, $a \notin BA(B \cup \{a\})$.

If Conjecture 1 is correct, ME satisfies all properties defined in Section 2.5, is a refinement of BA due to Propositions 4 and 5 and Theorem 4, and is a refinement of MC since, according to 3, every covering set is also an extending set. We refer to Figure 1 for an example tournament where ME happens to be *strictly* contained in MC.[9]

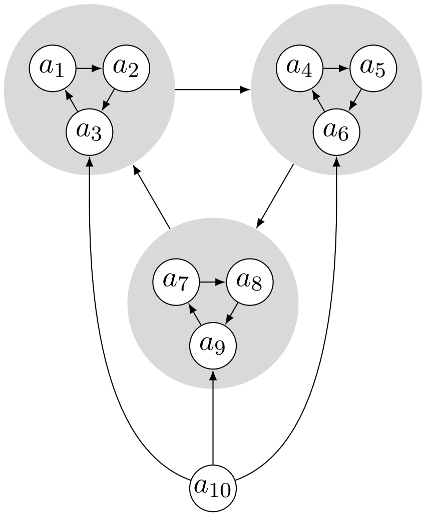

Figure 1: Example tournament $T = (A, \succ)$ where MC and ME differ ($MC(T) = A$ and $ME(T) = A \setminus \{a_{10}\}$). Omitted edges are assumed to point downwards by convention. Gray circles denote components, but a_{10} only dominates a_3, a_6, and a_9.

ME bears some resemblance to Schwartz's *tournament equilibrium set* TEQ [26], which is defined as the minimal *retentive* set of a tournament. There are some interesting similarities between retentiveness and stability and, as in the case of ME, the uniqueness of a minimal retentive set and thus the characteristics of TEQ remain an open problem [26, 18, 17]. It can be shown that Schwartz's conjecture is stronger than ours and has a number of interesting consequences such as that TEQ itself can be represented as a minimal stable set and is strictly contained in ME [7]. Formally,

$$TEQ = \widehat{TEQ} \subset ME.$$

[9]The smallest example of this kind requires only eight alternatives, but is less intuitive.

Hence, any counter-example to Conjecture 1 also constitutes a counter-example to Schwartz's conjecture, for which a recent computer analysis failed to find a counter-example in all tournaments of order 12 or less and a fairly large number of random tournaments [9].

A remarkable property of *ME* is that, just like *BA*, it is capable of ruling out alternatives in *regular* tournaments, i.e., it satisfies IRR [20]. No irregular tournament solution is known to satisfy all four basic properties. However, if Conjecture 1 were true, *ME* would be such a concept (see Table 1).

5 Conclusion

We proposed a unifying treatment of tournament solutions based on maximal qualified subsets and minimal stable sets. A central role in this theory may be ascribed to Conjecture 1, a statement of considerable mathematical depth that has a number of appealing consequences on minimal stable sets, some of which have been proved already.

(i) Every tournament T admits a unique *minimal dominant set* $TC(T)$ (as shown by 16). TC satisfies all basic properties and is the finest solution concept satisfying SSP and *CNL*-exclusivity. $TC \subseteq CNL$.

(ii) Every tournament T admits a unique *minimal covering set* $MC(T)$ (as shown by 11). MC satisfies all basic properties and is the finest solution concept satisfying SSP and UC-exclusivity. $MC \subseteq UC$ and $MC \subseteq TC$.

(iii) Every tournament T admits a unique *minimal extending set* $ME(T)$ (open problem). ME satisfies all basic properties and is the finest solution concept satisfying SSP and BA-exclusivity. $ME \subseteq BA$ and $ME \subseteq MC$.

Schwartz's conjecture, a stronger version of Conjecture 1, furthermore implies similarly desirable statements about TEQ.

Table 1 and Figure 2 summarize the properties and set-theoretic relationships of the considered tournament solutions, respectively. As mentioned in the introduction, our framework can be adapted to capture quantitative tournament solutions such as the Copeland set and the bipartisan set.

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Solution Concept		Origin	MON	IUA	WSP	SSP	COM	IRR
$S_{\mathcal{M}_2}$	(CNL)		✓	✓	✓	–	–	–
$S_{\mathcal{M}}$	(UC)	Fishburn [12], Miller [21]	✓	–	✓	–	✓	–
$S_{\mathcal{M}^*}$	(BA)	Banks [2]	✓	–	✓	–	✓	✓
$\widehat{S}_{\mathcal{M}_2}$	(TC)	Good [16], Smith [28]	✓	✓	✓	✓	–	–
$\widehat{S}_{\mathcal{M}}$	(MC)	Dutta [11]	✓	✓	✓	✓	✓	–
$\widehat{S}_{\mathcal{M}^*}$	(ME)		✓[a]	✓[a]	✓[a]	✓[a]	✓	✓

[a] This statement relies on Conjecture 1.

Table 1: Properties of tournament solutions (MON: monotonicity, IUA: independence of unchosen alternatives, WSP: weak superset property, SSP: strong superset property, COM: composition-consistency, IRR: irregularity). See Laslier [20] for all results not shown in this paper.

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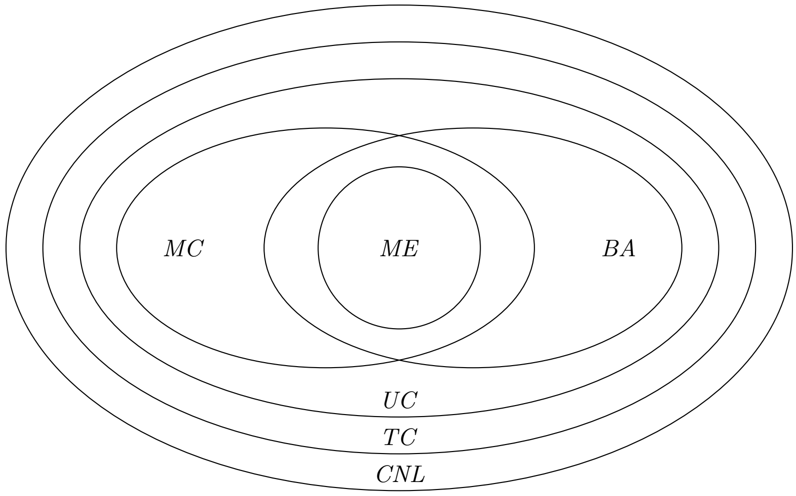

Figure 2: Set-theoretic relationships between tournament solutions. *BA* and *MC* are not contained in each other, but they always intersect. The inclusion of *ME* in *MC* and *BA* relies on Conjecture 1. If Schwartz's conjecture holds, *ME* can be further refined to *TEQ*.

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