d-dimensional Stable Matching with Cyclic Preferences

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Abstract

Gale and Shapley [D. Gale and L. S. Shapley. College admissions and the stability of marriage. *The American Mathematical Monthly*, 69(1):9–15, 1962] have shown that in marriage markets, where men and women have preferences over potential partners of the other gender, a stable matching always exists. In this paper, we study a more general framework with d different genders due to Knuth [D. E. Knuth. *Mariages stables*. Les Presses de l'Université de Montréal, 1976]. The genders are ordered in a directed cycle and agents only have preferences over agents of the subsequent gender. Agents are then matched into families, which contain exactly one agent of each gender. We show that there always exists a stable matching if there are at most $d+1$ agents per gender, thereby generalizing and extending previous results. The proof is constructive and computationally efficient.

JEL classification: C78
Keywords: Stable matching; Stable marriage; Cyclic preferences

1. Introduction

In 1962, Gale and Shapley formally introduced the *stable marriage problem*: Out of two sets M and W consisting of men and women, respectively, we want to form man-woman-pairs. Given that every man has preferences over the women and *vice versa*, the question is whether there exists a *matching* of all men and women that is *stable* in the sense that there is no man m and woman w that prefer being together over being with their current partner. Gale and Shapley identified this problem as special case of the *college admission problem* and solved both of these constructively, i.e., they not only showed that such a matching exists, but also how to find it in linear time.

We consider a generalization of the problem of stable marriage to more than two genders: the *d-dimensional stable marriage problem* (d-DSM). Knuth (1976), who first posed this as an interesting open problem for future research, gave the example of finding families consisting of men, women, and dogs.

The idea was picked up and first examined by Alkan (1988) who showed that stable matchings need not exist when we consider $d \geq 3$ genders and three agents per gender. Here, agents are allowed to express preferences over all $(d-1)$-tuples they could possibly join.

Ng and Hirschberg (1991) later proved that given three genders and preferences as above, even determining whether a stable matching exists is NP-complete. They moreover mention the special case where preferences are of the type that men only care about women, women about dogs and dogs about men as an open problem and attribute its origin to Knuth. We refer to this kind of preferences as *cyclic*. In accordance with Manlove (2013), we will use the acronym d-DSM-CYC for d-DSM with cyclic preferences.

The existence of stable matchings in d-DSM-CYC was first studied by Boros et al. (2004) who argued that a stable matching always exists as long as $n \leq d$ where n represents the number of agents per gender. Eriksson et al. (2006) partially extended this theorem by proving the existence of a stable matching for all instances of 3-DSM-CYC with $n = 4$.

Woeginger (2013) recently discussed some interesting open problems regarding stability in coalition formation. Among others, he mentioned the question whether every instance of 3-DSM-CYC (for general n) admits a stable matching and classified this problem as "hard and outstanding".

Even though not being able to solve this very problem, we want to continue the direction of research for the general case of d-DSM-CYC. We show that a stable matching does indeed exist for all instances of d-DSM-CYC with $n \leq d+1$, hereby generalizing and strengthening previous results.

Apart from the work mentioned above, 3-DSM-CYC was also examined with respect to computational complexity; there are some results concerning stronger notions of stability or possible indifferences and inacceptabilities (see, e.g., Biró and McDermid, 2010; Huang, 2010; Cui and Jia, 2013). A more extensive overview is given by Manlove (2013).

2. Preliminaries

We have d different sets of agents that are denoted $G^1, \ldots, G^d$ and contain n agents each. We will use male pronouns for all agents. A *family* $F = (f_1, \ldots, f_d)$ consists of one agent out of every set, i.e., $F \in G^1 \times \cdots \times G^d = G$. n disjoint families together form a *matching* $\mathcal{M}$, $\mathcal{M} \subseteq G$. We say that an agent *is matched to* another agent (in $\mathcal{M}$) if they are part of the same family $F \in \mathcal{M}$. Moreover, denote by $\mathcal{M}(x)$ the family x is part of, $\mathcal{M}(x) = \{F \in \mathcal{M} \mid x \in F\}$.

All agents x are endowed with preference relations $\geq_x$ according to a cyclic manner: agents in G^1 have complete, antisymmetric, and transitive preferences over the agents in G^2. Similarly, agents in G^j have preferences over the agents in $G^{(j \mod d)+1}$ for $j \in [d]$ where $[d] = \{1, 2, \ldots, d\}$. Otherwise, agents are completely indifferent. Consequently, we can derive a complete preference order

$\succsim_x \in G \times G$ over all families for each agent x. Let $x \in G^j$, $j \in [d]$, then $F \succsim_x F'$ if and only if $f_{(j \mod d)+1} \geq_x f'_{(j \mod d)+1}$. Note, however, that this complete order contains a lot of indifferences as for an agent only one member of his family is relevant, irrespective of the family's size. Denote by $\succ_x$ the strict part of $\succsim_x$, i.e., $F \succ_x F'$ iff $F \succsim_x F'$ but not $F' \succsim_x F$.

We say that an agent x is *m-content*, $m \in [n]$, if he is matched to his mth preference according to $\geq_x$. A family is called an m-family if $(d-1)$ of its participants are 1-content whereas the remaining agent is m-content. It is easy to see that every instance of d-DSM-CYC has to admit an m-family. Given an instance of d-DSM-CYC, we say that we are in the *m-case* if there is an m-family but no m'-family for all $m' \in [m-1]$.

For a matching $\mathcal{M}$, a family $F_b \notin \mathcal{M}$ is *blocking* if $F_b \succ_x \mathcal{M}(x)$ for each agent x in F_b. Whenever there does not exist a blocking family, a matching is called *stable*.[1] Agents that are 1-content can obviously never be part of a blocking family.

For better illustration, consider the following brief example: Let $G^1 = \{x_1, x_2, x_3\}$, $G^2 = \{y_1, y_2, y_3\}$, $G^3 = \{z_1, z_2, z_3\}$ and the corresponding preferences

$$\begin{array}{lll} y_1 \geq_{x_1} y_2 \geq_{x_1} y_3, & z_1 \geq_{y_1} z_3 \geq_{y_1} z_2, & x_3 \geq_{z_1} x_2 \geq_{z_1} x_1, \\ y_1 \geq_{x_2} y_3 \geq_{x_2} y_2, & z_1 \geq_{y_2} z_2 \geq_{y_2} z_3, & x_1 \geq_{z_2} x_2 \geq_{z_2} x_3, \\ y_2 \geq_{x_3} y_3 \geq_{x_3} y_1, & z_3 \geq_{y_3} z_1 \geq_{y_3} z_2, & x_3 \geq_{z_3} x_1 \geq_{z_3} x_2. \end{array}$$

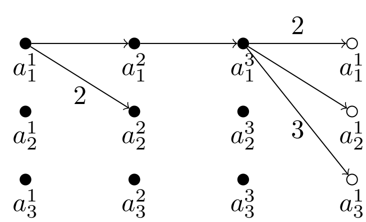
Here, $F = (x_1, y_1, z_1)$ forms a 3-family as x_1 and y_1 are 1-content while z_1 is 3-content. $\mathcal{M} = \{(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)\}$ is a possible matching. Note that $F_b = (x_3, y_2, z_1)$ is a blocking family as all agents it contains are strictly better off in F_b compared to their families in $\mathcal{M}$. Thus, $\mathcal{M}$ is not stable.

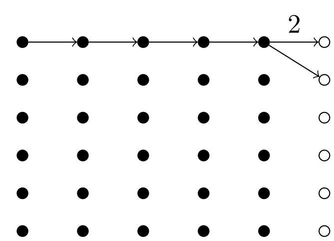
In the following, we make use of two kinds of permutations: For better representation, we want that in an m-family, the first $(d-1)$ agents are 1-content while the dth agent is m-content. We therefore map all G^j to $A^{\pi(j)}$ in a way such that the cyclic nature of the preferences is conserved. In all proofs to come we will always assume that this permutation has already taken place, i.e., when picking an m-family at the beginning of a proof, agents in $A^1, \ldots, A^{d-1}$ are assumed to be 1-content. Note that we will completely ignore the actual genders G^j for the rest of this work and limit ourselves to the neutral and more expedient sets A^j.

In addition, we will permute the alternatives within all A^j, $j \in [d]$. The reason for this is that we want families $F \in \mathcal{M}$ to consist of all ith agents of A^j, $i \in [n], j \in [d]$. For convenience, the naming, i.e., ordering, of agents in A^j, $j \in [d]$, will usually not be done at the beginning of a proof but rather on the way as long as we do not lose generality.

[1] Note that in the literature, this stability is sometimes also referred to as *weak stability*. It corresponds to what is usually termed *core stability* in hedonic games.

For depicting (parts of) instances of d-DSM-CYC we use dot diagrams similar to those of Eriksson et al. (2006). More precisely, we consider directed weighted graphs (V, E, w) with vertex set $V = \bigcup_{j\in[d]} A^j$, there exists an arc from v_1 to v_2 whenever v_2 is ranked in $\geq_{v_1}$ and for existing arcs (v_1, v_2), $w(v_1, v_2)$ is the rank of v_2 in v_1's preferences.[2] Agents in the same A^j are placed below one another and the different A^j are ordered numerically with an additional copy of A^1 on the very right in order to avoid arcs pointing from the rightmost column of vertices to the leftmost. To prevent confusion, these copies of agents in A^1 are depicted as (unfilled) circles only. Arcs with weight one are represented by regular arrows, for all other cases the corresponding weight is noted next to the arrow. Consider the diagram given below on the left: a_1^1's favorite is a_1^2, his second preference is a_2^2. (a_1^1, a_1^2, a_1^3) form a 2-family.
For all future diagrams, for the sake of clarity, we will forgo displaying the agents' names just as in the example on the right.





3. Existence of Stable Matchings

In this section we show that all instances of d-DSM-CYC with $n \leq d + 1$ admit a stable matching. Boros et al. (2004) show the same statement for $n \leq d$. Eriksson et al. (2006) strengthen the previous result for the special case of $n = d = 3$ by further characterizing the stable matchings. Our first lemma shows that this slightly stronger version does also hold for the general case where $n = d$.

Lemma 1. *For every instance of d-DSM-CYC where $n = d$ and for any $x \in \bigcup_{j\in[d]} G^j$, there is a stable matching such that either*

(i) x is 1-content, or

(ii) x is 2-content and x's favorite is 1-content.

Proof. Without loss of generality let $x \in A^1$. We distinguish different m-cases, namely $m = 1$ and $m \geq 2$. Recall that in the m-case, there is an m-family and no m'-family, $m' < m$.

1. 1-case.

[2] Formally, the rank of v_2 in v_1's preferences is defined as $1 + |\{v \in V \setminus \{v_2\} \mid v \geq_{v_1} v_2\}|$.

(i) x is part of a 1-family F_1. Let the agents in F_1 be $a_1^1, \dots, a_1^d$. Fix a random order over the remaining agents in $A^1 \setminus \{a_1^1\}$ and name them accordingly. Now, perform Algorithm 1 as explained below with $i_0 = 2$.

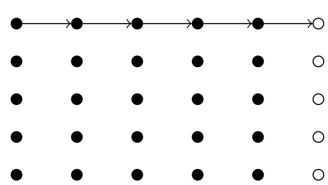
Algorithm 1 Find families for a matching

for $i = i_0, \dots, n$ **do**
  for $j = 1, \dots, d-1$ **do**
    Let a_i^j choose his favorite among $A^{j+1} \setminus \bigcup_{k \in [i-1]} \{a_k^{j+1}\}$ and name this agent a_i^{j+1}.
  end for
  Let $F_i = (a_i^1, \dots, a_i^d)$.
end for

(ii) x is not part of a 1-family. Choose one 1-family at random and fix it as $F_1 = (a_1^1, \dots, a_1^d)$. Let $x = a_2^1$ and name the remaining agents in $A^1 \setminus \{a_1^1, a_2^1\}$ according to a random order. Once more, use Algorithm 1 with $i_0 = 2$.

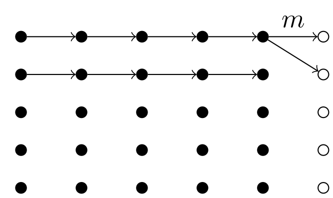
For both cases let $\mathcal{M} = \bigcup_{i \in [n]} \{F_i\}$. Claim: $\mathcal{M}$ is stable.
First note that members of F_1 are 1-content, thus cannot be part of a blocking family. Moreover, for all $j \in [d-1]$ we have $a_i^{(j \mod d)+1} \geq_{a_i^j} a_k^{(j \mod d)+1}$ for all $i < k \leq n$. Intuitively this means that all agents are either 1-content or would prefer to be matched to an agent *above* the current one (referring to the dot diagram), but never to an agent *below*.

Suppose a_k^1 was part of a blocking family F_b. We obtain for the corresponding $a_{k'}^d \in F_b$ that $1 \leq k' \leq k - (d-1)$. As $1 \leq k \leq n = d$, the only possibilities left are $k = n$ and $k' = 1$. Since a_1^d cannot be part of a blocking family, no such F_b can exist and $\mathcal{M}$ is stable. It trivially holds that x is either 1-content or 2-content and x's favorite is 1-content.



2. m-case, $m \geq 2$. Let $x = a_1^1$ and for $j \in [d-1]$ let a_1^j choose his favorite in A^{j+1} and name this agent a_1^{j+1}. Note that the favorite agent of a_1^d is distinct from a_1^1, hence, we can name him a_2^1. We order the remaining agents in $A^1 \setminus \{a_1^1, a_2^1\}$ at random and name them accordingly. Now, execute Algorithm 1 with $i_0 = 2$.
Claim: $\mathcal{M} = \bigcup_{i \in [n]} \{F_i\}$ is stable. First note that none of the a_2^j, $j \in [d-1]$, can have a_1^{j+1} as favorite, thus they are 1-content and cannot be part of a blocking family F_b. The same obviously also holds for a_1^j, $j \in [d-1]$.

Suppose a^1_k was part of a blocking family F_b. We obtain for the corresponding $a^{d-1}_{k'} \in F_b$ that $1 \leq k' \leq k-(d-2)$. As $1 \leq k \leq n = d$, the only possibilities left are $k' = 1$ or $k' = 2$. Since a^{d-1}_1 as well as a^{d-1}_2 cannot be part of a blocking family, no such F_b can exist and $\mathcal{M}$ is stable. It trivially holds that x is either 1-content or 2-content and his favorite is 1-content.



□

We now move to instances of d-DSM-CYC where $n \leq d+1$; our main theorem, Theorem 1, states that every such instance admits a stable matching. For convenience reasons, we split Theorem 1 into four lemmas that can be proven separately. More precisely, we are going to analyze different m-cases: $m = 1$, $m = 2$, $3 \leq m \leq n/2 + 1$, and $n/2 + 1 < m \leq n$.

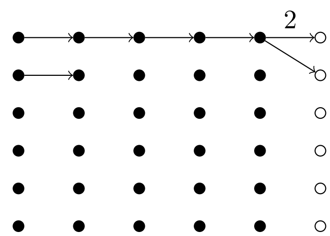
Lemma 2. *For every instance of d-DSM-CYC with $n = d + 1$ where there is a 1-family there exists a stable matching.*

Proof. Let F_1 be a 1-family and note that all members of F_1 are 1-content and thus cannot be part of a blocking family. Ignore them for the moment. Now, the remaining agents form an instance of d-DSM-CYC with $n' = d$. There exists a stable matching $\mathcal{M}'$ for this instance by Lemma 1. Together with F_1 we obtain $\mathcal{M} = \mathcal{M}' \cup \{F_1\}$ which is obviously stable. □

Lemma 3. *For every instance of d-DSM-CYC with $n = d + 1$, $d \geq 3$ where we are in the 2-case there exists a stable matching.*

Proof. Select a 2-family and name them $F_1 = (a^1_1, \ldots, a^d_1)$. Ignore them for the moment and consider the remaining instance of d-DSM-CYC with $n' = d$. By Lemma 1 there is a stable matching $\mathcal{M}'$ for this instance where a^d_1's favorite (without loss of generality a^1_2) is either 1-content or a^1_2 is 2-content and his favorite is 1-content.

1. a^1_2 is 1-content in the n'-instance. As a^1_2's favorite in the n-instance cannot be a^2_1, a^1_2 has to be 1-content in the n-instance as well. We deduce that if a^d_1 would be part of a blocking family, so would have to be a^1_2 as this is the only option for a^d_1 to improve. Since this is not possible, no a^j_1, $j \in [d]$, can be part of a blocking family. Together with the fact that $\mathcal{M}'$ is stable we get that $\mathcal{M} = \mathcal{M}' \cup \{F_1\}$ also is stable.



2. a_2^1 is 2-content and his favorite is 1-content in the n'-instance. Follow the argumentation above and note that if a_1^d would be part of a blocking family so would have to be a_2^1. In the n-instance, a_2^1 has to be at least 3-content where a_1^2 cannot be his first preference. Let a_2^1's favorite be a_3^2. Note that a_3^2's favorite cannot be a_1^3 as $(a_2^1, a_3^2, a_1^3, \dots, a_1^d)$ would form a 1-family. Hence, a_3^2 has to be 1-content in the n-instance as well and therefore cannot be part of a blocking family. Consequently, $\mathcal{M} = \mathcal{M}' \cup \{F_1\}$ is a stable matching.

□

Lemma 4. *Every instance of d-DSM-CYC with $n = d + 1$ where we are in the m-case, $d \geq 4$, $3 \leq m \leq n/2 + 1$, admits a stable matching.*

Proof. Select an m-family and name them $F_1 = (a_1^1, \dots, a_1^d)$. Ignore them for the moment and consider the remaining instance of d-DSM-CYC with $n' = d$. We know that we can find a stable matching for the n'-instance by Lemma 1. As in Lemma 1, distinguish the two cases whether or not there is a 1-family in the n'-instance. Note that if such a 1-family exists, it cannot contain any agent a_1^d prefers to a_1^1.

1. There exists a 1-family in the n'-instance. Fix this 1-family as $F_2 = (a_2^1, \dots, a_2^d)$. As we are in the m-case, it holds that at least two of the a_2^j, $j \in [d]$, prefer $a_1^{(j \mod d)+1}$ to $a_2^{(j \mod d)+1}$. We represent this in the dot diagram by dotted arrows, indicating either 1- or 2-content.

   Take the remaining $(m-1)$ agents a_1^d prefers to a_1^1 and name them $a_3^1, \dots, a_{m+1}^1$, agents in $A^1 \setminus \bigcup_{i \in [m+1]} \{a_i^1\}$ are ordered and named at random. Once more, use Algorithm 1 with $i_0 = 3$ to obtain families $F_3, \dots, F_n$ and the matching $\mathcal{M}' = \bigcup_{2 \leq i \leq n} \{F_i\}$. Note that if for some $2 \leq j \leq d-1$ we have that $a_1^{j+1} >_{a_2^j} a_2^{j+1}$, then all $a_3^{j'}$ with $j' \in [j-1]$ have to be 1-content.

2. There exists no 1-family in the n'-instance. Fix a_1^d's favorite to be a_2^1. For $j \in [d-1]$ let a_2^j choose his favorite from $A^{j+1} \setminus \{a_1^{j+1}\}$ and name this agent a_2^{j+1}. $F_2 = (a_2^1, \dots, a_2^d)$ forms an m'-family, $m' \geq m$. Let a_2^d's favorite in the n'-instance be a_3^1 and take the remaining $(m-2)$ agents a_1^d prefers to

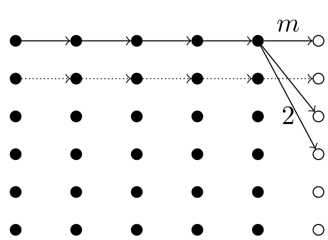
a_1^1 and name them $a_4^1, \dots, a_{m+1}^1$.[3] All agents in $A^1 \setminus \bigcup_{i \in [m+1]} \{a_i^1\}$ can be ordered and named at random. Just as in case 1 we use Algorithm 1 with $i_0 = 3$ to obtain families $F_3, \dots, F_n$ and the matching $\mathcal{M}' = \bigcup_{2 \le i \le n} \{F_i\}$.

Claim: $\mathcal{M} = \mathcal{M}' \cup \{F_1\}$ is stable. Assume for contradiction that there was a blocking family F_b. Suppose a_1^d was part of F_b, then so would have to be one of a_i^1, $2 \le i \le m+1$ depending on which case we consider. Substituting m, we get $2 \le i \le {}^n\!/\!_2 + 2 = {}^d\!/\!_2 + {}^5\!/\!_2$. We find that a_1^d can only be contained in F_b if

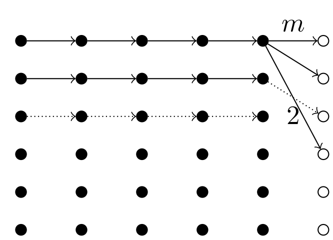
$$d - 1 \le i - 2 \le {}^d\!/\!_2 + {}^1\!/\!_2 \quad \Leftrightarrow \quad d \le 3.$$

This is a contradiction to $d \ge 4$. Intuitively speaking, $(d-1)$ corresponds to the number of necessary *steps to the right* in our dot diagram, i.e., the number of agents apart from a_1^d that are to be part of F_b. $(i-2)$ on the other hand represents the possible *steps upwards* when starting in row i, i.e., the maximal number of improvements taking those agents together. We have $(i-2)$ and not $(i-1)$ as in the first case, a 2-content a_2^j implies 1-content $a_3^{j'}$, $j' \in [j-1]$, and in the second case all a_2^j, $j \in [d-1]$ are 1-content.

As $\mathcal{M}'$ is stable by arguments similar to those used when proving Lemma 1, we deduce that $\mathcal{M}$ is stable as well.



1-family in the n'-instance



No 1-family in the n'-instance

□

Lemma 5. *For every instance of d-DSM-CYC with $n = d + 1$ where we are in the m-case, ${}^n\!/\!_2 + 1 < m \le n$, there exists a stable matching.*

Proof. Fix an m-family and name them $F_1 = (a_1^1, \dots, a_1^d)$. Let a_1^d's favorite be a_2^1. For $j \in [d-1]$ let a_2^j choose his favorite in $A^{j+1} \setminus \{a_1^{j+1}\}$ and name this agent a_2^{j+1}. Now, $F_2 = (a_2^1, \dots, a_2^d)$ forms an m'-family, $m' \ge m$.

Among the remaining $(n-2)$ agents in $A^1 \setminus \{a_1^1, a_2^1\}$, there are $(m-2)$ agents that a_1^d prefers to a_1^1. Also, there are at least $(m-2)$ agents that a_2^d prefers to a_2^1 (depending on m' and whether or not a_2^d prefers a_1^1 to a_2^1). If there are

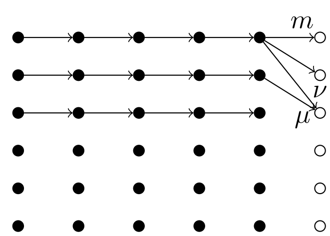
[3]Some details of the proof change if a_1^d also prefers a_3^1 to a_1^1. The line of argumentation, however, stays the same and we omit this special case in the interest of space.

more than $(m-2)$, consider a_2^d's $(m-2)$ most preferred ones out of those. As $m > n/2 + 1$ we have

$$(m-2) + (m-2) \geq (n/2 + 3/2 - 2) + (n/2 + 3/2 - 2) = n - 1 > n - 2.$$

Thus, by the pigeonhole principle, there has to exist at least one agent whom both a_1^d and a_2^d prefer to their current match (referring to the dot diagram below, $\nu, \mu < m$). Name this agent a_3^1. For $j \in [d-1]$ let a_3^j choose his favorite in $A^{j+1} \setminus \{a_1^{j+1}, a_2^{j+1}\}$ and name this agent a_3^{j+1}. We let $(a_3^1, \ldots, a_3^d) = F_3$. Order the remaining agents in $A^1 \setminus \{a_1^1, a_2^1, a_3^1\}$ at random, name them accordingly and find families F_i, $4 \leq i \leq n$, by executing Algorithm 1 with $i_0 = 4$.

Claim: the matching $\mathcal{M} = \bigcup_{i \in [n]} \{F_i\}$ is stable. First note that for $j \in [d-1], i \in \{2, 3\}$, all a_i^j are 1-content as we are in the m-case, so they cannot be part of a blocking family. Assume for contradiction that some a_k^1 would be part of a blocking family F_b. Then we get for the corresponding $a_{k'}^{d-1}$ in F_b that $k - (d-2) \geq k'$. As $k \in [n]$ and $n = d + 1$ we deduce $k' \leq 3$. Recall that no $a_{k'}^{d-1}$ can be part of a blocking family for $k' \leq 3$, hence, no such F_b can exist. Consequently, $\mathcal{M}$ is stable.



□

It is worth noting that there actually exists a stable matching for every instance of d-DSM-CYC if we are in the n-case, irrespective of the relation between n and d. Generalizing arguments used by Eriksson et al. (2006), we deduce that in this case no two agents in G^j can have the same favorite, i.e., matching all agents in G^j to their respective favorite and assigning the remaining agents at random yields a stable matching.

With those lemmas at hand we are now able to formally give our main theorem:

Theorem 1. *Every instance of d-DSM-CYC with $n \leq d + 1$ admits a stable matching.*

Proof. For $n \leq d$, the statement is shown by Boros et al. (2004) or Lemma 1, equivalently.
Concerning $n = d + 1$, it is trivial for $d = 1$. For $d = 2$ and $d = 3$, it is shown by Gale and Shapley (1962) and Eriksson et al. (2006). For $d \geq 4$, Theorem 1 holds by Lemmas 2 to 5. □

Note that in their current form, Lemmas 2 to 5 do not imply the result by Eriksson et al. (2006). Instead, we use their result in order to also include the case $d = 3$ for our Theorem 1. Though our proof could be modified so to also capture that particular case, we decided against it in the interest of both space and comprehensiveness.

We see that the high-level idea of all proofs is to look for the 'best' families, i.e., the m-families with minimal m. Those we fix, name the remaining agents in A^1 according to a certain order depending on m and execute Algorithm 1, basically a greedy-algorithm.

Boros et al. (2004) questioned whether a procedure of that kind works for $n > d$; we have shown that for $n = d + 1$ it indeed does. However, note that this is not the case for $n > d+1$. As illustration consider the instance of d-DSM-CYC with $d = 2$, $n = 4$ given by Boros et al. (2004) where in the only stable matching no agent is matched to his first choice.

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