The Distribution of Optimal Strategies in Symmetric Zero-sum Games

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Given a skew-symmetric matrix, the corresponding two-player symmetric zero-sum game is defined as follows: one player, the row player, chooses a row and the other player, the column player, chooses a column. The payoff of the row player is given by the corresponding matrix entry, the column player receives the negative of the row player. A randomized strategy is optimal if it guarantees an expected payoff of at least 0 for a player independently of the strategy of the other player. We determine the probability that an optimal strategy randomizes over a given set of actions when the game is drawn from a distribution that satisfies certain regularity conditions. Our result applies to a wide range of distributions and implies a result by Fisher and Reeves (1995) on tournament games.

1 Introduction

A (two-player) zero-sum game is played on a matrix where the row player chooses a row and the column player chooses a column. The payoff of the row player is given by the corresponding matrix entry, the column player receives the negative of the row player. Both players may randomize over their actions. Von Neumann's minimax theorem shows that every zero-sum game admits a value, that is, the row player can guarantee an expected payoff for himself that is equal to the negative of the expected payoff that the column player can guarantee for himself. A strategy that maximizes the minimal guaranteed expected payoff of a player is a maximin strategy for this player. Pairs of maximin strategies correspond to Nash equilibria of the game. We will refer to maximin strategies as *optimal strategies*.

A zero-sum game is *symmetric* if the corresponding payoff matrix is skew-symmetric. Thus, both players have the same set of actions and every maximin strategy of the row player is also a maximin strategy of the column player and *vice versa*. Moreover, both

players can achieve a payoff of at least 0 by playing the same strategy as the other player. This also implies that the value of a symmetric zero-sum game is 0. Symmetric zero-sum games can be associated with weighted digraphs where the vertices correspond to actions and the weights of the edges are the payoffs from choosing the corresponding actions. In this paper we show that, for every set of actions S, the probability that a symmetric zero-sum game admits an optimal strategy with support S is $2^{-(n-1)}$ if S has odd cardinality and 0 otherwise, where n is the total number of actions. In particular, this probability only depends on the parity of S. For the proof of this result we assume that the distribution of games is symmetric and regular. A distribution is *symmetric* if it is invariant under negation of all payoffs when the row player chooses an action from a certain set and the column player chooses an action from the complement set. Intuitively, this condition prescribes that, in the graph representation of the game, reversing all edges between S and its complement set does not change the probability of this game being chosen. A distribution is *regular* if a randomly chosen game almost surely admits a unique optimal strategy. We assume throughout that games are drawn from a symmetric, regular distribution.

Related questions have been studied for various classes of games. Wilson (1971) showed that the number of Nash equilibria is finite and odd for almost all n-person normal form games. A different proof of the same statement was given by Harsanyi (1973a). McLennan (2005) derived a formula for the expected number of Nash equilibria in which agents play a certain set of actions with positive probability in normal form games. His model assumes that the payoffs of the agents are independent and distributed uniformly over the unit sphere. If games are distributed such that Nash equilibria are almost surely unique, the expected number of Nash equilibria with given support is equal to the probability that the game admits a Nash equilibrium with this support. Thus, our result can also be phrased as determining the expected number of Nash equilibria with given support.

In zero-sum games every convex combination of Nash equilibria is again a Nash equilibrium. Hence there is either a unique Nash equilibrium or infinitely many. However, Wilson's theorem does not imply that Nash equilibria are almost surely unique in low dimensional subclasses of normal form games, for example, zero-sum games, symmetric zero-sum games, or tournament games.[1] Fisher and Ryan (1992) showed that every tournament game admits a unique optimal strategy and, hence, a unique Nash equilibrium. This result was generalized by Laffond et al. (1997) for symmetric zero-sum games where all payoffs are odd integers, and by Le Breton (2005) for symmetric zero-sum games where all payoffs satisfy a more general congruency condition. However, optimal strategies are not generally unique in zero-sum games. Still, Jonasson (2004) showed that they are almost surely unique if the payoffs are given by continuous i.i.d. random variables that are symmetric about 0. Moreover, he proved that the expected fraction of actions in the support of an optimal strategy is close to $1/2$ when the number of actions goes to infinity.

The proof of our main result (Corollary 1) is divided into three statements. In Theorem 2 we determine the probability that an optimal strategy puts positive probability on all

[1] Tournament games are symmetric zero-sum games in which all off-diagonal payoffs are either 1 or -1.

actions, that is, the probability that a game admits a totally mixed optimal strategy. Lemma 3 establishes that the distribution of the subgame where both players are restricted to the same set of actions is symmetric and regular if the distribution of the full game is symmetric and regular. As a consequence of these two statements we get the probability that such a subgame admits a totally mixed optimal strategy. Lastly, in Theorem 3 we determine the probability that an optimal strategy of a subgame is optimal in the full game. The probability that a game admits an optimal strategy with given support may then easily be derived. In this sense the structure of the proof is very similar to McLennan's (2005) argument.

Finally, we will argue that symmetric, regular distributions occur naturally. For example, if the payoffs of the game follow independent normal distributions the distribution of games is symmetric and regular. More generally, we will show that every absolutely continuous distribution is regular (Theorem 4). As noted before, every tournament game admits a unique optimal strategy. Thus, the uniform distribution over all tournament games is symmetric and regular. As a consequence, Corollary 1 implies a result of Fisher and Reeves (1995), who determine the probability that the optimal strategy of a random tournament game uses k actions with non-zero probability.

2 Preliminaries

A *zero-sum game* G is a matrix in $\mathbb{R}^{M\times N}$, where M and N are the sets of actions for the row and column player, respectively. We will write $|N| = n$ for short. The matrix entry G_{ij} represents the payoff of the *row* player if he choses row i and the column player choses column j. The set of all probability distributions over a finite set S is denoted by $\Delta(S)$, that is, $\Delta(S) = \{p \in \mathbb{R}^S \colon p \geq 0 \text{ and } \sum_{i\in S} p_i = 1\}$. A *(randomized) strategy* for the row player or the column player is a probability distribution on M or N, respectively. The *support* p_+ of a strategy $p \in \Delta(N)$ is the set of actions to which p assigns positive probability, that is, $p_+ = \{i \in N \colon p_i > 0\}$. For vectors $v \in \mathbb{R}^N$, we additionally define $v_- = \{i \in N \colon v_i < 0\}$. A strategy q^* is a *maximin strategy* for the row player if it maximizes his minimum expected payoff, that is,

$$\min_{p\in\Delta(N)} q^{*T} G p \geq \max_{q\in\Delta(M)} \min_{p\in\Delta(N)} q^T G p.$$

Maximin strategies for the column player are defined analogously. By the minimax theorem the minimum expected payoff of the row player when he plays a maximin strategy is equal to the negative of the minimum expected payoff of the column player when he plays a maximin strategy. This payoff is called the *value* of the game. We say that p^* is an *optimal strategy of* G if it is a maximin strategy for the column player. Note that the set of optimal strategies is convex, since it is the set of solutions to a linear program.

A zero-sum game is *symmetric* if G is skew-symmetric, that is, $G = -G^T$. For brevity, we will simply use *game* to refer to a symmetric zero-sum game for the remainder of the paper. The set of all games is denoted by $\mathcal{G}$. For a set of actions $S \subseteq N$, a game $G \in \mathcal{G}$,

and a vector $v \in \mathbb{R}^N$, we denote by $G_S = (G_{ij})_{i,j \in S}$ and $v_S = (v_i)_{i \in S}$ the sub-matrix and sub-vector induced by S, respectively. To simplify the proofs, we introduce special notation for particular classes of games. The set of games where G_S has multiple optimal strategies is denoted by $\mathcal{G}_S^{>1}$, that is,

$$\mathcal{G}_S^{>1} = \{G \in \mathcal{G} \colon G_S \text{ has two distinct optimal strategies}\}$$

Note that $\mathcal{G}_N^{>1}$ contains all games with two distinct optimal strategies. We write $\mathcal{G}^{>1}$ short for $\mathcal{G}_N^{>1}$.

A strategy is *totally mixed* if all actions are played with strictly positive probability. The set of all games where G_S has a totally mixed optimal strategy is denoted by $\mathcal{G}_S$, that is,

$$\mathcal{G}_S = \{G \in \mathcal{G} \colon G_S \text{ has an optimal strategy } p \text{ with } p_+ = S\}$$

Lastly, we define the set of all games that admit an optimal strategy with support S, that is,

$$\mathcal{G}_S^* = \{G \in \mathcal{G} \colon G \text{ has an optimal strategy } p \text{ with } p_+ = S\}.$$

Since every optimal strategy of the full game is also an optimal strategy of the subgame induced by its support, $\mathcal{G}_S^*$ is a subset of $\mathcal{G}_S$.

We assume that games are drawn from a probability distribution $\mathcal{X}$. By X we denote a random variable with distribution $\mathcal{X}$, that is, $X \sim \mathcal{X}$. For a set of games $\mathcal{G}' \subseteq \mathcal{G}$, let $P_X(\mathcal{G}')$ be the probability that a realization of X is in $\mathcal{G}'$. To establish our results, we require that $\mathcal{X}$ satisfies two regularity conditions. For $S \subseteq N$, we define the automorphism Φ_S on $\mathcal{G}$ such that, for all $i, j \in N$,

$$(\Phi_S(G))_{ij} = \begin{cases} G_{ij} & \text{if } i,j \in S \text{ or } i,j \in N \setminus S, \text{ and} \\ -G_{ij} & \text{otherwise.} \end{cases}$$

Then $\mathcal{X}$ is *symmetric* if it is invariant under Φ_S for every $S \subseteq N$, that is, $P_X(\mathcal{G}') = P_X(\Phi_S(\mathcal{G}'))$ for every $\mathcal{G}' \subseteq \mathcal{G}$. Observe that, for all $S, T \subseteq N$, we have $\Phi_S \circ \Phi_T = \Phi_{S \Delta T}$, where Δ is the symmetric difference of S and T. Furthermore, $\Phi_S = \Phi_{N \setminus S}$ for all $S \subseteq N$. As a consequence, $(\{\Phi_S \colon S \subseteq N\}, \circ)$ is a group with neutral element $\Phi_\emptyset$ such that every element is self-inverse. The fact that every element is self-inverse implies that this group is abelian. Moreover, we require $\mathcal{X}$ to be *regular* in the sense that X almost surely admits a unique optimal strategy or, formally, $P_X(\mathcal{G}^{>1}) = 0$.

3 The Result

The main result is obtained in Corollary 1 and states the following: if games are drawn from a symmetric, regular probability distribution then, for every set of actions S, the probability that a symmetric zero-sum game admits an optimal strategy with support S is $2^{-(n-1)}$ if S has odd cardinality and 0 if S has even cardinality. Corollary 1 is an obvious

consequence of Lemma 3 and Theorems 2 and 3. First we prove three lemmas, which are interesting on their own. The proof of Lemma 1 makes use of the equalizer theorem, which states that every action that yields payoff 0 against all optimal strategies is played with positive probability in some optimal strategy.

Theorem 1 (Raghavan, 1994)**.** *Let G be a game and $i \in N$. If $(Gp)_i = 0$ for all optimal strategies p of G, then there is an optimal strategy p' of G with $p'_i > 0$.*

Following Harsanyi (1973b), an optimal strategy is *quasi-strict*[2] if it yields strictly positive payoff against every action that is not in its support. It is a well-known fact that if a game only admits quasi-strict optimal strategies, then it in fact has a unique optimal strategy. Lemma 1 shows that the converse is also true, that is, if a game has an optimal strategy that is not quasi-strict, then it cannot be the unique optimal strategy of the game.

Lemma 1. *Let G be a game and p an optimal strategy of G. If $(Gp)_i = 0$ for some $i \notin p_+$, then G has multiple optimal strategies.*

Proof. Assume for contradiction that p is the unique optimal strategy of G and $(Gp)_i = 0$ for some $i \notin p_+$. Then it follows from Theorem 1 that G has an optimal strategy p' with $p'_i > 0$ and, in particular, $p' \neq p$. This contradicts uniqueness of p. □

The next lemma shows that every strategy that is the unique optimal of some game puts positive probability on an odd number of actions. This does not hold for non-symmetric zero-sum games. For example, the game known as matching pennies has a unique optimal strategy of size 2.

Lemma 2. *Let G be a game and p be the unique optimal strategy of G. Then the support of p has odd cardinality.*

Proof. Assume for contradiction that p_+ has even cardinality. Let $p_+ = S$. Since p is the unique optimal strategy of G, it follows from Lemma 1 that $(Gp)_i < 0$ for all $i \notin S$. By definition of S, $|S \setminus \{i\}|$ is odd for $i \in S$. Hence, $G_{S\setminus\{i\}}$ does not have full rank,[3] that is, there is $v \in \mathbb{R}^n \setminus \{0\}$ with $v_- \cup v_+ \subseteq S \setminus \{i\}$ and $G_{S\setminus\{i\}} v_{S\setminus\{i\}} = 0$. Assume without loss of generality that $(Gv)_i \leq 0$ (otherwise we take $-v$). Then, for $\epsilon > 0$ small enough, we have that $p^\epsilon = (1-\epsilon)p + \epsilon v \geq 0$ and $Gp^\epsilon \leq 0$, that is, $p^\epsilon / |p^\epsilon|$ is an optimal strategy of G. This contradicts uniqueness of p. □

Now we prove an equation that will be useful for the upcoming proofs. For $G \in \mathcal{G}$, $v \in \mathbb{R}^n$, and $S \subseteq N$, we have

$$\Phi_S(G)\Phi_S(v) = \Phi_S(Gv). \tag{$\star$}$$

[2] Harsanyi introduced the concept of quasi-strong equilibria, which however was referred to as quasi-strict equilibria in subsequent papers to avoid confusion with Aumann's notion of strong equilibria (Aumann, 1959).

[3] A skew-symmetric matrix of odd size G cannot have full rank, since $\det(G) = \det(G^T) = \det(-G) = (-1)^n \det(G) = -\det(G)$ and, hence, $\det(G) = 0$.

This equality is quickly checked by verifying

$$\Phi_S(G)\Phi_S(v) = \left(\begin{array}{c:c} \overbrace{G_{ij}}^{S} & \overbrace{-G_{ij}}^{N\setminus S} \\ \hdashline -G_{ij} & G_{ij} \end{array}\right) \cdot \left(\begin{array}{c} -v_j \\ \hdashline v_j \end{array}\right) = \left(\begin{array}{c} -(Gv)_i \\ \hdashline (Gv)_i \end{array}\right) = \Phi_S(Gv).$$

For regular distributions, it follows quickly from Lemma 2 that the probability that a game has an optimal strategy with even support size is 0. If the distribution is also symmetric, it turns out that the probability that a game has an optimal strategy with given support of odd size is independent of the chosen support. This is again specific to symmetric zero-sum games and does not hold in general for zero-sum games.

Theorem 2. *Let $\mathcal{X}$ be symmetric and regular. Then the probability that X has a totally mixed optimal strategy is*

$$\begin{cases} 0 & \text{if } n \text{ is even, and} \\ 2^{-(n-1)} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. First we consider the case when n is even. Assume that a game $G \in \mathcal{G}$ has a totally mixed optimal strategy. It follows from Lemma 2 that G has multiple optimal strategies. Thus, $\mathcal{G}_N \subseteq \mathcal{G}^{>1}$, which implies that $P_X(\mathcal{G}_N) = 0$.

Now assume that n is odd. For all $S \subseteq N$, let $\mathcal{G}^{=}_{\overline{S}}$ be the set of games with a vector v in the null space such that $v_+ = N \setminus S$. Note that $\mathcal{G}^{=}_{\overline{\emptyset}}$ is the set of all games with a totally mixed optimal strategy. Every game is in $\mathcal{G}^{=}_{\overline{S}}$ for some $S \subseteq N$, since a skew-symmetric matrix of odd size cannot have full rank (cf. Footnote 3). For $S \subseteq N$, let $\mathcal{G}^0_S \subseteq \mathcal{G}^{=}_{\overline{S}}$ be the set of games with a vector v in the null space such that $v_+ = N \setminus S$ and $v_i = 0$ for some $i \in N$. It follows from $(\star)$ that $\Phi_S(G)\Phi_S(v) = 0$. Since $\Phi_S(v) \geq 0$ and $\Phi_S(v)_i = 0$, it follows from Lemma 1 that $\Phi_S(G)$ has multiple optimal strategies. Thus, $\Phi_S(\mathcal{G}^0_S) \subseteq \mathcal{G}^{>1}$. By symmetry of $\mathcal{X}$, we then have $P_X(\mathcal{G}^0_S) = P_X(\Phi_S(\mathcal{G}^0_S)) \leq P_X(\mathcal{G}^{>1}) = 0$. Hence, vectors in the null space almost surely have no entries equal to 0. This implies that $P_X(\mathcal{G}^{=}_{\overline{S}}) = P_X(\Phi_{S\Delta T}(\mathcal{G}^{=}_{\overline{S}})) = P_X(\mathcal{G}^{=}_{\overline{T}})$ for all $S, T \subseteq N$. Moreover, $\mathcal{G}^{=}_{\overline{S}}$ and $\mathcal{G}^{=}_{\overline{N\setminus S}}$ only differ by a null set, since $v_+ = N \setminus (-v)_+$ if v has no zero entries. Hence $P_X(\mathcal{G}^{=}_{\overline{S}}) = P_X(\mathcal{G}^{=}_{\overline{S}} \cap \mathcal{G}^{=}_{\overline{N\setminus S}})$ for all $S \subseteq N$. Now we show that X almost surely has rank $n-1$. From before we know that X has rank at most $n-1$. If X has rank less than $n-1$, there are distinct $v, w \in \mathbb{R}^n$ such that $Xv = Xw = 0$. But then $\lambda v + (1-\lambda)w$ is in the null space of X and has an entry equal to 0 for some $\lambda \in \mathbb{R}$. This is a probability zero event as shown above. Hence, X almost surely has rank $n-1$. This implies that $P_X(\mathcal{G}^{=}_{\overline{S}} \cap \mathcal{G}^{=}_{\overline{T}}) = 0$ for all $S, T \subseteq N$ with $S \neq T$ and $S \neq N \setminus T$. Together, we get $P_X(\mathcal{G}^{=}_{\overline{S}}) = 2^{-(n-1)}$ for all $S \subseteq N$. □

It was already observed by Kaplansky (1945) that a game of even size cannot have a unique, totally mixed optimal strategy, which follows from the fact that the rank of a skew-symmetric matrix is even.[4] Moreover, Kaplansky (1995) shows that a game admits

[4] The rank of a skew-symmetric matrix is even, since skew-symmetric matrices of odd size cannot have full rank (cf. Footnote 3).

a unique, totally mixed optimal strategy if and only if the principal Pfaffians[5] of the corresponding matrix alternate in sign. This result allows for a more algebraic but arguably less instructive proof of Theorem 2.

Lemma 3. *Let $S \subseteq N$. If $\mathcal{X}$ is symmetric and regular, then $\mathcal{X}_S$ is a symmetric and regular.*

Proof. Let $S \subseteq N$ and $\mathcal{X}$ be symmetric and regular. First we show that $\mathcal{X}_S$ is symmetric. To this end, let $T \subseteq S$ and $\mathcal{G}'_S \subseteq \mathcal{G}_S$. Then,

$$\begin{aligned} P_{X_S}(\mathcal{G}'_S) &= P_X(\{G \in \mathcal{G} \colon G_S \in \mathcal{G}'_S\}) = P_X(\Phi_T(\{G \in \mathcal{G} \colon G_S \in \mathcal{G}'_S\})) \\ &= P_X(\{\Phi_T(G) \colon G \in \mathcal{G} \text{ and } G_S \in \mathcal{G}'_S\}) = P_X(\{G \in \mathcal{G} \colon \phi_T(G_S) \in \mathcal{G}'_S\}) \\ &= P_X(\{G \in \mathcal{G} \colon G_S \in \phi_T(\mathcal{G}'_S)\}) = P_{X_S}(\phi_T(\mathcal{G}'_S)). \end{aligned}$$

The first and the last equality follow from the definition of X_S. The second equality holds by symmetry of $\mathcal{X}$. The third equality uses the definition of Φ_T as applied to sets of games. The forth equality holds since Φ_T is self-inverse and since Φ_T commutes with restriction to S. Lastly, the fifth equality again holds since Φ_T is self-inverse.

Now we show by induction on $|S|$ that $\mathcal{X}_S$ is regular. If $S = N$ the statement is clear by the hypothesis of the lemma. For the induction step, let $S \subsetneq N$ and assume that $\mathcal{X}_T$ is regular for all $T \subseteq N$ with $|T| > |S|$. Assume for contradiction that $\mathcal{X}_S$ is not regular, that is, $P_{X_S}(\mathcal{G}_S^{>1}) > 0$. Let $i \in N \setminus S$ and $S^i = S \cup \{i\}$. Then, we have that $P_{X_{S^i}}(\{G_{S^i} \in \mathcal{G}_{S^i} \colon G_S \in \mathcal{G}_S^{>1}\}) = P_{X_S}(\mathcal{G}_S^{>1}) > 0$. We define

$$\mathcal{G}_{S^i}^- = \{G \in \mathcal{G}_{S^i} \colon p \in \Delta(S^i) \text{ with } p_+ = S, G_S p_S \leq 0 \text{ and, } (Gp)_i \leq 0 \text{ for some } p \in \Delta(S^i)\},$$

with $\mathcal{G}_{S^i}^+$ defined by replacing the last $\leq$ by $\geq$. Since $\mathcal{X}_T$ is symmetric for every $T \subseteq N$, it follows that $P_{X_{S^i}}(\mathcal{G}_{S^i}^-) = P_{X_{S^i}}(\mathcal{G}^+)$. Moreover, $\mathcal{G}_{S^i}^- \cup \mathcal{G}_{S^i}^+ = \{G_{S^i} \in \mathcal{G}_{S^i} \colon G_S \in \mathcal{G}_S^{>1}\}$ and, hence, $P_{X_{S^i}}(\mathcal{G}_{S^i}^-) > 0$. Now let $G \in \mathcal{G}_{S^i}^-$. If there is $p \in \Delta(S^i)$ such that $G_S p_S \leq 0$ and $(Gp)_i = 0$, then if follows from Lemma 1 that G has multiple optimal strategies. If $(Gp)_i < 0$, there is $q \in \Delta(S_i)$ such that $q_S \neq p_S$ and $G_S q_S \leq 0$. Such a q exists since $G_S \in \mathcal{G}_S^{>1}$ by definition. But then $(1-\lambda)p_{S^i} + \lambda q_{S^i}$ is another optimal strategy of G for small $\lambda > 0$. In any case, G has two distinct optimal strategies. Thus, we have

$$P_{X_{S^i}}(\mathcal{G}_{S^i}^{>1}) \geq P_{X_{S^i}}(\mathcal{G}_{S^i}^-) > 0,$$

which contradicts the induction hypothesis that $\mathcal{X}_{S^i}$ is regular. □

By combining the last two statements we get the probability that X_S admits a totally mixed optimal strategy. In the next theorem we determine the probability that X has an optimal strategy with support S given that X_S has a totally mixed optimal strategy.

Theorem 3. *Let $\mathcal{X}$ be symmetric and regular and $S \subseteq N$. Then $P_X(\mathcal{G}_S^* \mid \mathcal{G}_S) = 2^{-(n-|S|)}$.*

[5]The ith principal Pfaffian is the Pfaffian of the matrix obtained by deleting the ith row and ith.

Proof. Let $\mathcal{X}$ be symmetric and regular and $S \subseteq N$. Recall that $\mathcal{G}_S$ is the set of all games where G_S has a totally mixed optimal strategy. Moreover, we define $\mathcal{G}_S(T)$ to be the set of all games where G_S has a totally mixed optimal strategy such that the set of actions yielding positive payoff corresponds exactly to the rows in T, that is,

$$\mathcal{G}_S(T) = \{G \in \mathcal{G} \colon p_S \text{ is a totally mixed optimal strategy of } G_S \text{ and } (Gp)_+ = T \\ \text{for some } p \in \Delta(N)\}.$$

Note that $\mathcal{G}_S(\emptyset) = \mathcal{G}_S^*$ and $\mathcal{G}_S(T)$ is non-empty only if $T \subseteq N \setminus S$. It follows from $(\star)$ that $\Phi_T(\mathcal{G}_S(T)) \subseteq \mathcal{G}_S^*$ for all $T \subseteq N \setminus S$. For $G \in \mathcal{G}_S^* \setminus \Phi_T(\mathcal{G}_S(T))$ we have that $(Gp)_i = 0$ for some $i \in T$. Then it follows from Lemma 1 that G has multiple optimal strategies. Thus, by symmetry of $\mathcal{X}$, we have $P_X(\mathcal{G}_S(T)) = P_X(\Phi_T(\mathcal{G}_S(T))) = P_X(\mathcal{G}_S^*)$ for all $T \subseteq N \setminus S$. Since, by Lemma 3, X_S almost surely has a unique optimal strategy, we also have that $P_X(\mathcal{G}_S(T) \cap \mathcal{G}_S(T')) = 0$ for all $T, T' \subseteq N \setminus S$. Since $N \setminus S$ has $2^{n-|S|}$ distinct subsets, we get $P_X(\mathcal{G}_S^* \mid \mathcal{G}_S) = 2^{-(n-|S|)}$. □

The main result easily follows from Lemma 3 and Theorems 2 and 3.

Corollary 1. *Let $\mathcal{X}$ be symmetric and regular. Then, for every $S \subseteq N$, the probability that X has an optimal strategy with support S is*

$$\begin{cases} 0 & \text{if } |S| \text{ is even, and} \\ 2^{-(n-1)} & \text{if } |S| \text{ is odd.} \end{cases}$$

Observe that N has 2^{n-1} subsets of odd size. Hence, the probabilities above sum to 1.

As we have noted before, Corollary 1 does not hold for non-symmetric zero-sum games. Experiments by Faris and Maier (1987) suggest that the size of the optimal strategy of a zero-sum game chosen uniformly at random approximately follows a binomial distribution with parameters n and $1/2$. Finding the exact distribution of optimal strategies for this case is still an open problem.

Lastly, we show that a distribution is regular if it is absolutely continuous (w.r.t. the Lebesgue measure). In particular, a distribution is absolutely continuous if all payoffs are independent, absolutely continuous random variables, for example, normal random variables or uniform random variables on an interval.

Theorem 4. *If $\mathcal{X}$ is absolutely continuous, then it is regular.*

Proof. Let $\mathcal{X}$ be absolutely continuous. Moreover, let $G \in \mathcal{G}^{>1}$. We will show that G has a singular square sub-matrix of even size. As discussed before, every game that has multiple optimal strategies admits an optimal strategy that is not quasi-strict. Let p be an optimal strategy of G that is not quasi-strict, that is, there is $i \notin p_+$ such that $(Gp)_i = 0$. Let $p_+ = S$. Then, $G_S p_S = 0$ and $G_{S \cup \{i\}} p_{S \cup \{i\}} = 0$. So either G_S or $G_{S \cup \{i\}}$ is a singular square sub-matrix of even size.

Since $\mathcal{X}$ is absolutely continuous, every even-sized square sub-matrix of X is almost surely regular. Thus, X almost surely admits a unique optimal strategy, that is, $\mathcal{X}$ is regular. □

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