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# Minimal Retentive Sets in Tournaments

**Felix Brandt · Markus Brill · Felix Fischer · Paul Harrenstein**

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**Abstract** *Tournament solutions*, i.e., functions that associate with each complete and asymmetric relation on a set of alternatives a non-empty subset of the alternatives, play an important role in the mathematical social sciences at large. For any given tournament solution $S$, there is another tournament solution $\mathring{S}$ which returns the union of all inclusion-minimal sets that satisfy $S$*-retentiveness*, a natural stability criterion with respect to $S$. Schwartz's *tournament equilibrium set* (*TEQ*) is defined as $TEQ = \mathring{TEQ}$. Due to this unwieldy recursive definition, preciously little is known about *TEQ*. Contingent on a well-known conjecture about *TEQ*, we show that $\mathring{S}$ inherits a number of important and desirable properties from $S$. We thus obtain an infinite hierarchy of attractive and efficiently computable tournament solutions that "approximate" *TEQ*, which itself is computationally intractable. This hierarchy contains well-known tournament solutions such as the top cycle (*TC*) and the minimal covering set (*MC*). We further prove a weaker version of the conjecture mentioned above, which establishes $\mathring{TC}$ as an attractive new tournament solution.

**Keywords** Tournament Solutions · Retentiveness · Tournament Equilibrium Set

## 1 Introduction

Many problems in the mathematical social sciences can be addressed using tournament solutions, i.e., functions that associate with each complete and asymmetric dominance relation on a set of alternatives a non-empty subset of the alternatives. For

F. Brandt · M. Brill · P. Harrenstein
Institut für Informatik, Technische Universität München, 85748 Garching, Germany
Tel.: +49-89-289 17512
Fax: +49-89-289 17535
E-mail: {brandtf, brill, harrenst}@in.tum.de

F. Fischer
School of Engineering and Applied Sciences, Harvard University, Cambridge, MA 02138, USA
E-mail: fischerf@seas.harvard.edu

instance, tournament solutions play an important role in social choice theory, where the binary relation is typically defined via pairwise majority voting (Moulin, 1986; Laslier, 1997). Other application areas include multi-criteria decision analysis (Arrow and Raynaud, 1986; Bouyssou et al, 2006), zero-sum games (Fisher and Ryan, 1995; Laffond et al, 1993a; Duggan and Le Breton, 1996), coalition formation (Brandt and Harrenstein, 2010), and argumentation theory (Dung, 1995; Dunne, 2007).

Examples of well-studied tournament solutions are the *top cycle* (*TC*), the *Copeland set* (*CO*), the *minimal covering set* (*MC*), the *Banks set* (*BA*), and the *Slater set* (see Laslier, 1997). Recent years have also witnessed an increasing interest in these concepts by the computer science community, particularly with respect to their computational complexity. For example, the Copeland set and the minimal covering set of a tournament can be computed efficiently, i.e., in polynomial time (Brandt et al, 2009; Brandt and Fischer, 2008), whereas computing the Banks set and the Slater set is computationally intractable (Woeginger, 2003; Alon, 2006; Conitzer, 2006).

The *tournament equilibrium set* (*TEQ*), introduced by Schwartz (1990), ranks among the most intriguing, but also among the most enigmatic, tournament solutions. Schwartz defined *TEQ* on the basis of the concept of *retentiveness*. For a given tournament solution $S$, a set $B$ of alternatives is said to be $S$*-retentive* if $S$ selects from each dominator set of some alternative in $B$ a subset of alternatives that is contained in $B$. The requirement of retentiveness can be argued for from the perspective of cooperative majority voting, where the voters have to come to an eventual agreement as to which alternative to elect (see Schwartz, 1990, for more details). Additionally, retentiveness strongly resembles the game-theoretic notion of closure under best-response behavior (Basu and Weibull, 1991).

Schwartz defines *TEQ* as the union of all inclusion-minimal *TEQ*-retentive sets. This is a proper recursive definition, as the cardinality of the set of dominators of an alternative in a particular set is always smaller than the cardinality of the set itself. Unfortunately, and somewhat surprisingly, it is unknown whether *TEQ* satisfies a number of important properties proposed in the literature on tournament solutions, namely *monotonicity*, *independence of unchosen alternatives*, and the *weak superset property*. However, Laffond et al (1993b) and Houy (2009b,a) have shown that *TEQ* satisfies any one of these properties if and only if it satisfies all of them. They have moreover shown that *TEQ* satisfying any of the properties is equivalent to a conjecture by Schwartz (1990) that every tournament contains a *unique* minimal *TEQ*-retentive set. Schwartz's conjecture also implies that *TEQ* is strictly contained in the minimal covering set. Apart from these implications, the only known facts about *TEQ* are that it is contained in the Banks set (Schwartz, 1990), satisfies composition-consistency (Laffond et al, 1996), and is NP-hard to compute (Brandt et al, 2010).[1]

In this article, we view the matter from a more general perspective and study tournament solutions that are defined via Schwartz's notion of retentiveness. More specifically, for any given tournament solution $S$, we define another tournament solution $\mathring{S}$ which yields the union of all minimal $S$-retentive sets. Thus, e.g., *TEQ* equals $\mathring{TEQ}$ by definition and the *top cycle* coincides with $\mathring{TRIV}$, where *TRIV* is the trivial

[1] NP-hardness is commonly seen as strong evidence that a problem cannot be solved efficiently. The interested reader is referred to the articles by Woeginger (2003) and Brandt and Fischer (2008) for a more detailed discussion.

tournament solution that always returns all alternatives. With every tournament solution $S$ we associate an infinite sequence $(S_1, S_2, \dots)$ of tournament solutions such that $S = S_1$ and $S_{i+1} = \mathring{S}_i$ for all $i \geq 1$.

Our contribution concentrates on three main issues of such sequences and the solution concepts therein:

- the *inheritance of desirable properties*,
- their *asymptotic behavior*, and
- the *uniqueness of minimal retentive sets*.

First, while it is unknown whether *TEQ* itself has the desirable properties mentioned above, we do know that some less sophisticated tournament solutions such as *TRIV* do. Thus, in Section 4, we investigate which properties are inherited from $S$ to $\mathring{S}$ and vice versa. We find that the former is the case for most of the properties mentioned above provided that $S$ always admit a unique minimal $S$-retentive set, whereas the latter also holds without this assumption. Composition-consistency is a notable exception: we prove that *TEQ* is the *only* composition-consistent tournament solution defined via retentiveness.

Secondly, we find that for every $S$ the sequence $(S_1, S_2, \dots)$ converges to *TEQ*. In Section 5, we investigate the properties of these sequences in more detail. Assuming Schwartz's conjecture, we show that all tournament solutions in the sequence associated with the trivial tournament solution *TRIV* are strictly contained in each other, strictly contain *TEQ*, and, by the inheritance results of Section 4, share most of the desirable properties of *TEQ*. Efficient computability turns out to be inherited from $S$ to $\mathring{S}$ even without assuming Schwartz's conjecture. While this does not imply that *TEQ* itself is efficiently computable, the tournament solutions in the sequence for *TRIV* provide better and better efficiently computable approximations of *TEQ*. We also establish tight bounds on the minimal number $k$ such that $S_k$ is guaranteed to coincide with *TEQ*, relative to the size of the tournament in question.

The sequence associated with each tournament solution gives rise to a corresponding sequence of weaker versions of Schwartz's conjecture, which we saw were important for the inheritance of desirable properties. The first such statement, which concerns the sequence for *TRIV* and alleges that every tournament has a unique minimal *TRIV*-retentive set, was proved by Good (1971). In Section 6 we show the second statement: there is a unique minimal *TC*-retentive set in every tournament. We conclude the article by showing that uniqueness of minimal $S$-retentive sets is not guaranteed for all well-known tournament solutions $S$. In particular, we identify a tournament with disjoint minimal Copeland-retentive sets.

## 2 Preliminaries

In this section, we provide the terminology and notation required for our results. For a more extensive treatment of tournament solutions and their properties the reader is referred to Laslier (1997).

## 2.1 Tournaments

Let $X$ be a universe of alternatives. The set of all non-empty finite subsets of $X$ will be denoted by $\mathcal{F}(X)$. A (finite) *tournament* $T$ is a pair $(A, \succ)$, where $A \in \mathcal{F}(X)$ and $\succ$ is an asymmetric and complete (and thus irreflexive) binary relation on $X$, usually referred to as the *dominance relation*.[2] Intuitively, $a \succ b$ signifies that alternative $a$ is preferable to alternative $b$. The dominance relation can be extended to sets of alternatives by writing $A \succ B$ when $a \succ b$ for all $a \in A$ and $b \in B$. We further write $\mathcal{T}(X)$ for the set of all tournaments on $X$.

For a tournament $(A, \succ)$, an alternative $a \in A$, and a subset $B \subseteq A$ of alternatives, we denote by $D_{B,\succ}(a)$ the *dominion* of $a$ in $B$, i.e.,

$$D_{B,\succ}(a) = \{\, b \in B : a \succ b \,\},$$

and by $\overline{D}_{B,\succ}(a)$ the *dominators* of $a$ in $B$, i.e.,

$$\overline{D}_{B,\succ}(a) = \{\, b \in B : b \succ a \,\}.$$

Whenever the dominance relation $\succ$ is known from the context or $B$ is the set of all alternatives $A$, the respective subscript will be omitted to improve readability.

For a tournament $T = (A, \succ)$ and a subset $B \subseteq A$ of alternatives, we further write $T|_B = (B, \{(a, b) \in B \times B : a \succ b\})$ for the restriction of $T$ to $B$. The *order* $|T|$ of a tournament $T = (A, \succ)$ refers to the cardinality of $A$. A *tournament isomorphism* of two tournaments $T = (A, \succ)$ and $T' = (A', \succ')$ is a bijection $\pi\colon A \to A'$ such that for all $a, b \in A$, $a \succ b$ if and only if $\pi(a) \succ' \pi(b)$. A tournament $(A, \succ)$ can be conveniently represented as a directed graph with vertex set $A$ and edge set $\{(a, b) : a \succ b\}$ (see Figure 1 for an example).

## 2.2 Components and Decompositions

An important structural notion in the context of tournaments is that of a *component*. A component is a subset of alternatives that bear the same relationship to all alternatives not in the set.

**Definition 1** Let $T = (A, \succ)$ be a tournament. A non-empty subset $B$ of $A$ is a *component* of $T$ if for all $a \in A \setminus B$, either $B \succ \{a\}$ or $\{a\} \succ B$. A *decomposition* of $T$ is a set of pairwise disjoint components $\{B_1, \dots, B_k\}$ of $T$ such that $A = \bigcup_{i=1}^{k} B_i$.

For a given tournament $\tilde{T}$, a new tournament $T$ can be constructed by replacing each alternative with a component. For notational convenience, we tacitly assume that $\mathbb{N} \subseteq X$.

**Definition 2** Let $B_1, \dots, B_k \subseteq X$ be pairwise disjoint sets and consider tournaments $\tilde{T} = (\{1, \dots, k\}, \tilde{\succ})$ and $T_1 = (B_1, \succ_1), \dots, T_k = (B_k, \succ_k)$. The *product* of $T_1, \dots, T_k$ with respect to $\tilde{T}$, denoted by $\Pi(\tilde{T}, T_1, \dots, T_k)$, is the tournament $(A, \succ)$ such that $A = \bigcup_{i=1}^{k} B_i$ and for all $b_1 \in B_i, b_2 \in B_j$,

$$b_1 \succ b_2 \quad \text{if and only if} \quad i = j \text{ and } b_1 \succ_i b_2, \text{ or } i \neq j \text{ and } i \mathrel{\tilde{\succ}} j.$$

[2] This definition slightly diverges from the common graph-theoretic definition where $\succ$ is defined on $A$ rather than $X$. However, it facilitates the sound definition of tournament solutions.

### 2.3 Tournament Solutions

A *Condorcet winner* in a tournament is an alternative that dominates every other alternative. Let $Cond(T)$ denote the set of Condorcet winners of $T = (A, \succ)$, i.e., $Cond(T) = \{a \in A : a \succ b \text{ for all } b \in A \setminus \{a\}\}$. Due to the asymmetry of the dominance relation, every tournament contains at most one Condorcet winner.

Since the dominance relation may contain cycles and thus fail to have a Condorcet winner, a variety of concepts have been suggested to take over the role of singling out the "best" alternatives of a tournament. Formally, a *tournament solution $S$* is defined as a function that associates with each tournament $T = (A, \succ)$ a non-empty subset $S(T)$ of $A$.

Following Laslier (1997), we require a tournament solution to be independent of alternatives outside the tournament, invariant under tournament isomorphisms, and to choose the Condorcet winner whenever it exists.

**Definition 3** A *tournament solution* is a function $S : \mathcal{T}(X) \to \mathcal{F}(X)$ such that

($i$) $S(T) = S(T')$ for all tournaments $T = (A, \succ)$ and $T' = (A, \succ')$ such that $T|_A = T'|_A$;
($ii$) $S((\pi(A), \succ')) = \pi(S((A, \succ)))$ for all tournaments $(A, \succ)$, $(A', \succ')$, and every tournament isomorphism $\pi : A \to A'$ of $(A, \succ)$ and $(A', \succ')$; and
($iii$) $Cond(T) \subseteq S(T) \subseteq A$ for all tournaments $T = (A, \succ)$.

Laslier (1997) is slightly more stringent in that he requires the Condorcet winner to be the *only* element in $S(T)$ whenever it exists. We will call a tournament solution *proper* if it satisfies this additional requirement.

The conditions of Definition 3 are trivially satisfied if one invariably selects the set of all alternatives. The corresponding tournament solution *TRIV* is obtained by letting $TRIV((A, \succ)) = A$ for every tournament $(A, \succ)$. Among the tournament solutions considered in this article, *TRIV* is the only one that is not proper. The *top cycle* $TC(T)$ of a tournament $T = (A, \succ)$ is defined as the smallest set $B \subseteq A$ such that $B \succ A \setminus B$. Uniqueness of such a set is straightforward and was first shown by Good (1971). The *Copeland set* $CO(T)$ consists of all alternatives whose dominion is of maximal size, i.e., $CO(T) = \arg\max_{a \in A} |D(a)|$.

For two tournament solutions $S$ and $S'$, we write $S' \subseteq S$, and say that $S'$ is a *refinement* of $S$, if $S'(T) \subseteq S(T)$ for all tournaments $T$. For example, it can easily be checked that $CO \subseteq TC \subseteq TRIV$. To avoid cluttered notation, we write $S(A, \succ)$ instead of $S((A, \succ))$ for a tournament $(A, \succ)$. Furthermore, we frequently write $S(B)$ instead of $S(B, \succ)$ for a subset $B \subseteq A$ of alternatives, if the dominance relation $\succ$ is known from the context.

## 3 Retentive Sets

Motivated by cooperative majority voting, Schwartz (1990) introduced a tournament solution based on a notion he calls *retentiveness*. The intuition underlying retentive sets is that alternative $a$ is only "properly" dominated by alternative $b$ if $b$ is chosen among $a$'s dominators by some underlying tournament solution $S$. A set of alternatives is then called $S$-retentive if none of its elements is properly dominated by some alternative outside the set.

**Definition 4** Let $S$ be a tournament solution and $T = (A, >)$ a tournament. Then, $B \subseteq A$ is $S$*-retentive* in $T$ if $B \neq \emptyset$ and $S(\overline{D}(b)) \subseteq B$ for all $b \in B$ such that $\overline{D}(b) \neq \emptyset$. The set of $S$-retentive sets for a given tournament $T = (A, >)$ will be denoted by $\mathcal{R}_S(T)$, i.e., $\mathcal{R}_S(T) = \{B \subseteq A : B \text{ is } S\text{-retentive in } T\}$.

Fix an arbitrary tournament solution $S$. Since the set $A$ of all alternatives is trivially $S$-retentive in $(A, >)$, $S$-retentive sets are guaranteed to exist. If a Condorcet winner exists, it must clearly be contained in any $S$-retentive set. The union of all (inclusion-)minimal $S$-retentive sets thus defines a tournament solution.

**Definition 5** Let $S$ be a tournament solution. Then, the tournament solution $\mathring{S}$ is given by

$$\mathring{S}(T) = \bigcup \min_{\subseteq}(\mathcal{R}_S(T)).$$

Consider for example the tournament solution *TRIV*, which always selects the set of all alternatives. It is easily verified that there always exists a *unique* minimal *TRIV*-retentive set, and that in fact $\mathring{TRIV} = TC$.

For a tournament solution $S$, we say that $\mathcal{R}_S$ is *pairwise intersecting* if for each tournament $T$ and for all sets $B, C \in \mathcal{R}_S(T)$, $B \cap C \neq \emptyset$. Observe that the non-empty intersection of two $S$-retentive sets is itself $S$-retentive. We thus have the following.

**Proposition 1** *For every tournament solution $S$, $\mathcal{R}_S$ admits a unique minimal element if and only if $\mathcal{R}_S$ is pairwise intersecting.*

Schwartz introduced retentiveness in order to recursively define the *tournament equilibrium set* (*TEQ*) as the union of minimal *TEQ*-retentive sets. This recursion is well-defined because the order of the dominator set of any alternative is strictly smaller than the order of the original tournament.

**Definition 6** (Schwartz, 1990) The *tournament equilibrium set* (TEQ) is defined recursively as $TEQ = \mathring{TEQ}$.

In other words, *TEQ* is the unique fixed point of the $\circ$-operator. Schwartz conjectured that every tournament admits a *unique* minimal *TEQ*-retentive set.

**Conjecture 1** (Schwartz, 1990) *$\mathcal{R}_{TEQ}$ is pairwise intersecting.*

Despite several attempts to prove or disprove this statement (e.g., Laffond et al, 1993b; Houy, 2009b,a), Schwartz's conjecture has remained an open problem. A recent computer analysis showed that no counter-example can be found among tournaments of order twelve or less nor within a fairly large number of random tournaments (Brandt et al, 2010).

It turns out that the existence of a *unique* minimal $S$-retentive set is quintessential for showing that $\mathring{S}$ satisfies several important properties to be defined in the next section.

The ∘-operator can also be applied iteratively. Inductively define

$$S^{(0)} = S \quad \text{and} \quad S^{(k+1)} = \mathring{S}^{(k)},$$

and consider the sequence $(S^{(n)})_{n\in\mathbb{N}_0} = (S^{(0)}, S^{(1)}, S^{(2)}, \ldots)$. We say that $(S^{(n)})_{n\in\mathbb{N}_0}$ *converges* to a tournament solution $S'$ if for each tournament $T$, there exists $k_T \in \mathbb{N}_0$ such that $S^{(n)}(T) = S'(T)$ for all $n \geq k_T$. It turns out that the limit of all these sequences is *TEQ*.

**Theorem 1** *Every tournament solution converges to TEQ.*

*Proof* Let $S$ be a tournament solution. We show by induction on $n$ that

$$S^{(n-1)}(T) = TEQ(T).$$

for all tournaments $T$ of order $|T| \leq n$. The case $n = 1$ is trivial. For the induction step, let $T = (A, \succ)$ be a tournament of order $|A| = n + 1$. We have to show that $S^{(n)}(T) = TEQ(T)$. Since $S^{(n)}$ is defined as the union of all minimal $S^{(n-1)}$-retentive sets, it suffices to show that a subset $B \subseteq A$ is $S^{(n-1)}$-retentive if and only if it is *TEQ*-retentive. We have the following chain of equivalences:

$$\begin{aligned} B \text{ is } S^{(n-1)}\text{-retentive} \quad &\text{iff} \quad \text{for all } b \in B,\, S^{(n-1)}(\overline{D}(b)) \subseteq B \\ &\text{iff} \quad \text{for all } b \in B,\, TEQ(\overline{D}(b)) \subseteq B \\ &\text{iff} \quad B \text{ is } TEQ\text{-retentive}. \end{aligned}$$

In particular, the second equivalence follows from the induction hypothesis, since obviously $|\overline{D}(a)| \leq n$ for all $a \in A$. □

## 4 Inheritance of Properties

In order to compare tournament solutions with one another, a number of desirable properties for tournament solutions have been identified. In this section, we review five of the most common properties—*monotonicity*, *independence of unchosen alternatives*, *the weak and strong superset properties*, and $\widehat{\gamma}$—and investigate which of them are inherited from $S$ to $\mathring{S}$ or from $\mathring{S}$ to $S$.[3]

A tournament solution is monotonic if a chosen alternative remains in the choice set when it is strengthened in the sense of dominating more alternatives, while everything else remains unchanged. It is independent of unchosen alternatives if the choice set is invariant under any modification of the dominance relation among the alternatives that are not chosen. A tournament solution satisfies the weak superset property if no new alternatives are chosen when unchosen alternatives are removed, and the strong superset property if in this case the choice set remains unchanged. Finally, $\widehat{\gamma}$ requires that if a the same set of alternatives is selected in two subtournaments $(B_1, \succ)$ and $(B_2, \succ)$ of the same tournament $(A, \succ)$, then this set is also selected

[3] Our terminology differs slightly from those of Laslier (1997) and others. *Independence of unchosen alternatives* is also called *independence of the losers* or *independence of non-winners*. The *weak superset property* has been referred to as $\epsilon^+$ or as the *Aïzerman property*.

in the tournament $(B_1 \cup B_2, \succ)$. As such, $\widehat{\gamma}$ is a variant of the better-known expansion property $\gamma$, which, together with Sen's $\alpha$, figures prominently in the characterization of rationalizable choice functions (Brandt and Harrenstein, 2011). Formally, we have the following definitions.

**Definition 7** Let $S$ be a tournament solution.

($i$) $S$ satisfies *monotonicity* (MON) if $a \in S(T)$ implies $a \in S(T')$ for all tournaments $T = (A, \succ)$, $T' = (A, \succ')$, and $a \in A$ such that $T|_{A\setminus\{a\}} = T'|_{A\setminus\{a\}}$ and $D_{\succ}(a) \subseteq D_{\succ'}(a)$.
($ii$) $S$ satisfies *independence of unchosen alternatives* (IUA) if $S(T) = S(T')$ for all tournaments $T = (A, \succ)$ and $T' = (A, \succ')$ such that $T|_{S(T)\cup\{a\}} = T'|_{S(T)\cup\{a\}}$ for all $a \in A$.
($iii$) $S$ satisfies the *weak superset property* (WSP) if $S(B) \subseteq S(A)$ for all tournaments $(A, \succ)$ and $B \subseteq A$ such that $S(A) \subseteq B$.
($iv$) $S$ satisfies the *strong superset property* (SSP) if $S(B) = S(A)$ for all tournaments $(A, \succ)$ and $B \subseteq A$ such that $S(A) \subseteq B$.
($v$) $S$ satisfies $\widehat{\gamma}$ if $S(B_1) = S(B_2)$ implies $S(B_1 \cup B_2) = S(B_1) = S(B_2)$ for all tournaments $(A, \succ)$ and all $B_1, B_2 \subseteq A$.

The five properties just defined—MON, IUA, WSP, SSP, and $\widehat{\gamma}$—will be called *basic properties* of tournament solutions. Observe that SSP implies WSP. Furthermore, the conjunction of MON and SSP implies IUA. To prove that a tournament solution satisfies all basic properties it is therefore sufficient to show that it satisfies MON, SSP, and $\widehat{\gamma}$.

We know from the work of Laffond et al (1993b) and Houy (2009b,a) that *TEQ* satisfies any $\mathsf{P} \in \{\mathsf{MON}, \mathsf{IUA}, \mathsf{WSP}, \mathsf{SSP}\}$ if and only if Conjecture 1 is true. The same can be shown for $\widehat{\gamma}$, so we have the following.

**Proposition 2** *TEQ satisfies any of the basic properties if and only if Conjecture 1 is true.*

An additional property considered in this article is *composition-consistency*. A tournament solution is composition-consistent if it chooses the "best" alternatives from the "best" components.

**Definition 8** A tournament solution $S$ is *composition-consistent* (COM) if for all tournaments $T, T_1, \dots, T_k$, and $\tilde{T} = (\{1, \dots, k\}, \tilde{\succ})$ such that $T = \Pi(\tilde{T}, T_1, \dots, T_k)$,

$$S(T) = \bigcup_{i \in S(\tilde{T})} S(T_i).$$

While *TRIV* trivially satisfies all of these properties, more discriminative tournament solutions often fail to satisfy some of them. For example, the Copeland set and the Slater set only satisfy MON and the Banks set only satisfies MON, WSP, and COM. The minimal covering set satisfies all of the properties. The same holds for *TEQ* if Conjecture 1 is true.

We begin by looking at a particular type of decomposable tournament that will be useful in the following. Let $C_3 = (\{1, 2, 3\}, \succ)$ with $1 \succ 2 \succ 3 \succ 1$, and let $I_a$ be

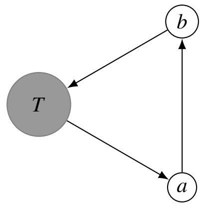


**Fig. 1** Tournament $C(T, I_a, I_b)$ for a given tournament $T$. The gray circle represents a component isomorphic to the original tournament $T$. An edge incident to a component signifies that there is an edge of the same direction incident to each alternative in the component.

the unique tournament on $\{a\}$. For three tournaments $T_1$, $T_2$, and $T_3$ on disjoint sets of alternatives, let $C(T_1, T_2, T_3) = \prod(C_3; T_1, T_2, T_3)$. Figure 1 illustrates the structure of $C(T, I_a, I_b)$ for a given tournament $T$.

We have the following lemma.

**Lemma 1** *Let $S$ be a proper tournament solution. Then, for each tournament $T = (A, \succ)$ and $a, b \notin A$,*

$$\mathring{S}(C(T, I_a, I_b)) = \{a, b\} \cup S(T).$$

*Proof* Let $B = \mathring{S}(C(T, I_a, I_b))$ and observe that $B \cap A \neq \emptyset$, because neither $\{a, b\}$ nor any of its subsets is $S$-retentive. Since $a$ is the Condorcet winner in $\overline{D}(b) = \{a\}$ and $b$ is the Condorcet winner in $\overline{D}(c)$ for any $c \in B \cap A$, by $S$-retentiveness of $B$ we have that $a \in B$ and $b \in B$. Also by retentiveness of $B$, we have $S(\overline{D}(a)) = S(T) \subseteq B$. We have thus shown that every $S$-retentive set must contain $\{a, b\} \cup S(T)$, and that $\{a, b\} \cup S(T)$ is itself $S$-retentive. □

We are now ready to show that a number of desirable properties are inherited from $\mathring{S}$ to $S$.

**Theorem 2** *Let $S$ be a proper tournament solution. Then each of the five basic properties is satisfied by $S$ if it is satisfied by $\mathring{S}$.*

*Proof* We show the following: if $S$ violates one of the five basic properties MON, IUA, WSP, SSP, or $\widehat{\gamma}$, then $\mathring{S}$ violates the same property. Observe that if $S$ violates any of these properties, this is witnessed by a tournament $T = (A, \succ)$ that serves as a counter-example. In the case of SSP (or WSP), there exists a set $B \subset A$ such that $S(A) \subseteq B \subset A$ and $S(B) \neq S(A)$ (or $S(B) \not\subseteq S(A)$, respectively). In the case of MON, there exists $a \in S(T)$ such that $a \notin S(T')$ for a tournament $T' = (A, \succ')$ that satisfies $T|_{A\setminus\{a\}} = T'|_{A\setminus\{a\}}$ and $D_{\succ}(a) \subseteq D_{\succ'}(a)$. In the case of IUA, $S(T) \neq S(T')$ for a tournament $T' = (A, \succ')$ that satisfies $T|_{S(T)\cup\{a\}} = T'|_{S(T)\cup\{a\}}$ for all $a \in A$. In the case of $\widehat{\gamma}$, there exist subsets $B_1, B_2 \subseteq A$ such that $S(B_1) = S(B_2)$ and $S(B_1 \cup B_2) \neq S(B_1)$.

It thus suffices to show how a counter-example $T$ for $S$ can be transformed into a counter-example $T'$ for $\mathring{S}$. Let $a, b \notin A$ and define $T' = C(T, I_a, I_b)$. Lemma 1 implies that $\mathring{S}(T') = \{a, b\} \cup S(T)$. Hence, tournament $T'$ constitutes a counter-example for $\mathring{S}$. □

If $\mathcal{R}_S$ is pairwise intersecting, a similar statement holds for the opposite direction. The proof of the following result can be found in the appendix. The conjunction of properties P and Q is denoted by $P \wedge Q$.

**Theorem 3** *Let $S$ be a proper tournament solution such that $\mathcal{R}_S$ is pairwise intersecting, and let* P *be any of the properties* SSP, WSP, IUA, MON $\wedge$ SSP, *or* $\widehat{\gamma} \wedge$ SSP. *Then,* P *is satisfied by $S$ if and only if it is satisfied by $\mathring{S}$.*

We proceed by showing that, among all tournament solutions that are defined as a minimal retentive set with respect to some proper tournament solution, *TEQ* is the only one that is composition-consistent.

**Proposition 3** *Let $S$ be a proper tournament solution. Then, $\mathring{S}$ satisfies* COM *if and only if $S = TEQ$.*

*Proof* It is well-known that *TEQ* is composition-consistent (Laffond et al, 1996). For the direction from left to right, let $S$ be a tournament solution different from *TEQ*, and assume that $\mathring{S}$ is composition-consistent. Since *TEQ* is the only tournament solution $S'$ such that $S' = \mathring{S}'$, there has to exist a tournament $T = (A, \succ)$ such that $S(T) \neq \mathring{S}(T)$. Let $a, b \notin A$, and define $T^* = C(T, I_a, I_b)$. By Lemma 1,

$$\mathring{S}(T^*) = \{a, b\} \cup S(T).$$

On the other hand, by composition-consistency of $\mathring{S}$,

$$\mathring{S}(T^*) = \mathring{S}(T) \cup \mathring{S}(I_a) \cup \mathring{S}(I_b) = \{a, b\} \cup \mathring{S}(T).$$

It follows that $S(T) = \mathring{S}(T)$, a contradiction. □

*Remark 1* The *composition-consistent hull* of a tournament solution $S$, denoted by $S^*$, is defined as the inclusion-minimal tournament solution that is composition-consistent and contains $S$ (Laffond et al, 1996). It can be shown that $(\mathring{S})^* = S^*$ for all tournament solutions $S$ that satisfy $\mathring{S} \subseteq S$.

*Remark 2* Although *TRIV* is not proper, it is easily seen that all the statements in this section also hold for *TRIV*. This is due to the fact that Lemma 1 trivially holds for $S = TRIV$.

We proceed by identifying properties of $S^{(k)}$ that are equivalent to Conjecture 1. The following lemma will be useful.

**Lemma 2** *Let $S_1$ and $S_2$ be tournament solutions such that $S_1 \subseteq S_2$ and $\mathcal{R}_{S_1}$ is pairwise intersecting. Then, $\mathcal{R}_{S_2}$ is pairwise intersecting and $\mathring{S}_1 \subseteq \mathring{S}_2$.*

*Proof* First observe that $S_1 \subseteq S_2$ implies that every $S_2$-retentive set is $S_1$-retentive. Now assume for contradiction that $\mathcal{R}_{S_2}$ is not pairwise intersecting and consider a tournament $(A, \succ)$ with two disjoint $S_2$-retentive sets $B, C \subseteq A$. Then, by the above observation, $B$ and $C$ are $S_1$-retentive, which contradicts the assumption that $\mathcal{R}_{S_1}$ is pairwise intersecting.

Furthermore, for every tournament $T$, $\mathring{S}_2(T)$ is $S_1$-retentive and thus contains the unique minimal $S_1$-retentive set, i.e., $\mathring{S}_1(T) \subseteq \mathring{S}_2(T)$. □

**Theorem 4** *Let $S$ be a tournament solution with $TEQ \subseteq S$ that satisfies* $\mathsf{WSP}$ *or* $\mathsf{IUA}$*. Then, the following statements are equivalent:*

*(i)* *For all $k \in \mathbb{N}_0$, $\mathcal{R}_{S^{(k)}}$ is pairwise intersecting.*
*(ii)* *For all $k \in \mathbb{N}_0$, $S^{(k)}$ satisfies each of the following properties if $S$ does:* $\mathsf{SSP}$, $\mathsf{WSP}$, $\mathsf{IUA}$, $\mathsf{MON} \wedge \mathsf{SSP}$, $\widehat{\gamma} \wedge \mathsf{SSP}$.
*(iii)* *Conjecture 1 is true.*

*Proof* To see that (*i*) implies (*ii*), assume that $\mathcal{R}_{S^{(k)}}$ is pairwise intersecting. Then, by Theorem 3, the properties SSP, WSP, IUA, MON $\wedge$ SSP, and $\widehat{\gamma} \wedge$ SSP are inherited from $S^{(k)}$ to $S^{(k+1)}$ for all $k \in \mathbb{N}_0$.

For the implication from (*ii*) to (*iii*), let P $\in$ {WSP, IUA} be satisfied by $S$. By (*ii*), $S^{(k)}$ satisfies P for all $k \in \mathbb{N}_0$. Assume for contradiction that Conjecture 1 does not hold. We know from Proposition 2 that this assumption is equivalent to *TEQ* not satisfying P. Let $T^*$ be a counter-example, i.e., a tournament showing that *TEQ* indeed violates P, and let $n$ be the order of $T^*$. In the proof of Theorem 1, we have shown that $S^{(n-1)}(T) = TEQ(T)$ for all tournaments $T$ of order at most $n$. Thus $T^*$ shows that $S^{(n-1)}$ violates P, contradicting (*ii*).

Finally, for the implication from (*iii*) to (*i*), assume that Conjecture 1 is true. We first prove by induction on $k$ that $TEQ \subseteq S^{(k)}$ for all $k \in \mathbb{N}_0$. The case $k = 0$ holds by assumption. Now let $T$ be a tournament and suppose that $TEQ(T) \subseteq S^{(k)}(T)$ for some $k \in \mathbb{N}_0$. By definition, $S^{(k+1)}(T)$ is $S^{(k)}$-retentive. We can thus apply the induction hypothesis to obtain that $S^{(k+1)}(T)$ is *TEQ*-retentive. Having assumed that the minimal *TEQ*-retentive set is unique, it is contained in any *TEQ*-retentive set, and we have that $TEQ(T) \subseteq S^{(k+1)}(T)$. This proves that $TEQ \subseteq S^{(k)}$ for all $k \in \mathbb{N}_0$. We can now apply Lemma 2 with $S_1 = TEQ$ and $S_2 = S^{(k)}$ to show that $\mathcal{R}_{S^{(k)}}$ is pairwise intersecting for all $k \in \mathbb{N}_0$. □

Among the tournament solutions that satisfy the requirements of Theorem 4 are *TRIV*, *TC*, the uncovered set *UC*, and the Banks set *BA* (see, e.g., Laslier, 1997, for definitions of the latter two).

## 5 Convergence to TEQ

By Theorem 1, every tournament solution converges to *TEQ*. Particularly well-behaved types of convergence are those that either yield larger and larger subsets of *TEQ* or smaller and smaller supersets of *TEQ*. The problem with the former type is that no refinement of *TEQ* is known and it is doubtful whether any such refinement would be efficiently computable. The latter type, however, turns out to be particularly useful.

Call a sequence $(S^{(n)})_{n \in \mathbb{N}_0}$ *contracting* if for all $k \in \mathbb{N}_0$, $S^{(k+1)} \subseteq S^{(k)}$. Intuitively, the elements of such a sequence constitute better and better "approximations" of *TEQ*. The following proposition identifies a sufficient condition for a sequence to be contracting.

**Proposition 4** *Let $S$ be a tournament solution with $TEQ \subseteq S$. If Conjecture 1 is true and $\mathring{S} \subseteq S$, then $S^{(k+1)} \subseteq S^{(k)}$ for all $k \in \mathbb{N}_0$.*

*Proof* We prove the statement by induction on $k$. Let $T$ be an arbitrary tournament. $\mathring{S}(T) \subseteq S(T)$ holds by assumption. Now suppose that $S^{(k)}(T) \subseteq S^{(k-1)}(T)$ for some $k \in \mathbb{N}_0$. As in the proof of Theorem 4, one can show that $TEQ \subseteq S^{(k)}$. Applying Lemma 2 with $S_1 = TEQ$ and $S_2 = S^{(k)}$ yields that $\mathcal{R}_{S^{(k)}}$ is pairwise intersecting. Therefore, we can apply Lemma 2 again, this time with $S_1 = S^{(k)}$ and $S_2 = S^{(k-1)}$, which gives $S^{(k+1)} \subseteq S^{(k)}$. □

**Corollary 1** *If Conjecture 1 is true, the tournament solutions TRIV, TC, MC, UC, and BA give rise to contracting sequences.*

*Proof* As *TRIV* obviously satisfies the assumptions of Proposition 4, $(TRIV^{(n)})_{n\in\mathbb{N}_0}$ and $(TC^{(n)})_{n\in\mathbb{N}_0}$ are contracting. *MC* satisfies the assumptions if Conjecture 1 holds, because under this assumption $TEQ \subseteq MC$ (Laffond et al, 1993b) and $\mathring{MC} \subseteq MC$ (Brandt, 2011). $TEQ \subseteq BA$ was shown by Schwartz (1990), and $TEQ \subseteq UC$ follows from $BA \subseteq UC$. It thus remains to be shown that $\mathring{UC} \subseteq UC$ and $\mathring{BA} \subseteq BA$.

A tournament solution $S$ satisfies *strong retentiveness* if the choice set of every dominator set is contained in the original choice set, i.e., if $S(\overline{D}(a)) \subseteq S(A)$ for all $a \in A$ (Brandt, 2011). It is easy to see that $\mathring{S} \subseteq S$ for every tournament solution $S$ that satisfies strong retentiveness. Indeed, for an arbitrary tournament $T$, strong retentiveness implies that $S(T)$ is $S$-retentive and that there do not exist any $S$-retentive sets disjoint from $S(T)$.[4] Since both *UC* and *BA* satisfy strong retentiveness (Brandt, 2011), this completes the proof. □

*Remark 3* One might wonder if *MC* is contained in the sequence $(TRIV^{(n)})_{n\in\mathbb{N}_0}$. It is easy to see that this is not the case: while *MC* is known to be composition-consistent (see Laffond et al, 1996), Proposition 3 establishes that this is not the case for any $TC^{(k)} = TRIV^{(k+1)}$ with $k \geq 0$.

*Remark 4* For a given tournament solution $S$, one may further want to compare the sequence $(S^{(n)})_{n\in\mathbb{N}_0}$ with the corresponding sequence $(S^n)_{n\in\mathbb{N}}$ generated by the repeated application of $S$. Formally, $S^k(T) = S(S^{k-1}(T))$ where $S^1(T) = S(T)$. Since SSP implies that $S^n = S$ for all $n \in \mathbb{N}$, *UC* and *BA* are the only tournament solutions covered by Corollary 1 for which such a comparison makes sense. It turns out that for both *UC* and *BA*, the sequences $(S^{(n)})_{n\in\mathbb{N}_0}$ and $(S^n)_{n\in\mathbb{N}}$ are incomparable in the sense that for all $n \in \mathbb{N}$, neither $S^{(n)} \subseteq S^2$ nor $S^n \subseteq \mathring{S}$.

## 5.1 Iterations to Convergence

We may ask how many iterated applications of the ∘-operator are needed until we arrive at *TEQ*. While we have seen that every tournament solution converges to *TEQ*, it turns out that no solution other than *TEQ* itself does so in a finite number of steps.

For a tournament solution $S$, let $k_S(n)$ be the smallest $k \in \mathbb{N}_0$ such that $S^{(k)}(T) = TEQ(T)$ for all tournaments $T$ of order at most $n$.[5]

[4] The statement was independently shown by Moser (2009).

[5] It can easily be shown that $S^{(\ell)}(T) = TEQ(T)$ for all $\ell \geq k_S(n)$.

**Proposition 5** *Let $S \neq TEQ$ be a proper tournament solution and let $n_0$ be the order of a smallest tournament $T$ with $S(T) \neq TEQ(T)$. Then, for every $n \in \mathbb{N}_0$,*

$$k_S(n) = \max\left(\left\lfloor \frac{n - n_0}{2} \right\rfloor + 1\,,\, 0\right).$$

*Proof* Let $f(n) = \max\left(\left\lfloor \frac{n-n_0}{2} \right\rfloor + 1, 0\right)$. Our goal is to prove that $f(n)$ is both an upper bound and a lower bound on $k_S(n)$.

For the former, we show that $S^{(f(n))}(T) = TEQ(T)$ for all tournaments $T$ of order at most $n$. To this end, let $T = (A, \succ)$ be an arbitrary tournament. Denote by $k_S(T)$ the smallest number $k$ such that $S^{(k)}(T) = TEQ(T)$. Thus, $k_S(n) = \max_{T:|T|\leq n} k_S(T)$.

A *Condorcet loser* in $T$ is an alternative $a \in A$ such that $\overline{D}(a) = A \setminus \{a\}$. We claim that the following statements hold for every proper tournament solution $S$ and every tournament $T$ of order $n$:

($i$) If $T$ has a Condorcet loser, then $k_S(T) \leq k_S(n-1)$.
($ii$) If $T$ has no Condorcet loser, then $k_S(T) \leq k_S(n-2) + 1$.

For ($i$), let $a$ be a Condorcet loser in $T = (A, \succ)$. Then,

$$S^{(k_S(n-1))}(T) = S^{(k_S(n-1))}(A \setminus \{a\}) = TEQ(A \setminus \{a\}) = TEQ(T).$$

The first and the third equality follow from the observations that no minimal retentive set contains $a$ and that a set $B \subseteq A \setminus \{a\}$ is retentive in $T$ if and only if it is retentive in $(A \setminus \{a\}, \succ)$. The second equality is a direct consequence of the definition of $k_S$. For ($ii$), assume that $T = (A, \succ)$ does not have a Condorcet loser. It follows that $|\overline{D}(a)| \leq n-2$ for all $a \in A$. Similar reasoning as in the proof of Theorem 1 implies that a set $B \subseteq A$ is $S^{(k_S(n-2))}$-retentive if and only if $B$ is $TEQ$-retentive. Thus, $S^{(k_S(n-2)+1)}(T) = TEQ(T)$.

We are now ready to show that $k_S(n) \leq f(n)$ by induction on $n$. For $n \leq n_0$, $k_S(n) = 0$. Now assume that $k_S(m) \leq f(m)$ holds for every $m < n$, and consider a tournament $T$ of order $n$. If $T$ has a Condorcet loser, ($i$) implies that $k_S(T) \leq k_S(n-1) \leq f(n-1)$, where the latter inequality follows from the induction hypothesis. If, on the other hand, $T$ does not have a Condorcet loser, ($ii$) implies that $k_S(T) \leq k_S(n-2)+1 \leq f(n-2) + 1$. Thus, $k_S(n) \leq \max(f(n-1), f(n-2)+1) = f(n-2)+1$. A simple calculation shows that $f(n-2) + 1 = f(n)$ as desired.

In order to show that $k_S(n) \geq f(n)$, we inductively define a family of tournaments $T_0, T_1, T_2, \ldots$ such that $S^{(f(|T_k|)-1)}(T_k) \neq TEQ(T_k)$. Let $T_0 = (A_0, \succ)$ be a smallest tournament such that $S(T_0) \neq TEQ(T_0)$. By definition, $|A_0| = n_0$. Given $T_{k-1} = (A_{k-1}, \succ)$, let $T_k = C(T_{k-1}, I_{a_k}, I_{b_k})$, where $a_k, b_k \notin A_{k-1}$ are two new alternatives. Observe that $A_k = A_0 \cup \bigcup_{\ell=1}^{k} \{a_\ell, b_\ell\}$. The structure of $T_k$ is illustrated in Figure 2. Repeated application of Lemma 1 yields

$$\begin{aligned} S^{(k)}(T_k) &= \{a_k, b_k\} \cup S^{(k-1)}(T_{k-1}) \\ &= \{a_k, b_k\} \cup \{a_{k-1}, b_{k-1}\} \cup S^{(k-2)}(T_{k-2}) \\ &= \ldots \\ &= \bigcup_{\ell=1}^{k} \{a_\ell, b_\ell\} \cup S(T_0). \end{aligned}$$

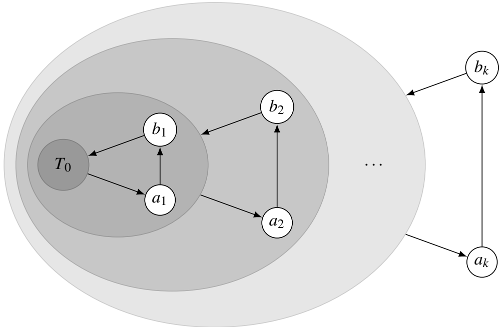


**Fig. 2** Tournament $T_k$ used in the proof of Proposition 5.

Since $S(T_0) \neq TEQ(T_0)$, we have that $S^{(k)}(T_k) \neq TEQ^{(k)}(T_k) = TEQ(T_k)$.

We have thus shown that $k_S(n_k) > k$, where $n_k = |A_k|$ is the order of tournament $T_k$. By definition of $T_k$, $n_k = n_0 + 2k$, so $k_S(n_k) > k$ implies $k_S(n) > \frac{n-n_0}{2}$ for all $n \geq n_0$ such that $n - n_0$ is even. For the case when $n - n_0$ is odd, i.e., when $n = n_0 + 2k + 1$ for some $k \in \mathbb{N}_0$, consider the tournament $T'_k = (A_{k+1} \setminus \{b_{k+1}\}, \succ)$ with $T'_k|_{A_{k+1}\setminus\{b_{k+1}\}} = T_{k+1}|_{A_{k+1}\setminus\{b_{k+1}\}}$. This tournament of order $n$ has $a_{k+1}$ as a Condorcet loser. Thus, $S^{(k)}(T'_k) = S^{(k)}(T_k) \neq TEQ(T_k) = TEQ(T'_k)$. This implies that $k_S(n_0+2k+1) > k$, or, equivalently, $k_S(n) > \lfloor \frac{n-n_0}{2} \rfloor$. □

*Remark 5* Like the results in Section 4 (cf. Remark 2), Proposition 5 also holds for *TRIV*, even though *TRIV* is not a proper tournament solution. Since *TRIV* and *TEQ* differ for every tournament with two alternatives, we immediately have $k_{TRIV}(n) = \lfloor \frac{n}{2} \rfloor$. Furthermore, Dutta (1990) constructed a tournament $T$ of order 8 for which $TEQ(T) \neq MC(T)$, and thus $k_{MC}(n) \geq \max(\lfloor \frac{n}{2} \rfloor - 3, 0)$.

*Remark 6* Interestingly, the tournaments $T_k$ constructed in the proof of Proposition 5 show that it might be impossible to *recognize* convergence within less than $k_S(n_k)$ iterations, as, for example, $TRIV^{(m)}(T_k) = TRIV^{(m')}(T_k)$ for all $m, m' < k_S(n_k)$.

*Remark 7* An easy corollary of the proof of Proposition 5 is that $k_S(n) \leq \lfloor \frac{n}{2} \rfloor$ for all tournament solutions $S$. To see this, observe that the proof of the upper bound also works for non-proper tournament solutions and that $n_0 \geq 2$.

## 5.2 Computational Aspects

The sequences $(TRIV^{(n)})_{n\in\mathbb{N}_0}$ and $(MC^{(n)})_{n\in\mathbb{N}_0}$ appear particularly interesting: under the assumption that Conjecture 1 is true, these sequences are contracting, and their members satisfy all basic properties. In addition, *TRIV* and *MC* can be computed efficiently, and we might ask whether this is also true for $TRIV^{(n)}$ and $MC^{(n)}$ when $n \geq 1$. This turns out to be the case, as a consequence of the following more general result.

**Proposition 6** *$\mathring{S}$ is efficiently computable if and only if $S$ is efficiently computable.*

*Proof* We show that the computation of $S$ and the computation of $\mathring{S}$ are equivalent under polynomial-time reductions.

To see that $\mathring{S}$ can be reduced to $S$, consider an arbitrary tournament $T = (A, \succ)$ and define the relation $R = \{(a, b) : a \in S(\overline{D}(b))\}$. It is easily verified that $\mathring{S}(T)$ is the union of all minimal $R$-undominated sets[6] or, equivalently, the maximal elements of the asymmetric part of the transitive closure of $R$. Observing that both $R$ and the minimal $R$-undominated sets can be computed in polynomial time (see, e.g., Brandt et al, 2009, for the latter) completes the reduction.

For the reduction from $S$ to $\mathring{S}$, consider a tournament $T = (A, \succ)$ and define $T^* = C(T, I_a, I_b)$ for $a, b \notin A$. By Lemma 1, $S(T) = \mathring{S}(T^*) \setminus \{a, b\}$. Clearly, $T^*$ can be computed in polynomial time from $T$, and $S(T)$ can be computed in polynomial time from $\mathring{S}(T^*)$. □

This result does not imply that *TEQ* can be computed efficiently, despite the fact that both *TRIV* and *MC* converge to *TEQ*. To compute $S^{(n)}(T)$ for a particular tournament $T$, it might be necessary to compute $S^{(n-1)}$ recursively for a linear number of dominator sets. By Proposition 5, the depth of the recursion might also be linear in the order of $T$, which leads to an exponential number of steps. Brandt et al (2010) have in fact shown that it is NP-hard to decide whether a given alternative is in *TEQ*, which is seen as strong evidence that *TEQ* cannot be computed efficiently by any algorithm. Nevertheless, if Conjecture 1 is true, Propositions 4 and 6 identify sequences of efficiently computable tournament solutions that provide better and better approximations of TEQ.

## 6 Uniqueness of Minimal Retentive Sets

We know from Proposition 2 that Conjecture 1 is equivalent to *TEQ* satisfying any of the basic properties. The attractiveness of *TEQ* thus hinges on the resolution of Conjecture 1. In Section 4 we have looked more generally at tournament solutions $\mathring{S}$ defined as the union of all minimal $S$-retentive sets for an arbitrary tournament solution $S$. Uniqueness of minimal retentive sets again turned out to play an important role: if $\mathcal{R}_S$ is pairwise intersecting, then $\mathring{S}$ inherits many desirable properties from $S$. It is therefore an interesting question which tournament solutions are pairwise intersecting, and also a surprisingly difficult one. In this section, we answer the question for the top cycle and the Copeland set.

### 6.1 The Minimal TC-Retentive Set

We now prove the weaker version of Conjecture 1 for the top cycle, thus establishing $\mathring{TC}$ as an efficiently computable refinement of $TC$ that satisfies all basic properties.

**Theorem 5** *$\mathcal{R}_{TC}$ is pairwise intersecting.*

[6] A set $B \subseteq A$ is $R$-undominated if $(a, b) \in R$ for no $b \in B$ and $a \in A \setminus B$.

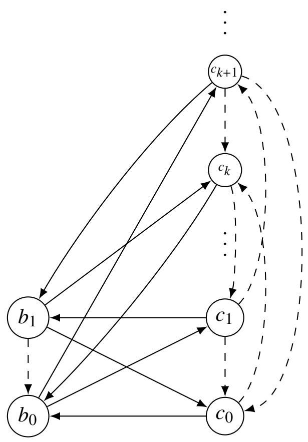


**Fig. 3** Structure of a tournament with two disjoint $TC$-retentive sets ($k$ is even). A dashed edge $(a, b)$ indicates that $a \in TC(\overline{D}(b))$.

*Proof* Consider an arbitrary tournament $(A, \succ)$, and assume for contradiction that $B$ and $C$ are two disjoint $TC$-retentive subsets of $A$. Let $b_0 \in B$ and $c_0 \in C$. Without loss of generality we may assume that $c_0 \succ b_0$. Then, $c_0 \in \overline{D}(b_0)$, and by $TC$-retentiveness of $B$ there has to be some $b_1 \in B$ with $b_1 \in TC(\overline{D}(b_0))$ and $b_1 \succ c_0$. We claim that for each $m \geq 1$ there are $c_1, \ldots, c_m \in C$ such that for all $i$ and $j$ with $0 \leq i < j \leq m$,

($i$) $c_{i+1} \in TC(\overline{D}(c_i))$;
($ii$) $b_0 \succ c_i$ and $c_i \succ b_1$ if $i$ is odd, and $b_1 \succ c_i$ and $c_i \succ b_0$ otherwise; and
($iii$) $c_j \succ c_i$ if $j - i$ is odd, and $c_i \succ c_j$ otherwise.

To see that this claim implies the theorem, consider $i$ and $j$ with $0 \leq i < j \leq m$. Since the dominance relation is irreflexive, and by ($iii$), $c_i$ and $c_j$ must be distinct alternatives. This in turn implies that the size of $C$ is unbounded, contradicting finiteness of $A$. The situation is illustrated in Figure 3.

The claim itself can be proved by induction on $m$. First consider the case $m = 1$. Since $b_1 \succ c_0$, and by $TC$-retentiveness of $C$, there has to be some $c_1 \in C$ with $c_1 \in TC(\overline{D}(c_0))$ and $c_1 \succ b_1$, showing ($i$). Furthermore, by $TC$-retentiveness of $B$, $c_1 \notin TC(\overline{D}(b_0))$ and thus $b_0 \succ c_1$. It is now easily verified that ($ii$) and ($iii$) hold as well.

Now assume that the claim holds for all $k \leq m$. We show that it also holds for $m + 1$.

Consider the case when $m + 1$ is even; the case when $m + 1$ is odd is analogous. By the induction hypothesis, $b_0 \succ c_m$. Hence, by $TC$-retentiveness of $C$, there has to exist some $c_{m+1} \in C$ with $c_{m+1} \in TC(\overline{D}(c_m))$ and $c_{m+1} \succ b_0$, which together with the induction hypothesis implies ($i$).

Moreover, since $b_1 \in TC(\overline{D}(b_0))$ and $c_{m+1} \in \overline{D}(b_0)$, $TC$-retentiveness of $B$ yields $b_1 \succ c_{m+1}$. With the induction hypothesis this proves ($ii$).

For ($iii$), consider an arbitrary $i \in \{1, \ldots, m\}$, and first assume that $i$ is odd. We have to prove that $c_{m+1} \succ c_i$. If $i = m$, this immediately follows from ($i$). If $i < m$, then by the induction hypothesis, $c_i \succ c_m$, $b_0 \succ c_i$, and $b_0 \succ c_m$. Hence, $\{c_{m+1}, c_i, b_0\} \subseteq \overline{D}(c_m)$. Moreover, as we have already shown, $c_{m+1} \succ b_0$. Assuming for contradiction that $c_i \succ c_{m+1}$, the three alternatives $c_{m+1}$, $c_i$, and $b_0$ would constitute a cycle in $\overline{D}(c_m)$. Since $c_{m+1} \in TC(\overline{D}(c_m))$, we would then have that $b_0 \in TC(\overline{D}(c_m))$, contradicting $TC$-retentiveness of $C$. Thus $c_i \not\succ c_{m+1}$. As $c_{m+1} \succ b_0$ and $b_0 \succ c_i$, also $c_{m+1} \neq c_i$. Completeness of $\succ$ implies $c_{m+1} \succ c_i$.

Now assume that $i$ is even. We have to prove that $c_i \succ c_{m+1}$. By the induction hypothesis, $c_m \succ c_i$ and $b_1 \succ c_i$. Assume for contradiction that $c_{m+1} \succ c_i$ and thus $c_{m+1} \in \overline{D}(c_i)$. Since $i+1$ is odd, we already know that $c_{m+1} \succ c_{i+1}$. Furthermore, $c_{i+1} \in TC(\overline{D}(c_i))$, and thus $c_{m+1} \in TC(\overline{D}(c_i))$. However, $b_1 \succ c_{m+1}$ and $b_1 \in \overline{D}(c_i)$ imply that $b_1 \in TC(\overline{D}(c_i))$, contradicting $TC$-retentiveness of $C$. Therefore $c_{m+1} \not\succ c_i$. Since $c_{m+1} \succ c_m$ and $c_m \succ c_i$, we have $c_{m+1} \neq c_i$ and may conclude that $c_i \succ c_{m+1}$. By virtue of the induction hypothesis we are done. □

**Corollary 2** *$\mathring{TC}$ is efficiently computable and satisfies all basic properties. Furthermore, $\mathring{TC} \subseteq TC$.*

*Proof* The result is immediate from Proposition 6, Theorem 3, and Lemma 2. □

## 6.2 Copeland-Retentive Sets May Be Disjoint

For the Copeland set the situation turns out to be quite different: minimal $CO$-retentive sets are not always unique. Our proof makes use of a special class of tournaments, so-called *cyclones*.

**Definition 9** Let $n$ be an odd integer and $A = \{a_0, \dots, a_{n-1}\}$ an ordered set of size $|A| = n$. The *cyclone on $A$* then is the tournament $(A, \succ)$ such that $a_i \succ a_j$ if and only if $j - i \bmod n \in \{1, \dots, \frac{n-1}{2}\}$.

We are now in a position to prove the following result.

**Proposition 7** *$\mathcal{R}_{CO}$ is not pairwise intersecting.*

*Proof* We construct a tournament $T$ with 70 alternatives that can be partitioned into eight subsets $A, B_0, \dots, B_6$. $A = \{a_0, \dots, a_6\}$ contains seven alternatives, whereas for each $k \in \{0, \dots, 6\}$, $B_k = \{b^k_0, \dots, b^k_8\}$ contains nine. First consider the tournament $\tilde{T} = (\{1, \dots, 14\}, \tilde{\succ})$, where $\tilde{T}|_{\{1,\dots,7\}}$ and $\tilde{T}|_{\{8,\dots,14\}}$ are cyclones on $\{1, \dots, 7\}$ and $\{8, \dots, 14\}$, respectively. For all $i$ and $j$ with $1 \leq i \leq 7$ and $8 \leq j \leq 14$, moreover, $j \succ i$ if and only if $j - i \in \{7, 10\}$. Now define

$$T = \Pi(\tilde{T}, I_{a_0}, \dots, I_{a_6}, T_0, \dots, T_6),$$

where for each $i \in \{0, \dots, 6\}$, $T_k$ is the cyclone on $B_k$. Thus $B_j \succ \{a_i\}$ if $j \in \{i, i + 3 \bmod 7\}$ and $\{a_i\} \succ B_j$ otherwise.

We claim that both $A = \{a_0, \dots, a_6\}$ and $B = B_0 \cup \dots \cup B_6$ are $CO$-retentive in $T$. For better readability, we will henceforth write $a_{x+y}$ for $a_{x+y \bmod 7}$, $B_{x+y}$ for $B_{x+y \bmod 7}$, and $b^k_{x+y}$ for $b^k_{x+y \bmod 9}$.

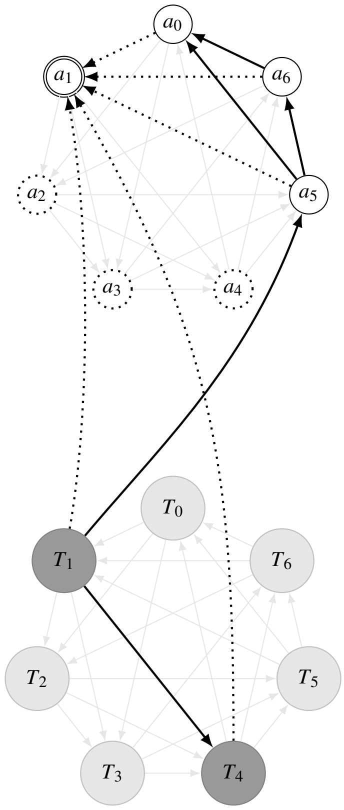


**Fig. 4** Partial representation of the tournament $T$ used in the proof of Proposition 7, illustrating that $A$ is $CO$-retentive. The case shown is the one where $a_i = a_1$. The dotted edges indicate the dominators of $a_1$, all missing edges in $(\overline{D}(a_1), \succ)$ point downward. It is easy to see that $a_6$ is the Copeland winner in $(\overline{D}(a_1), \succ)$.

For $CO$-retentiveness of $A$, fix an arbitrary $i \in \{0, \ldots, 6\}$ and consider $a_i \in A$. The dominators of $a_i$ are given by

$$\overline{D}(a_i) = \{a_{i+4}, a_{i+5}, a_{i+6}\} \cup B_i \cup B_{i+3}.$$

Figure 4 illustrates the case where $a_i = a_1$. It is now readily appreciated that in $(\overline{D}(a_1), \succ)$, $a_{i+5}$ is only dominated by $a_{i+4}$, whereas all other alternatives are dominated by at least two alternatives. Accordingly, $CO(\overline{D}(a_i)) = \{a_{i+5}\} \subseteq A$, which implies that $A$ is $CO$-retentive in $T$.

For $CO$-retentiveness of $B = B_0 \cup \cdots \cup B_6$, fix $k \in \{0, \ldots, 6\}$ and $i \in \{0, \ldots, 8\}$ arbitrarily and consider $b_i^k \in B_k$. The dominators of $b_i^k$ are given by

$$\overline{D}(b_i^k) = \{b_{i+5}^k, b_{i+6}^k, b_{i+7}^k, b_{i+8}^k\} \cup \{a_{k+1}, a_{k+2}, a_{k+3}, a_{k+5}, a_{k+6}\} \cup B_{k+4} \cup B_{k+5} \cup B_{k+6}.$$

Figure 5 illustrates the case where $b_i^k = b_2^1$. We now find that $CO(\overline{D}(b_i^k)) = B_{k+4}$: each alternative $b \in B_{k+4}$ has a Copeland score of $4 + 9 + 9 + 4 + 1 = 27$, whereas each of the alternatives in $\{a_{k+1}, a_{k+2}, a_{k+3}, a_{k+5}, a_{k+6}\}$ has a score of $2 + 4 + 9 + 9 = 24$ and all

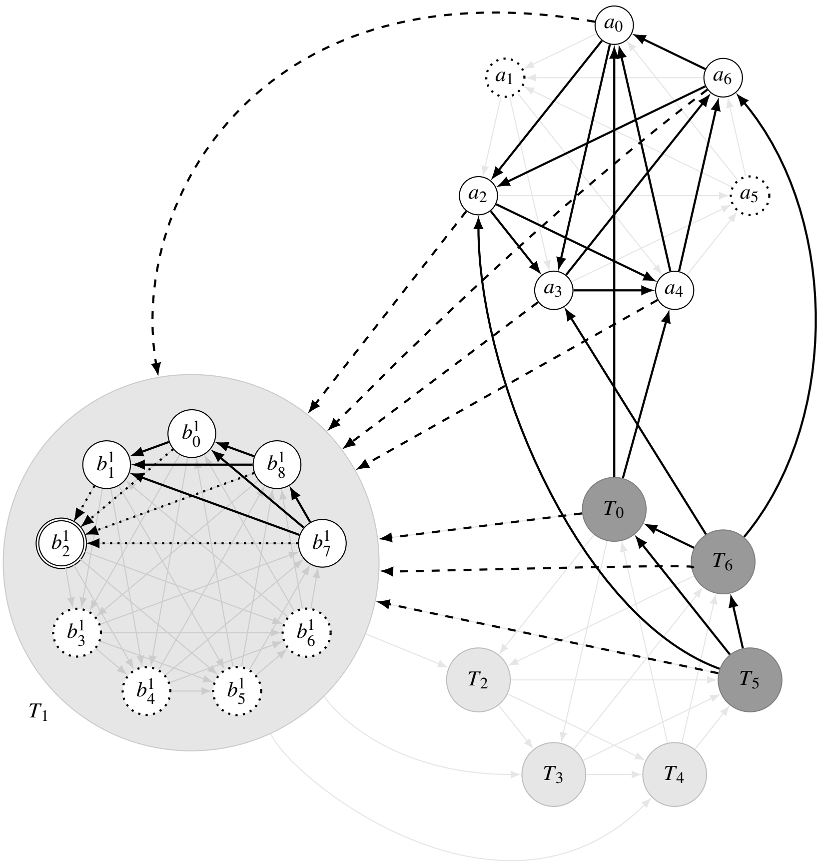


**Fig. 5** Partial representation of the tournament $T$ used in the proof of Proposition 7, illustrating that $B$ is $CO$-retentive. The case shown is the one where $b^k_i = b^1_2$. The dotted and dashed edges indicate the dominators of $b^1_2$. The dashed edges also represent (part of) the dominance relation inside $\overline{D}(b^1_2)$. All missing edges in $(\overline{D}(b^1_2), \succ)$ point downward. It is easy to see that the Copeland winners in $(\overline{D}(b^1_2), \succ)$ are exactly the alternatives in $T_5$.

other alternatives in $\overline{D}(b^k_i)$ have a score of at most 19. It follows that $B$ is $CO$-retentive in $T$. □

*Remark 8* The same construction can also be used to show that $\mathring{CO}$ is not monotonic, which establishes that monotonicity is not inherited in general. To see this, first observe that both $A$ and $B$ are *minimal* retentive sets in $T$, i.e., $\mathring{CO}(T) = A \cup B$. Now fix $k \in \{0, \dots, 6\}$ and $i \in \{0, \dots, 8\}$ arbitrarily and consider $b^k_i \in B_k$. Let $T'$ be the tournament that is identical to $T$ except that $b^k_i$ is strengthened against all alternatives in $B_{k+4}$. For example, let $k = 1$. Then $T' = (A \cup B, \succ')$ with $T'|_{A \cup B \setminus \{b^1_i\}} = T|_{A \cup B \setminus \{b^1_i\}}$ and

$\overline{D}_{>'}(b_i^1) = \overline{D}_{>}(b_i^1) \setminus B_5$. Since $T'|_{\overline{D}_{>'}(a)} = T|_{\overline{D}_{>}(a)}$ for all $a \in A$, the set $A$ is a minimal $CO$-retentive set in $T'$. On the other hand, $CO(\overline{D}_{>'}(b_i^1)) = \{a_2\}$, which means that $B$ is not $CO$-retentive in $T'$. Furthermore, no minimal $CO$-retentive set $X$ can contain $b_i^1$: every such set would also have to contain $CO(\overline{D}_{>'}(b_i^1)) = \{a_2\}$, and $X' = X \cap A$ would be a strictly smaller $CO$-retentive set. Thus $b_i^1 \notin \mathring{CO}(T')$.

## 7 Discussion

Assuming Schwartz's conjecture and starting with the trivial tournament solution, we have defined an infinite sequence of efficiently computable tournament solutions that are strictly contained in each other, strictly contain *TEQ*, and share most of its desirable properties. The implications of these findings are both of theoretical and practical nature.

From a practical point of view, we have outlined an anytime algorithm for computing *TEQ* that returns smaller and smaller supersets of *TEQ*, which furthermore satisfy standard properties suggested in the literature. Previous algorithms for *TEQ* (see, e.g., Brandt et al, 2010) are incapable of providing *any* useful information in general when stopped prematurely.

From a theoretical point of view, the new perspective on *TEQ* as the limit of an infinite sequence of tournament solutions may prove useful for showing Schwartz's conjecture. In particular, it yields an infinite sequence of increasingly difficult conjectures, each of them a weaker version of that of Schwartz. We proved the second conjecture in this sequence. Our inheritance results can be interpreted as alternative proofs for the fact that Schwartz's conjecture implies that *TEQ* satisfies all basic properties. A natural way to prove Schwartz's conjecture would be to prove all statements in the above sequence by induction, i.e., by showing that $\mathcal{R}_{\mathring{S}}$ is pairwise intersecting if $\mathcal{R}_S$ is. Both proving and disproving that $\mathcal{R}_S$ is pairwise intersecting for non-trivial tournament solutions $S$ turns out to be surprisingly difficult.

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## A Proof of Theorem 3

**Theorem 3** *Let $S$ be a proper tournament solution such that $\mathcal{R}_S$ is pairwise intersecting, and let* P *be any of the properties* SSP, WSP, IUA, MON $\wedge$ SSP, *or* $\widehat{\gamma} \wedge$ SSP. *Then,* P *is satisfied by $S$ if and only if it is satisfied by $\mathring{S}$.*

*Proof* Assume that $\mathcal{R}_S$ is pairwise intersecting. We need to show that each of the properties SSP, WSP, IUA, MON $\wedge$ SSP, and $\widehat{\gamma} \wedge$ SSP is satisfied by $S$ if and only if it is satisfied by $\mathring{S}$. The direction from right to left follows from Theorem 2. We now show that the properties are inherited from $S$ to $\mathring{S}$.

Assume that $S$ satisfies SSP. Let $T = (A, \succ)$ be a tournament, and consider an alternative $x \in A \setminus \mathring{S}(T)$. We need to show that $\mathring{S}(T') = \mathring{S}(T)$, where $T' = (A \setminus \{x\}, \succ)$. Since $\mathcal{R}_S$ is assumed to be pairwise intersecting, it suffices to show that for all $a \in \mathring{S}(T)$, $S(\overline{D}_A(a)) = S(\overline{D}_{A\setminus\{x\}}(a))$. To this end, consider an arbitrary $a \in \mathring{S}(T)$. If $x \notin \overline{D}_A(a)$, then obviously $\overline{D}_A(a) = \overline{D}_{A\setminus\{x\}}(a)$ and thus $S(\overline{D}_A(a)) = S(\overline{D}_{A\setminus\{x\}}(a))$. Assume on the other hand that $x \in \overline{D}_A(a)$. Since $a \in \mathring{S}(T)$ and $x \notin \mathring{S}(T)$, it follows that $x \notin S(\overline{D}_A(a))$. Now, since $S$ satisfies SSP, we obtain $S(\overline{D}_A(a)) = S(\overline{D}_{A\setminus\{x\}}(a))$ as desired.

Assume that $S$ satisfies WSP. Let $T = (A, \succ)$ be a tournament, and consider an alternative $x \in A \setminus \mathring{S}(T)$. We need to show that $\mathring{S}(T') \subseteq \mathring{S}(T)$, where $T' = (A \setminus \{x\}, \succ)$. Since $\mathcal{R}_S$ is assumed to be pairwise intersecting, it suffices to show that $\mathring{S}(T)$ is also $S$-retentive in $T'$. To this end, consider an arbitrary $a \in \mathring{S}(T)$. Since $S$ satisfies WSP, we have that $S(\overline{D}_{A\setminus\{x\}}(a)) \subseteq S(\overline{D}_A(a))$. Furthermore, by $S$-retentiveness of $\mathring{S}(T)$, $S(\overline{D}_A(a)) \subseteq \mathring{S}(T)$ and thus $S(\overline{D}_{A\setminus\{x\}}(a)) \subseteq \mathring{S}(T)$.

Assume that $S$ satisfies IUA. Let $T = (A, \succ)$ and $T' = (A, \succ')$ be tournaments with $x, y \in A \setminus \mathring{S}(T)$ and $T|_{A\setminus\{x,y\}} = T'|_{A\setminus\{x,y\}}$. We need to show that $\mathring{S}(T) = \mathring{S}(T')$. Since $\mathcal{R}_S$ is assumed to be pairwise intersecting, it suffices to show that for all $a \in \mathring{S}(T)$, $S(\overline{D}_{\succ}(a), \succ) = S(\overline{D}_{\succ'}(a), \succ')$. To this end, consider an arbitrary $a \in \mathring{S}(T)$. By assumption, $a \neq x$ and $a \neq y$. First consider the case when both $x \in \overline{D}_{\succ}(a)$ and $y \in \overline{D}_{\succ}(a)$. Then, $\overline{D}_{\succ}(a) = \overline{D}_{\succ'}(a)$ and, by $S$-retentiveness of $\mathring{S}(T)$, $x, y \notin S(\overline{D}_{\succ}(a), \succ)$. Since $S$ satisfies IUA, $S(\overline{D}_{\succ}(a), \succ) = S(\overline{D}_{\succ'}(a), \succ')$ as required. Now consider the case when $x \notin \overline{D}_{\succ}(a)$ or $y \notin \overline{D}_{\succ}(a)$. Then, $T|_{\overline{D}_{\succ}(a)} = T'|_{\overline{D}_{\succ'}(a)}$, and the claim follows immediately.

Assume that $S$ satisfies MON and SSP. We have already seen that SSP is inherited, so it remains to be shown that $\mathring{S}$ satisfies MON. Let $T = (A, \succ)$ be a tournament, and consider two alternatives $a, b \in A$ such that $a \in \mathring{S}(T)$ and $b \succ a$. Let $T' = (A, \succ')$ be the tournament with $T|_{A\setminus\{a\}} = T'|_{A\setminus\{a\}}$ and $D_{\succ'}(a) = D_{\succ}(a) \cup \{b\}$. We have to show that $a \in \mathring{S}(T')$. To this end, we claim that for all $c \in A \setminus \{a\}$,

$$a \notin S(\overline{D}_{\succ'}(c), \succ') \quad \text{implies} \quad S(\overline{D}_{\succ}(c), \succ) = S(\overline{D}_{\succ'}(c), \succ'). \tag{1}$$

Consider the case when $c \neq b$ and assume that $a \notin S(\overline{D}_{\succ'}(c), \succ')$. It follows from monotonicity of $S$ that $a \notin S(\overline{D}_{\succ}(c), \succ)$. To see this, observe that monotonicity of $S$ implies that $a \in S(\overline{D}_{\succ'}(c), \succ')$ whenever $a \in S(\overline{D}_{\succ}(c), \succ)$. Now, since $S$ satisfies SSP,

$$S(\overline{D}_{\succ'}(c), \succ') = S(\overline{D}_{\succ'}(c) \setminus \{a\}, \succ') \quad \text{and}$$
$$S(\overline{D}_{\succ}(c), \succ) = S(\overline{D}_{\succ}(c) \setminus \{a\}, \succ).$$

It is easily verified that $(\overline{D}_{\succ'}(c) \setminus \{a\}, \succ') = (\overline{D}_{\succ}(c) \setminus \{a\}, \succ)$, thus we have $S(\overline{D}_{\succ'}(c), \succ') = S(\overline{D}_{\succ}(c), \succ)$.

If $c = b$, then $a \notin S(\overline{D}_{\succ'}(b), \succ')$ together with SSP of $S$ implies $S(\overline{D}_{\succ'}(b), \succ') = S(\overline{D}_{\succ'}(b) \setminus \{a\}, \succ')$. Furthermore, by definition of $T$ and $T'$, $(\overline{D}_{\succ'}(b) \setminus \{a\}, \succ') = (\overline{D}_{\succ}(b), \succ)$ and thus $S(\overline{D}_{\succ'}(b), \succ') = S(\overline{D}_{\succ}(b), \succ)$. This proves (1).

We proceed to show that $a \in \mathring{S}(T')$. Assume for contradiction that this is not the case. We claim that this implies that

$$\mathring{S}(T') \text{ is } S\text{-retentive in } T. \tag{2}$$

To see this, consider $c \in \mathring{S}(T')$. We have to show that $S(\overline{D}_{\succ}(c), \succ) \subseteq \mathring{S}(T')$. Since, by assumption, $a \notin \mathring{S}(T')$, we have that $a \notin S(\overline{D}_{\succ'}(c), \succ')$. We can thus apply (1) and get

$$S(\overline{D}_{\succ}(c), \succ) = S(\overline{D}_{\succ'}(c), \succ') \text{ for all } c \in \mathring{S}(T'),$$

which, together with the $S$-retentiveness of $\mathring{S}(T')$ in $T'$, implies (2).

Having assumed that $\mathcal{R}_S$ is pairwise intersecting, it follows from (2) that $\mathring{S}(T) \subseteq \mathring{S}(T')$. Hence, $a \notin \mathring{S}(T)$, a contradiction. This shows that $\mathring{S}$ satisfies MON.

Finally assume that $S$ satisfies $\widehat{\gamma}$ and SSP. We already know from the above that $\mathring{S}$ satisfies SSP, so it remains to be shown that $\mathring{S}$ satisfies $\widehat{\gamma}$. Let $T = (A, \succ)$ be a tournament, and consider two subsets $B_1, B_2 \subseteq A$ such that $\mathring{S}(B_1) = \mathring{S}(B_2) = X$. We have to show that $\mathring{S}(B_1 \cup B_2) = X$. Since $\mathcal{R}_S$ is assumed to be pairwise intersecting, it suffices to show that for all $x \in X$, $S(\overline{D}_{B_1 \cup B_2}(x)) = S(\overline{D}_{B_1}(x))$. To this end, consider an arbitrary $x \in X$. As $\mathring{S}(B_1)$ and $\mathring{S}(B_2)$ are $S$-retentive in $B_1$ and $B_2$, respectively, we have $S(\overline{D}_{B_i}(x)) \subseteq X \subseteq B_1 \cap B_2$ for $i \in \{1, 2\}$. The fact that $S$ satisfies SSP now implies $S(\overline{D}_{B_1 \cap B_2}(x)) = S(\overline{D}_{B_1}(x))$ and $S(\overline{D}_{B_1 \cap B_2}(x)) = S(\overline{D}_{B_2}(x))$, and thus $S(\overline{D}_{B_1}(x)) = S(\overline{D}_{B_2}(x))$. Since $S$ satisfies $\widehat{\gamma}$, we have $S(\overline{D}_{B_1 \cup B_2}(x)) = S(\overline{D}_{B_1}(x) \cup \overline{D}_{B_2}(x)) = S(\overline{D}_{B_1}(x))$, as desired. □