

On the Structure of Stable Tournament Solutions

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A fundamental property of choice functions is stability, which, loosely speaking, prescribes that choice sets are invariant under adding and removing unchosen alternatives. We provide several structural insights that improve our understanding of stable choice functions. In particular, *(i)* we show that every stable choice function is generated by a unique simple choice function, which never excludes more than one alternative, *(ii)* we completely characterize which simple choice functions give rise to stable choice functions, and *(iii)* we prove a strong relationship between stability and a new property of tournament solutions called *local reversal symmetry*. Based on these findings, we provide the first concrete tournament—consisting of 24 alternatives—in which the tournament equilibrium set fails to be stable. Furthermore, we prove that there is no more discriminating stable tournament solution than the bipartisan set and that the bipartisan set is the unique most discriminating tournament solution which satisfies standard properties proposed in the literature.

1. Introduction

Given a set of alternatives and binary non-transitive preferences over these alternatives, how can we consistently choose the “best” elements from any feasible subset of alternatives? This question has been studied in detail in the literature on tournament solutions (see, e.g., Laslier, 1997; Hudry, 2009; Moser, 2015; Brandt et al., 2016a). The lack of transitivity is typically attributed to the independence of pairwise comparisons as they

arise in sports competitions, multi-criteria decision analysis, and preference aggregation.¹ In particular, the pairwise majority relation of a profile of transitive individual preference relations often forms the basis of the study of tournament solutions. This is justified by a theorem due to McGarvey (1953), which shows that every tournament can be induced by some underlying preference profile. Many tournament solutions therefore correspond to well-known social choice functions such as Copeland’s rule, Slater’s rule, the Banks set, and the bipartisan set.

Over the years, many desirable properties of tournament solutions have been proposed. Some of these properties, so-called *choice consistency conditions*, make no reference to the actual tournament but only relate choices from different subtournaments to each other. An important choice consistency condition, that goes under various names, requires that the choice set is invariant under the removal of unchosen alternatives. In conjunction with a dual condition on expanded feasible sets, this property is known as *stability* (Brandt and Harrenstein, 2011). Stability implies that choices are made in a robust and coherent way. Furthermore, stable choice functions can be rationalized by a preference relation on *sets* of alternatives.

Examples of stable tournament solutions are the *top cycle*, the *minimal covering set*, and the *bipartisan set*. The latter is elegantly defined via the support of the unique mixed maximin strategies of the zero-sum game given by the tournament’s skew-adjacency matrix. Curiously, for some tournament solutions, including the *tournament equilibrium set* and the *minimal extending set*, proving or disproving stability turned out to be exceedingly difficult. As a matter of fact, whether the tournament equilibrium set satisfies stability was open for more than two decades before the existence of counterexamples with about 10^{136} alternatives was shown using the probabilistic method.

Brandt (2011) systematically constructed stable tournament solutions by applying a well-defined operation to existing (non-stable) tournament solutions. Brandt’s study was restricted to a particular class of generating tournament solutions, namely tournament solutions that can be defined via qualified subsets (such as the *uncovered set* and the *Banks set*). For any such generator, Brandt (2011) gave sufficient conditions for the resulting tournament solution to be stable. Later, Brandt et al. (2016b) showed that for one particular generator, the Banks set, the sufficient conditions for stability are also necessary.

In this paper, we show that *every* stable choice function is generated by a unique underlying simple choice function, which never excludes more than one alternative. We go on to prove a general characterization of stable tournament solutions that is not restricted to generators defined via qualified subsets. As a corollary, we obtain that the sufficient conditions for generators defined via qualified subsets are also necessary. Finally, we prove a strong connection between stability and a new property of tournament solutions called *local reversal symmetry*. Local reversal symmetry requires that an alternative is chosen if and only if it is unchosen when all its incident edges are inverted. This result allows us to settle two important problems in the theory of tournament solutions.

¹Due to their generality, tournament solutions have also found applications in unrelated areas such as biology (Schjelderup-Ebbe, 1922; Landau, 1951; Slater, 1961; Allesina and Levine, 2011).

We provide the first concrete tournament—consisting of 24 alternatives—in which the tournament equilibrium set violates stability. Secondly, we prove that there is no more discriminating stable tournament solution than the bipartisan set. We also axiomatically characterize the bipartisan set by only using properties that have been previously proposed in the literature. We believe that these results serve as a strong argument in favor of the bipartisan set if choice consistency is desired.

2. Stable Sets and Stable Choice Functions

Let U be a universal set of alternatives. Any finite non-empty subset of U will be called a *feasible set*. Before we analyze tournament solutions in Section 4, we first consider a more general model of choice which does not impose any structure on feasible sets. A *choice function* is a function that maps every feasible set A to a non-empty subset of A called the *choice set* of A . For two choice functions S and S' , we write $S' \subseteq S$, and say that S' is a *refinement* of S and S a *coarsening* of S' , if $S'(A) \subseteq S(A)$ for all feasible sets A . A choice function S is called *trivial* if $S(A) = A$ for all feasible sets A .

Brandt (2011) proposed a general method for refining a choice function S by defining minimal sets that satisfy internal and external stability criteria with respect to S , similar to von-Neumann–Morgenstern stable sets in cooperative game theory.²

A subset of alternatives $X \subseteq A$ is called *S -stable* within feasible set A for choice function S if it consists precisely of those alternatives that are chosen in the presence of all alternatives in X . Formally, X is *S -stable* in A if

$$X = \{a \in A : a \in S(X \cup \{a\})\}.$$

Equivalently, X is *S -stable* if and only if

$$\begin{aligned} S(X) &= X, \text{ and} && \text{(internal stability)} \\ a \notin S(X \cup \{a\}) &\text{ for all } a \in A \setminus X. && \text{(external stability)} \end{aligned}$$

The intuition underlying this formulation is that there should be no reason to restrict the choice set by excluding some alternative from it (internal stability) and there should be an argument against each proposal to include an outside alternative into the choice set (external stability).

An *S -stable* set is *inclusion-minimal* (or simply *minimal*) if it does not contain another *S -stable* set. $\widehat{S}(A)$ is defined as the union of all minimal *S -stable* sets in A . \widehat{S} defines a choice function whenever every feasible set admits at least one *S -stable* set. In general, however, neither the existence of *S -stable* sets nor the uniqueness of minimal *S -stable* sets is guaranteed. We say that \widehat{S} is *well-defined* if every choice set admits exactly one minimal *S -stable* set. We can now define the central concept of this paper.

Definition 1. A choice function S is *stable* if \widehat{S} is well-defined and $S = \widehat{S}$.

²This is a generalization of earlier work by Dutta (1988), who defined the minimal covering set as the unique minimal set that is internally and externally stable with respect to the uncovered set (see Section 4).

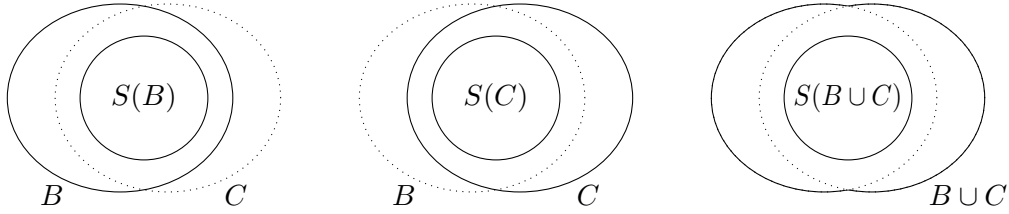


Figure 1: Visualization of stability. A stable choice function S chooses a set from both B (left) and C (middle) if and only if it chooses the same set from $B \cup C$ (right). The direction from the left and middle diagrams to the right diagram corresponds to $\hat{\gamma}$ while the converse direction corresponds to $\hat{\alpha}$.

Stability is connected to rationalizability and non-manipulability. In fact, every stable choice function can be rationalized via a preference relation on *sets* of alternatives (Brandt and Harrenstein, 2011) and, in the context of social choice, stability and monotonicity imply strategyproofness with respect to Kelly’s preference extension (Brandt, 2015).

The following example illustrates the preceding definitions. Consider universe $U = \{a, b, c\}$ and choice function S given by the table below (choices from singleton sets are trivial and therefore omitted).

| X | $S(X)$ | $\hat{S}(X)$ |
|---------------|---------------|--------------|
| $\{a, b\}$ | $\{a\}$ | $\{a\}$ |
| $\{b, c\}$ | $\{b\}$ | $\{b\}$ |
| $\{a, c\}$ | $\{a\}$ | $\{a\}$ |
| $\{a, b, c\}$ | $\{a, b, c\}$ | $\{a\}$ |

The feasible set $\{a, b, c\}$ admits exactly two S -stable sets, $\{a, b, c\}$ itself and $\{a\}$. The latter holds because $S(\{a\}) = \{a\}$ (internal stability) and $S(\{a, b\}) = S(\{a, c\}) = \{a\}$ (external stability). All other feasible sets X admit unique S -stable sets, which coincide with $S(X)$. Hence, \hat{S} is well-defined and given by the entries in the rightmost column of the table. Since $S \neq \hat{S}$, S fails to be stable. \hat{S} , on the other hand, satisfies stability.

Choice functions are usually evaluated by checking whether they satisfy choice consistency conditions that relate choices from different feasible sets to each other. The following two properties, $\hat{\alpha}$ and $\hat{\gamma}$, are set-based variants of Sen’s α and γ (Sen, 1971). $\hat{\alpha}$ is a rather prominent choice-theoretic condition, also known as Chernoff’s *postulate 5** (Chernoff, 1954), the *strong superset property* (Bordes, 1979), *outcast* (Aizerman and Aleskerov, 1995), and the *attention filter axiom* (Masatlioglu et al., 2012).³

Definition 2. A choice function S satisfies $\hat{\alpha}$ if for all feasible sets B and C ,

$$S(B) \subseteq C \subseteq B \text{ implies } S(C) = S(B). \quad (\hat{\alpha})$$

³We refer to Monjardet (2008) for a more thorough discussion of the origins of this condition.

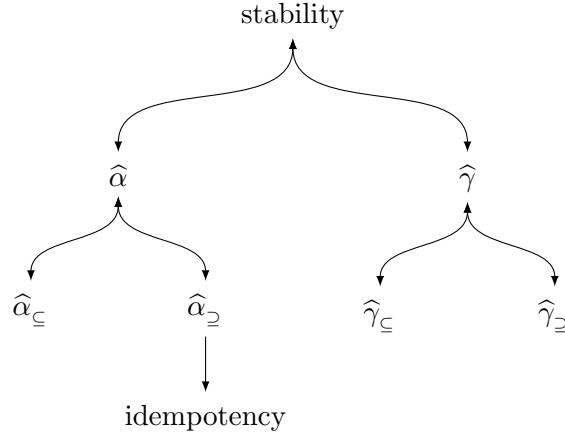


Figure 2: Logical relationships between choice-theoretic properties.

A choice function S satisfies $\hat{\gamma}$ if for all feasible sets B and C ,

$$S(B) = S(C) \text{ implies } S(B) = S(B \cup C). \quad (\hat{\gamma})$$

It has been shown that stability is equivalent to the conjunction of $\hat{\alpha}$ and $\hat{\gamma}$.

Theorem 1 (Brandt and Harrenstein, 2011). *A choice function is stable if and only if it satisfies $\hat{\alpha}$ and $\hat{\gamma}$.*

Hence, a choice function S is stable if and only if for all feasible sets B , C , and X with $X \subseteq B \cap C$,

$$X = S(B) \text{ and } X = S(C) \quad \text{if and only if} \quad X = S(B \cup C).$$

Stability, $\hat{\alpha}$, and $\hat{\gamma}$ are illustrated in Figure 1.

For a finer analysis, we split $\hat{\alpha}$ and $\hat{\gamma}$ into two conditions (Brandt and Harrenstein, 2011, Remark 1).

Definition 3. A choice function S satisfies

- $\hat{\alpha}_{\subseteq}$ if for all B, C , it holds that $S(B) \subseteq C \subseteq B$ implies $S(C) \subseteq S(B)$,⁴
- $\hat{\alpha}_{\supseteq}$ if for all B, C , it holds that $S(B) \subseteq C \subseteq B$ implies $S(C) \supseteq S(B)$,
- $\hat{\gamma}_{\subseteq}$ if for all B, C , it holds that $S(B) = S(C)$ implies $S(B) \subseteq S(B \cup C)$, and
- $\hat{\gamma}_{\supseteq}$ if for all B, C , it holds that $S(B) = S(C)$ implies $S(B) \supseteq S(B \cup C)$.

⁴ $\hat{\alpha}_{\subseteq}$ has also been called the *Aizerman property* or the *weak superset property* (e.g., Laslier, 1997; Brandt, 2011).

Obviously, for any choice function S we have

$$\begin{aligned} S \text{ satisfies } \hat{\alpha} & \quad \text{if and only if} \quad S \text{ satisfies } \hat{\alpha}_{\subseteq} \text{ and } \hat{\alpha}_{\supseteq}, \text{ and} \\ S \text{ satisfies } \hat{\gamma} & \quad \text{if and only if} \quad S \text{ satisfies } \hat{\gamma}_{\subseteq} \text{ and } \hat{\gamma}_{\supseteq}. \end{aligned}$$

A choice function is *idempotent* if the choice set is invariant under repeated application of the choice function, i.e., $S(S(A)) = S(A)$ for all feasible sets A . It is easily seen that $\hat{\alpha}_{\supseteq}$ is stronger than idempotency since $S(S(A)) \supseteq S(A)$ implies $S(S(A)) = S(A)$.

Figure 2 shows the logical relationships between stability and its weakenings.

3. Generators of Stable Choice Functions

We say that a choice function S' *generates* a stable choice function S if $S = \widehat{S}'$. Understanding stable choice functions can be reduced to understanding their generators. It turns out that important generators of stable choice functions are *simple* choice functions, i.e., choice functions S' with $|S'(A)| \geq |A| - 1$ for all A . In fact, every stable choice function S is generated by a unique simple choice function. To this end, we define the *root* of a choice function S as

$$\lceil S \rceil(A) = \begin{cases} S(A) & \text{if } |S(A)| = |A| - 1, \\ A & \text{otherwise.} \end{cases}$$

Not only does $\lceil S \rceil$ generate S , but any choice function sandwiched between S and $\lceil S \rceil$ is a generator of S .

Theorem 2. *Let S and S' be choice functions such that S is stable and $S \subseteq S' \subseteq \lceil S \rceil$. Then, \widehat{S}' is well-defined and $\widehat{S}' = S$. In particular, S is generated by the simple choice function $\lceil S \rceil$.*

Proof. We first show that any S -stable set is also S' -stable. Suppose that a set $X \subseteq A$ is S -stable in A . Then $S(X) = X$, and $a \notin S(X \cup \{a\})$ for all $a \in A \setminus X$. Since S satisfies $\hat{\alpha}$, we have $S(X \cup \{a\}) = X$ and therefore $\lceil S \rceil(X \cup \{a\}) = X$ for all $a \in A \setminus X$. Using the inclusion relationship $S \subseteq S' \subseteq \lceil S \rceil$, we find that $S'(X) = X$ and $S'(X \cup \{a\}) = X$ for all $a \in A \setminus X$. Hence, X is S' -stable in A .

Next, we show that every S' -stable set contains an S -stable set. Suppose that a set $X \subseteq A$ is S' -stable in A . Then $S'(X) = X$ and $a \notin S'(X \cup \{a\})$ for all $a \in A \setminus X$. Using the relation $S \subseteq S'$, we find that $a \notin S(X \cup \{a\})$ for all $a \in A \setminus X$. We will show that $S(X) \subseteq X$ is S -stable in A . Since S satisfies $\hat{\alpha}$, we have $S(S(X)) = S(X)$ and $S(X \cup \{a\}) = S(X)$ for all $a \in A \setminus X$. It remains to show that $b \notin S(S(X) \cup \{b\})$ for all $b \in A \setminus S(X)$. If $b \in A \setminus X$, we already have that $S(X \cup \{b\}) = S(X)$ and therefore $S(S(X) \cup \{b\}) = S(X)$ by $\hat{\alpha}$. Otherwise, if $b \in X \setminus S(X)$, $\hat{\alpha}$ again implies that $S(S(X) \cup \{b\}) = S(X)$.

Since S is stable, for any feasible set A there exists a unique minimal S -stable set in A , which is given by $S(A) = \widehat{S}(A)$. From what we have shown, this set is also S' -stable, and moreover any S' -stable set contains an S -stable set which in turn contains $S(A)$. Hence

$S(A)$ is also the unique minimal S' -stable set in A . This implies that \widehat{S}' is well-defined and $\widehat{S}' = \widehat{S} = S$. \square

Theorem 2 entails that in order to understand stable choice functions, we only need to understand the circumstances under which a single alternative is discarded.⁵

An important question is which simple choice functions are roots of stable choice functions. It follows from the definition of root functions that any root of a stable choice function needs to satisfy $\widehat{\alpha}$. This condition is, however, not sufficient as it is easy to construct a simple choice function S that satisfies $\widehat{\alpha}$ such that \widehat{S} violates $\widehat{\alpha}$. Nevertheless, the theorem implies that the number of stable choice functions can be bounded by counting the number of simple choice functions that satisfy $\widehat{\alpha}$. The number of simple choice functions for a universe of size $n \geq 2$ is only $\prod_{i=2}^n (i+1)^{\binom{n}{i}}$, compared to $\prod_{i=2}^n (2^i - 1)^{\binom{n}{i}}$ for arbitrary choice functions.

In order to give a complete characterization of choice functions that generate stable choice functions, we need to introduce a new property. A choice function S satisfies local $\widehat{\alpha}$ if minimal S -stable sets are invariant under removing outside alternatives.⁶

Definition 4. A choice function S satisfies *local* $\widehat{\alpha}$ if for any sets $X \subseteq Y \subseteq Z$ such that X is minimally S -stable in Z , we have that X is also minimally S -stable in Y .

Recall that a choice function S satisfies $\widehat{\alpha}_{\subseteq}$ if for any sets A, B such that $S(A) \subseteq B \subseteq A$, we have $S(B) \subseteq S(A)$. In particular, every simple choice function satisfies $\widehat{\alpha}_{\subseteq}$. We will provide a characterization of choice functions S satisfying $\widehat{\alpha}_{\subseteq}$ such that \widehat{S} is stable. First we need the following (known) lemma.

Lemma 1 (Brandt and Harrenstein, 2011). *Let S be a choice function such that \widehat{S} is well-defined. Then \widehat{S} satisfies $\widehat{\gamma}$.*

Theorem 3. *Let S be a choice function satisfying $\widehat{\alpha}_{\subseteq}$. Then \widehat{S} is stable if and only if \widehat{S} is well-defined and S satisfies local $\widehat{\alpha}$.*

Proof. For the direction from right to left, suppose that \widehat{S} is well-defined and S satisfies local $\widehat{\alpha}$. Then Lemma 1 implies that \widehat{S} satisfies $\widehat{\gamma}$. Moreover, it follows directly from local $\widehat{\alpha}$ and the fact that \widehat{S} is well-defined that \widehat{S} satisfies $\widehat{\alpha}$. Hence, \widehat{S} is stable.

For the converse direction, suppose that \widehat{S} is stable. We first show that \widehat{S} is well-defined. Every feasible set A contains at least one S -stable set because otherwise \widehat{S} is not a choice function. Next, suppose for contradiction that there exists a feasible set that contains two distinct minimal S -stable sets. Consider such a feasible set A of minimum

⁵Together with Theorem 3, Theorem 2 also implies that, for any stable tournament solution S , $[S]$ is a coarsest generator of S . When only considering generators that satisfy $\widehat{\alpha}_{\subseteq}$, $[S]$ is also *the* coarsest generator of S . In addition, since simple choice functions trivially satisfy $\widehat{\alpha}_{\subseteq}$, the two theorems imply that $[S]$ is the unique simple choice function generating S .

⁶It can be checked that we obtain an equivalent condition even if we require that *all* outside alternatives have to be removed. When defining local $\widehat{\alpha}$ in this way, it can be interpreted as some form of transitivity of stability: stable sets of minimally stable sets are also stable within the original feasible set (cf. Brandt, 2011, Lem. 3).

size, and pick any two distinct minimal S -stable sets in A , which we denote by B and C . If $|B \setminus C| = |C \setminus B| = 1$, then $\widehat{\alpha}_{\subseteq}$ implies $S(B \cup C) = B = C$, a contradiction. Otherwise, assume without loss of generality that $|C \setminus B| \geq 2$, and pick $x, y \in C \setminus B$ with $x \neq y$. Then $A \setminus \{x\}$ contains a unique minimal S -stable set. As B is also S -stable in $A \setminus \{x\}$, it follows that $\widehat{S}(A \setminus \{x\}) \subseteq B$. Since \widehat{S} satisfies $\widehat{\alpha}$, we have $\widehat{S}(A \setminus \{x\}) = \widehat{S}(A \setminus \{x, y\})$. Similarly, we have $\widehat{S}(A \setminus \{y\}) = \widehat{S}(A \setminus \{x, y\})$. But then $\widehat{\gamma}$ implies that $\widehat{S}(A) = \widehat{S}(A \setminus \{x, y\}) \subseteq A$, which contradicts the fact that C is minimal S -stable in A .

We now show that S satisfies local $\widehat{\alpha}$. Since \widehat{S} is well-defined, minimal S -stable sets are unique and given by \widehat{S} . Since \widehat{S} satisfies $\widehat{\alpha}$, minimal S -stable sets are invariant under deleting outside alternatives. Hence, S satisfies local $\widehat{\alpha}$, as desired. \square

Remark 1. Theorem 3 does not hold without the condition that S satisfies $\widehat{\alpha}_{\subseteq}$. To this end, let $U = \{a, b, c\}$, $S(\{a, b, c\}) = \{b\}$, and $S(X) = X$ for all other feasible sets X . Then both $\{a, b\}$ and $\{b, c\}$ are minimally S -stable in $\{a, b, c\}$, implying that \widehat{S} is not well-defined. On the other hand, \widehat{S} is trivial and therefore also stable. This example also shows that a generator of a stable choice function needs not be sandwiched between the choice function and its root.

Combining Theorem 3 with Theorem 1, we obtain the following characterization.

Corollary 1. *Let S be a choice function satisfying $\widehat{\alpha}_{\subseteq}$. Then,*

$$\begin{aligned} \widehat{S} \text{ is stable if and only if } & \widehat{S} \text{ is well-defined and } \widehat{S} = \widehat{S} \\ & \text{if and only if } \widehat{S} \text{ satisfies } \widehat{\alpha} \text{ and } \widehat{\gamma} \\ & \text{if and only if } \widehat{S} \text{ is well-defined and } S \text{ satisfies local } \widehat{\alpha}. \end{aligned}$$

Since simple choice functions trivially satisfy $\widehat{\alpha}_{\subseteq}$, this corollary completely characterizes which simple choice functions generate stable choice functions.

4. Tournament Solutions

We now turn to the important special case of choice functions whose output depends on a binary relation.

4.1. Preliminaries

A *tournament* T is a pair (A, \succ) , where A is a feasible set and \succ is a connex and asymmetric (and thus irreflexive) binary relation on A , usually referred to as the *dominance relation*. Intuitively, $a \succ b$ signifies that alternative a is preferable to alternative b . The dominance relation can be extended to sets of alternatives by writing $X \succ Y$ when $a \succ b$ for all $a \in X$ and $b \in Y$.

For a tournament $T = (A, \succ)$ and an alternative $a \in A$, we denote by

$$\overline{D}(a) = \{x \in A \mid x \succ a\}$$

the *dominators* of a and by

$$D(a) = \{ x \in A \mid a \succ x \}$$

the *dominion* of a . When varying the tournament, we will refer to $\overline{D}_{T'}(a)$ and $D_{T'}(a)$ for some tournament $T' = (A', \succ')$. An alternative a is said to *cover* another alternative b if $D(b) \subseteq D(a)$. It is said to be a *Condorcet winner* if it dominates all other alternatives, and a *Condorcet loser* if it is dominated by all other alternatives. The order of a tournament $T = (A, \succ)$ is denoted by $|T| = |A|$. A tournament is *regular* if the dominator set and the dominion set of each alternative are of the same size, i.e., for all $a \in A$ we have $|D(a)| = |\overline{D}(a)|$. It is easily seen that regular tournaments are always of odd order.

A *tournament solution* is a function that maps a tournament to a nonempty subset of its alternatives. We assume that tournament solutions are invariant under tournament isomorphisms. For every fixed tournament, a tournament solution yields a choice function. A tournament solution is *trivial* if it returns all alternatives of every tournament.

Three common tournament solutions are the top cycle, the uncovered set, and the Banks set. For a given tournament (A, \succ) , the *top cycle* (*TC*) is the (unique) smallest set $B \subseteq A$ such that $B \succ A \setminus B$, the *uncovered set* (*UC*) contains all alternatives that are not covered by another alternative, and the *Banks set* (*BA*) contains all alternatives that are Condorcet winners of inclusion-maximal transitive subtournaments.

For two tournament solutions S and S' , we write $S' \subseteq S$, and say that S' is a *refinement* of S and S a *coarsening* of S' , if $S'(T) \subseteq S(T)$ for all tournaments T . The following inclusions are well-known:

$$BA \subseteq UC \subseteq TC.$$

To simplify notation, we will often identify a (sub)tournament by its set of alternatives when the dominance relation is clear from the context. For example, for a tournament solution S and a subset of alternatives $X \subseteq A$ in a tournament $T = (A, \succ)$ we will write $S(X)$ for $S(T|_X)$.

The definitions of stability and other choice consistency conditions directly carry over from choice functions to tournament solutions by requiring that the given condition should be satisfied by every choice function induced by the tournament solution and a tournament. We additionally consider the following desirable properties of tournament solutions, all of which are standard conditions in the literature.

Monotonicity requires that a chosen alternative will still be chosen when its dominion is enlarged, while leaving everything else unchanged.

Definition 5. A tournament solution is *monotonic* if for all $T = (A, \succ)$, $T' = (A, \succ')$, $a \in A$ such that $\succ_{A \setminus \{a\}} = \succ'_{A \setminus \{a\}}$ and for all $b \in A \setminus \{a\}$, $a \succ' b$ whenever $a \succ b$,

$$a \in S(T) \quad \text{implies} \quad a \in S(T').$$

Regularity requires that all alternatives are chosen from regular tournaments.

Definition 6. A tournament solution is *regular* if $S(T) = A$ for all regular tournaments $T = (A, \succ)$.

Even though regularity is often considered in the context of tournament solutions, it does not possess the normative appeal of other conditions.

Finally, we consider a structural invariance property that is based on components of similar alternatives and, loosely speaking, requires that a tournament solution chooses the “best” alternatives from the “best” components. A *component* is a nonempty subset of alternatives $B \subseteq A$ that bear the same relationship to any alternative not in the set, i.e., for all $a \in A \setminus B$, either $B \succ \{a\}$ or $\{a\} \succ B$. A *decomposition* of T is a partition of A into components.

For a given tournament \tilde{T} , a new tournament T can be constructed by replacing each alternative with a component. Let B_1, \dots, B_k be pairwise disjoint sets of alternatives and consider tournaments $T_1 = (B_1, \succ_1), \dots, T_k = (B_k, \succ_k)$, and $\tilde{T} = (\{1, \dots, k\}, \tilde{\succ})$. The *product* of T_1, \dots, T_k with respect to \tilde{T} , denoted by $\prod(\tilde{T}, T_1, \dots, T_k)$, is the tournament $T = (A, \succ)$ such that $A = \bigcup_{i=1}^k B_i$ and for all $b_1 \in B_i, b_2 \in B_j$,

$$b_1 \succ b_2 \quad \text{if and only if} \quad i = j \text{ and } b_1 \succ_i b_2, \text{ or } i \neq j \text{ and } i \tilde{\succ} j.$$

Here, \tilde{T} is called the *summary* of T with respect to the above decomposition.

Definition 7. A tournament solution is *composition-consistent* if for all tournaments T, T_1, \dots, T_k and \tilde{T} such that $T = \prod(\tilde{T}, T_1, \dots, T_k)$,

$$S(T) = \bigcup_{i \in S(\tilde{T})} S(T_i).$$

All of the three tournament solutions we briefly introduced above satisfy monotonicity. *TC* and *UC* are regular, *UC* and *BA* are composition-consistent, and only *TC* is stable. For more thorough treatments of tournament solutions, see Laslier (1997) and Brandt et al. (2016a).

4.2. The Bipartisan set and the Tournament Equilibrium Set

We now define two tournament solutions that are central to this paper. The first one, the bipartisan set, generalizes the notion of a Condorcet winner to probability distributions over alternatives. The *skew-adjacency matrix* $G(T) = (g_{ab})_{a,b \in A}$ of a tournament T is defined by letting

$$g_{ab} = \begin{cases} 1 & \text{if } a \succ b \\ -1 & \text{if } b \succ a \\ 0 & \text{if } a = b. \end{cases}$$

The skew-adjacency matrix can be interpreted as a symmetric zero-sum game in which there are two players, one choosing rows and the other choosing columns, and in which the matrix entries are the payoffs of the row player. Laffond et al. (1993b) and Fisher and Ryan (1995) have shown independently that every such game admits a unique mixed Nash equilibrium, which moreover is symmetric. Let $p_T \in \Delta(A)$ denote the mixed

strategy played by both players in equilibrium. Then, p_T is the unique probability distribution such that

$$\sum_{a,b \in A} p_T(a)q(b)g_{ab} \geq 0 \quad \text{for all } q \in \Delta(A).$$

In other words, there is no other probability distribution that is more likely to yield a better alternative than p_T . Laffond et al. (1993b) defined the bipartisan set $BP(T)$ of T as the support of p_T .⁷

Definition 8. The *bipartisan set* (BP) of a given tournament $T = (A, \succ)$ is defined as

$$BP(T) = \{a \in A \mid p_T(a) > 0\}.$$

BP satisfies stability, monotonicity, regularity, and composition-consistency. Moreover, $BP \subseteq UC$ and BP can be computed in polynomial time by solving a linear feasibility problem.

The next tournament solution, the tournament equilibrium set, was defined by Schwartz (1990). Given a tournament $T = (A, \succ)$ and a tournament solution S , a nonempty subset of alternatives $X \subseteq A$ is called S -retentive if $S(\overline{D}(x)) \subseteq X$ for all $x \in X$ such that $\overline{D}(x) \neq \emptyset$.

Definition 9. The *tournament equilibrium set* (TEQ) of a given tournament $T = (A, \succ)$ is defined recursively as the union of all inclusion-minimal TEQ -retentive sets in T .

This is a proper recursive definition because the cardinality of the set of dominators of an alternative in a particular set is always smaller than the cardinality of the set itself. BP and TEQ coincide on all tournaments of order 5 or less (Brandt et al., 2015).⁸

Schwartz (1990) showed that $TEQ \subseteq BA$ and conjectured that every tournament contains a *unique* inclusion-minimal TEQ -retentive set, which was later shown to be equivalent to TEQ satisfying any one of a number of desirable properties for tournament solutions including stability and monotonicity (Laffond et al., 1993a; Houy, 2009a,b; Brandt et al., 2014; Brandt, 2011; Brandt and Harrenstein, 2011; Brandt, 2015). This conjecture was disproved by Brandt et al. (2013), who have non-constructively shown the existence of a counterexample with about 10^{136} alternatives using the probabilistic method. Since it was shown that TEQ satisfies the above mentioned desirable properties for all tournaments that are smaller than the smallest counterexample to Schwartz's conjecture, the search for smaller counterexamples remains an important problem. In fact, the counterexample found by Brandt et al. (2013) is so large that it has no practical consequences whatsoever for TEQ . Apart from concrete counterexamples, there is ongoing interest in why and under which circumstances TEQ and a related tournament solution called the *minimal extending set* $ME = \widehat{BA}$ violate stability (Mnich et al., 2015; Brandt et al., 2016b; Yang, 2016).

⁷The probability distribution p_T was independently analyzed by Kreweras (1965), Fishburn (1984), Felsenthal and Machover (1992), and others. An axiomatic characterization in the context of social choice was recently given by Brandl et al. (2016).

⁸It is open whether there are tournaments in which BP and TEQ are disjoint.

Computing the tournament equilibrium set of a given tournament was shown to be NP-hard and consequently there does not exist an efficient algorithm for this problem unless P equals NP (Brandt et al., 2010).

5. Stable Tournament Solutions and Their Generators

Tournament solutions comprise an important subclass of choice functions. In this section, we examine the consequences of the findings from Sections 2 and 3, in particular Theorems 1, 2, and 3, for tournament solutions.

Stability is a rather demanding property which is satisfied by only a few tournament solutions. Three well-known tournament solutions that satisfy stability are the top cycle TC , the minimal covering set MC defined by $MC = \widehat{UC}$, and the bipartisan set BP , which is a refinement of MC . By virtue of Theorem 2, any stable tournament solution is generated by its root $\lceil S \rceil$. For example, $\lceil TC \rceil$ is a tournament solution that excludes an alternative if and only if it is the only alternative not contained in the top cycle (and hence a Condorcet loser). Similarly, one can obtain the roots of other stable tournament solutions such as MC and BP . In some cases, the generator that is typically considered for a stable tournament solution is different from its root; for example, MC is traditionally generated by UC , a refinement of $\lceil MC \rceil$. Since tournament solutions are invariant under tournament isomorphisms, a simple tournament solution may only exclude an alternative a if any automorphism of T maps a to itself. Note that if a tournament solution S is stable, $\lceil S \rceil$ is different from S unless S is the trivial tournament solution.

It follows from Theorem 1 that stable tournament solutions satisfy both $\widehat{\alpha}$ and $\widehat{\gamma}$. It can be shown that $\widehat{\alpha}$ and $\widehat{\gamma}$ are independent from each other even in the context of tournament solutions.

Remark 2. There are tournament solutions that satisfy only one of $\widehat{\alpha}$ and $\widehat{\gamma}$. Examples are given in Appendix A.1.

We have shown in Theorem 2 that stable tournament solutions are generated by unique simple tournament solutions. If we furthermore restrict our attention to *monotonic* stable tournament solutions, the following theorem shows that we only need to consider root solutions that are monotonic.

Theorem 4. *A stable tournament solution S is monotonic if and only if $\lceil S \rceil$ is monotonic.*

Proof. First, note that monotonicity is equivalent to requiring that unchosen alternatives remain unchosen when they are weakened. Now, for the direction from left to right, suppose that S is monotonic. Let $T = (A, \succ)$, $B = \lceil S \rceil(T)$, and $a \in A \setminus B$. Since $\lceil S \rceil$ is simple, we have $\lceil S \rceil(T) = B \setminus \{a\}$, and therefore $S(T) = B \setminus \{a\}$. Using the fact that S is stable and thus satisfies $\widehat{\alpha}$, we find that $S(T|_{B \setminus \{a\}}) = B \setminus \{a\}$. Let T' be a tournament obtained by weakening a with respect to some alternative in B . Monotonicity of S entails that $a \notin S(T')$. Since $T|_{B \setminus \{a\}} = T'|_{B \setminus \{a\}}$, we have $S(T'|_{B \setminus \{a\}}) = B \setminus \{a\}$, and $\widehat{\alpha}$

implies that $S(T') = B \setminus \{a\}$ and $\lceil S \rceil(T') = B \setminus \{a\}$ as well. This means that a remains unchosen by $\lceil S \rceil$ in T' , as desired.

The converse direction even holds for all generators of S (see Brandt, 2011, Prop. 5). \square

Analogous results do *not* hold for composition-consistency or regularity.

Theorem 3 characterizes stable choice functions \widehat{S} using well-definedness of \widehat{S} and local $\widehat{\alpha}$ of S . These two properties are independent from each other (and therefore necessary for the characterization) even in the context of tournament solutions.

Remark 3. There is a tournament solution S that satisfies local $\widehat{\alpha}$, but \widehat{S} violates $\widehat{\alpha}$. There is a tournament solution S for which \widehat{S} is well-defined, but \widehat{S} is not stable. Examples are given in Appendix A.2.

Theorem 3 generalizes previous statements about stable tournament solutions. Brandt (2011) studied a particular class of generators defined via qualified subsets and shows the direction from right to left of Theorem 3 for these generators (Brandt, 2011, Thm. 4).⁹ Later, Brandt et al. (2016b) proved Theorem 3 for one particular generator BA (Brandt et al., 2016b, Cor. 2).

6. Local Reversal Symmetry

In this section, we introduce a new property of tournament solutions called local reversal symmetry (*LRS*).¹⁰ While intuitive by itself, *LRS* is strongly connected to stability and can be leveraged to disprove that *TEQ* is stable and to prove that no refinement of *BP* is stable.

For a tournament T , let T^a be the tournament whose dominance relation is *locally reversed* at alternative a , i.e., $T^a = (A, \succ^a)$ with

$$i \succ^a j \quad \text{if and only if} \quad (i \succ j \text{ and } a \notin \{i, j\}) \text{ or } (j \succ i \text{ and } a \in \{i, j\}).$$

The effect of local reversals is illustrated in Figure 3. Note that $T = (T^a)^a$ and $(T^a)^b = (T^b)^a$ for all alternatives a and b .

Definition 10. A tournament solution S satisfies *local reversal symmetry (LRS)* if for all tournaments T and alternatives a ,

$$a \in S(T) \text{ if and only if } a \notin S(T^a).$$

⁹Brandt's proof relies on a lemma that essentially showed that the generators he considers always satisfy local $\widehat{\alpha}$.

¹⁰The name of this axiom is inspired by a social choice criterion called *reversal symmetry*. Reversal symmetry prescribes that a uniquely chosen alternative has to be unchosen when the preferences of all voters are reversed (Saari and Barney, 2003). A stronger axiom, called *ballot reversal symmetry*, which demands that the choice set is inverted when all preferences are reversed was recently introduced by Duddy et al. (2014).

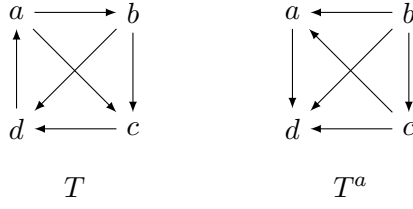


Figure 3: Local reversals of tournament T at alternative a result in T^a . $BP(T) = TEQ(T) = \{a, b, d\}$ and $BP(T^a) = TEQ(T^a) = \{b\}$.

LRS can be naturally split into two properties, LRS_{IN} and LRS_{OUT} . LRS_{IN} corresponds to the direction from right to left in the above equivalence and requires that unchosen alternatives are chosen when all incident edges are reversed. LRS_{OUT} corresponds to the direction from left to right and requires that chosen alternatives are not chosen when all incident edges are reversed.

It follows directly from the definition that LRS_{IN} (resp. LRS_{OUT}) of a tournament solution S carries over to any tournament solution that is a coarsening (resp. refinement) of S .

Lemma 2. *Let S and S' be two tournament solutions such that $S \subseteq S'$. If S satisfies LRS_{IN} , then so does S' . Conversely, if S' satisfies LRS_{OUT} , then so does S .*

There is an unexpected strong relationship between the purely choice-theoretic condition of stability and LRS .

Theorem 5. *Every stable tournament solution satisfies LRS_{IN} .*

Proof. Suppose for contradiction that S is stable but violates LRS_{IN} . Then there exists a tournament $T = (A, \succ)$ and an alternative $a \in A$ such that $a \notin S(T)$ and $a \notin S(T^a)$.

Let $T' = (A', \succ')$, where $A' = X \cup Y$ and each of $T'|_X$ and $T'|_Y$ is isomorphic to $T|_{A \setminus \{a\}}$. Also, partition $X = X_1 \cup X_2$ and $Y = Y_1 \cup Y_2$, where X_1 and Y_1 consist of the alternatives that are mapped to $\overline{D}_T(a)$ by the isomorphism. To complete the definition of T' , we add the relations $X_1 \succ' Y_2$, $Y_2 \succ' X_2$, $X_2 \succ' Y_1$, and $Y_1 \succ' X_1$. The structure of tournament T' is depicted in Figure 4.

We claim that both X and Y are externally S -stable in T' . To this end, we note that for every alternative $x \in X$ (resp. $y \in Y$) the subtournament $T|_{Y \cup \{x\}}$ (resp. $T|_{X \cup \{y\}}$) is isomorphic either to T or to T^a , with x (resp. y) being mapped to a . By assumption, a is neither chosen in T nor in T^a , and therefore X and Y are both externally S -stable in T' .

Now, suppose that $S(X \cup \{y\}) = X' \subseteq X$ for some $y \in Y$. Note that $X' \neq \emptyset$ because tournament solutions always return non-empty sets. Since S satisfies $\hat{\alpha}$, we have $S(X) = X'$. Hence, $S(X) = X' = S(X \cup \{y\})$ for all $y \in Y$. Since S satisfies $\hat{\gamma}$, we also have $S(X \cup Y) = X'$. Similarly, we can deduce that $S(X \cup Y) = Y'$ for some $\emptyset \neq Y' \subseteq Y$. This yields the desired contradiction. \square

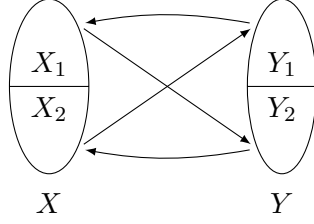


Figure 4: Construction of a tournament T' with two S -stable sets X and Y used in the proof of Theorem 5.

6.1. Disproving Stability

As discussed in Section 4, disproving that a tournament solution satisfies stability can be very difficult. By virtue of Theorem 5, it now suffices to show that the tournament solution violates LRS_{IN} . For TEQ , this leads to the first concrete tournament in which TEQ violates stability. With the help of a computer, we found a minimal tournament in which TEQ violates LRS_{IN} using exhaustive search. This tournament is of order 13 and thereby lies exactly at the boundary of the class of tournaments for which exhaustive search is possible. Using the construction explained in the proof of Theorem 5, we thus obtain a tournament of order 24 in which TEQ violates $\hat{\gamma}$. This tournament consists of two disjoint isomorphic subtournaments of order 12 both of which are TEQ -retentive.

Theorem 6. TEQ violates LRS_{IN} .

Proof. We define a tournament $T = (\{x_1, \dots, x_{13}\}, \succ)$ such that $x_{13} \notin TEQ(T)$ and $x_{13} \notin TEQ(T^{x_{13}})$. The dominator sets in T are defined as follows:

$$\begin{aligned}
\bar{D}(x_1) &= \{x_4, x_5, x_6, x_8, x_9, x_{12}\}, & \bar{D}(x_2) &= \{x_1, x_6, x_7, x_{10}, x_{12}\}, \\
\bar{D}(x_3) &= \{x_1, x_2, x_6, x_7, x_9, x_{10}\}, & \bar{D}(x_4) &= \{x_2, x_3, x_7, x_8, x_{11}\}, \\
\bar{D}(x_5) &= \{x_2, x_3, x_4, x_8, x_{10}, x_{11}\}, & \bar{D}(x_6) &= \{x_4, x_5, x_9, x_{11}, x_{12}\}, \\
\bar{D}(x_7) &= \{x_1, x_5, x_6, x_{11}, x_{12}, x_{13}\}, & \bar{D}(x_8) &= \{x_2, x_3, x_6, x_7, x_{12}, x_{13}\}, \\
\bar{D}(x_9) &= \{x_2, x_4, x_5, x_7, x_8, x_{13}\}, & \bar{D}(x_{10}) &= \{x_1, x_4, x_6, x_7, x_8, x_9, x_{13}\}, \\
\bar{D}(x_{11}) &= \{x_1, x_2, x_3, x_8, x_9, x_{10}, x_{13}\}, & \bar{D}(x_{12}) &= \{x_3, x_4, x_5, x_9, x_{10}, x_{11}, x_{13}\}, \\
\bar{D}(x_{13}) &= \{x_1, x_2, x_3, x_4, x_5, x_6\}.
\end{aligned}$$

A rather tedious check reveals that

$$\begin{aligned}
TEQ(\bar{D}(x_1)) &= \{x_4, x_8, x_{12}\}, & TEQ(\bar{D}(x_2)) &= \{x_6, x_{10}, x_{12}\}, \\
TEQ(\bar{D}(x_3)) &= \{x_6, x_7, x_9\}, & TEQ(\bar{D}(x_4)) &= \{x_2, x_7, x_{11}\}, \\
TEQ(\bar{D}(x_5)) &= \{x_2, x_8, x_{10}\}, & TEQ(\bar{D}(x_6)) &= \{x_4, x_9, x_{11}\}, \\
TEQ(\bar{D}(x_7)) &= \{x_1, x_5, x_{11}\}, & TEQ(\bar{D}(x_8)) &= \{x_3, x_6, x_{12}\}, \\
TEQ(\bar{D}(x_9)) &= \{x_2, x_5, x_7\}, & TEQ(\bar{D}(x_{10})) &= \{x_4, x_6, x_7\}, \\
TEQ(\bar{D}(x_{11})) &= \{x_1, x_2, x_8\}, \text{ and } & TEQ(\bar{D}(x_{12})) &= \{x_3, x_4, x_9\}.
\end{aligned}$$

It can then be checked that $TEQ(T) = TEQ(T^{x_{13}}) = \{x_1, \dots, x_{12}\}$. \square

Let n_{TEQ} be the greatest natural number n such that all tournaments of order n or less admit a unique inclusion-minimal *TEQ*-retentive set. Together with earlier results by Brandt et al. (2010) and Yang (2016), we now have that $14 \leq n_{TEQ} \leq 23$.

The tournament used in the preceding proof does not show that *ME* (or *BA*) violate LRS_{IN} . A computer search for such tournaments was unsuccessful. While it is known that *ME* violates stability, a concrete counterexample thus remains elusive.

6.2. Most Discriminating Stable Tournament Solutions

An important property of tournament solutions that is not captured by any of the axioms introduced in Section 4.1 is discriminative power.¹¹ It is known that *BA* and *MC* (and by the known inclusions also *UC* and *TC*) almost always select all alternatives when tournaments are drawn uniformly at random and the number of alternatives goes to infinity (Fey, 2008; Scott and Fey, 2012).¹² Experimental results suggest that the same is true for *TEQ*. Other tournament solutions, which are known to return small choice sets, fail to satisfy stability and composition-consistency. A challenging question is how discriminating tournament solutions can be while still satisfying desirable axioms.

LRS reveals an illuminating dichotomy in this context for common tournament solutions. We state without proof that discriminating tournament solutions such as Copeland's rule, Slater's rule, and Markov's rule satisfy LRS_{OUT} and violate LRS_{IN} . On the other hand, coarse tournament solutions such as *TC*, *UC*, and *MC* satisfy LRS_{IN} and violate LRS_{OUT} . The bipartisan set hits the sweet spot because it is the only one among the commonly considered tournament solutions that satisfies LRS_{IN} and LRS_{OUT} (and hence *LRS*).

Theorem 7. *BP satisfies LRS.*

Proof. Since *BP* is stable, Theorem 5 implies that *BP* satisfies LRS_{IN} . Now, assume for contradiction that *BP* violates LRS_{OUT} , i.e., there is a tournament $T = (A, \succ)$ and an alternative a such that $a \in BP(T)$ and $a \in BP(T^a)$. For a probability distribution p and a subset of alternatives $B \subseteq A$, let $p(B) = \sum_{x \in B} p(x)$. Consider the optimal mixed strategy $p_{T|_{A \setminus \{a\}}}$ in tournament $T|_{A \setminus \{a\}}$. It is known from Laslier (1997, Prop. 6.4.8) that $a \in BP(T)$ if and only if $p_{T|_{A \setminus \{a\}}}(D(a)) > p_{T|_{A \setminus \{a\}}}(\overline{D}(a))$. For T^a , we thus have that $p_{T^a|_{A \setminus \{a\}}}(D(a)) > p_{T^a|_{A \setminus \{a\}}}(\overline{D}(a))$. This is a contradiction because $D_T(a) = \overline{D}_{T^a}(a)$ and $\overline{D}_T(a) = D_{T^a}(a)$. \square

The relationship between *LRS* and the discriminative power of tournament solutions is no coincidence. To see this, consider all *labeled* tournaments of fixed order and an arbitrary alternative a . LRS_{IN} demands that a is chosen in *at least* one of T and T^a

¹¹To see that discriminative power is not captured by the axioms, observe that the trivial tournament solution satisfies stability, monotonicity, regularity, and composition-consistency.

¹²However, these analytic results stand in sharp contrast to empirical observations that Condorcet winners are likely to exist in real-world settings, which implies that tournament solutions are much more discriminative than results for the uniform distribution suggest (Brandt and Seedig, 2016).

while LRS_{OUT} requires that a is chosen in *at most* one of T and T^a . We thus obtain the following consequences.

- A tournament solution satisfying LRS_{IN} chooses on average at least half of the alternatives.
- A tournament solution satisfying LRS_{OUT} chooses on average at most half of the alternatives.
- A tournament solution satisfying LRS chooses on average half of the alternatives.

Hence, the well-known fact that BP chooses on average half of the alternatives (Fisher and Reeves, 1995) follows from Theorem 7. Also, all coarsenings of BP such as MC , UC , and TC satisfy LRS_{IN} by virtue of Lemma 2. On the other hand, since these tournament solutions are all different from BP , they choose on average more than half of the alternatives and hence cannot satisfy LRS_{OUT} .

These results already hint at BP being perhaps a “most discriminating” stable tournament solution. In order to make this precise, we formally define the discriminative power of a tournament solution. For two tournament solutions S and S' , we say that S is *more discriminating* than S' if there is $n \in \mathbb{N}$ such that the average number of alternatives chosen by S is lower than that of S' over all labeled tournaments of order n . Note that this definition is very weak because we only have an existential, not a universal, quantifier for n . It is therefore even possible that two tournament solutions are more discriminating than each other. However, this only strengthens the following results. Combining Theorems 5 and 7 immediately yields the following theorem.

Theorem 8. *A stable tournament solution satisfies LRS if and only if there is no more discriminating stable tournament solution.*

Proof. First consider the direction from left to right. Let S be a tournament solution that satisfies LRS . Due to the observations made above, S chooses on average half of the alternatives. Since any stable tournament solution satisfies LRS_{IN} by Theorem 5, it chooses on average at least half of the alternatives and therefore cannot be more discriminating than S .

Now consider the direction from right to left. Let S be a most discriminating stable tournament solution. Again, since any stable tournament solution satisfies LRS_{IN} , S chooses on average at least half of the alternatives. On the other hand, BP is a stable tournament solution that chooses on average exactly half of the alternatives. This means that S must also choose on average half of the alternatives, implying that it also satisfies LRS_{OUT} and hence LRS . \square

Corollary 2. *There is no more discriminating stable tournament solution than BP . In particular, there is no stable refinement of BP .*

Given Corollary 2, a natural question is whether every stable tournament solution that satisfies mild additional properties such as monotonicity is a coarsening of BP . We give an example in Appendix A.4 which shows that this is not true.

Finally, we provide two axiomatic characterizations of BP by leveraging other traditional properties. These characterizations leverage the following lemma, which entails that, in order to show that two stable tournament solutions that satisfy LRS are identical, it suffices to show that their roots are contained in each other.

Lemma 3. *Let S and S' be two stable tournament solutions satisfying LRS . Then $\lceil S \rceil \subseteq \lceil S' \rceil$ if and only if $S = S'$.*

Proof. Suppose that $\lceil S \rceil \subseteq \lceil S' \rceil$, and consider any tournament T . We will show that $S(T) \subseteq S'(T)$. If $S'(T) = T$, this is already the case. Otherwise, we have $S'(S'(T) \cup \{a\}) = S'(T)$ for each $a \notin S'(T)$. By definition of the root function, $\lceil S' \rceil(S'(T) \cup \{a\}) = S'(T)$. Since the root function excludes at most one alternative from any tournament, we also have $\lceil S \rceil(S'(T) \cup \{a\}) = S'(T)$ by our assumption $\lceil S \rceil \subseteq \lceil S' \rceil$. Hence $S(S'(T) \cup \{a\}) = S'(T)$ as well. Using $\hat{\gamma}$ of S , we find that $S(T) = S'(T)$. So $S(T) \subseteq S'(T)$ for every tournament T . However, since S and S' satisfy LRS , and therefore choose on average half of the alternatives, we must have $S = S'$.

Finally, if $S = S'$, then clearly $\lceil S \rceil = \lceil S' \rceil$ and so $\lceil S \rceil \subseteq \lceil S' \rceil$. \square

Theorem 9. *BP is the only tournament solution that satisfies stability, composition-consistency, monotonicity, regularity, and LRS .*

Proof. Let S be a tournament solution satisfying the five aforementioned properties. Since S and BP are stable and satisfy LRS , by Lemma 3 it suffices to show that $\lceil S \rceil \subseteq \lceil BP \rceil$. This is equivalent to showing that when $\lceil BP \rceil$ excludes an alternative from a tournament, then $\lceil S \rceil$ excludes the same alternative. In other words, we need to show that when BP excludes exactly one alternative a , then S also only excludes a .

Let T be a tournament in which BP excludes exactly one alternative a . As defined in Section 4.2, $BP(T)$ corresponds to the support of the unique Nash equilibrium of $G(T)$. Laffond et al. (1993b) and Fisher and Ryan (1995) have shown that this support is always of odd size and that the equilibrium weights associated to the alternatives in $BP(T)$ are odd numbers. Hence, using composition-consistency of BP and the fact that the value of a symmetric zero-sum game is zero, T can be transformed into a new (possibly larger) tournament $T_1 = (A, \succ)$ by replacing each alternative except a with a regular tournament of odd order such that $T_1|_{A \setminus \{a\}}$ is regular. Moreover, in T_1 , $|\overline{D}(a)| > \frac{|A|}{2}$.

We will now show that $a \notin S(T_1)$. Since S is monotonic, it suffices to prove this when we strengthen a arbitrarily against alternatives in T_1 until $|\overline{D}(a)| = \frac{|A|+1}{2}$.

Let $X = D(a)$ and $Y = \overline{D}(a)$, and let T_2 be a tournament obtained by adding a new alternative b to T_1 so that $X \succ \{b\} \succ Y$ and $a \succ b$. Note that T_2 is again a regular tournament, so $S(T_2) = A \cup \{b\}$. In particular, $b \in S(T_2)$.

Let $T_3 = (T_2)^b$ be the tournament obtained from T_2 by reversing all edges incident to b . By LRS_{OUT} , we have $b \notin S(T_3)$. If it were the case that $a \in S(T_3)$, then it should remain chosen when we reverse the edge between a and b . However, alternative a in the tournament after reversing the edge is isomorphic to alternative b in T_3 , and so we must

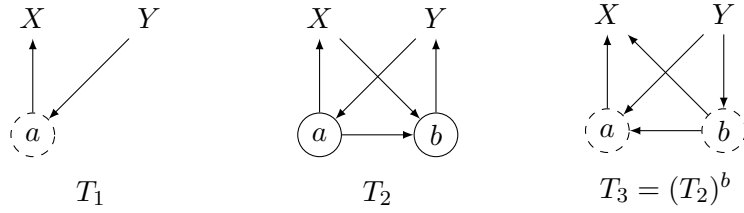


Figure 5: Illustration of the proof of Theorem 9. Circled alternatives are contained in the choice set $S(\cdot)$. Alternatives circled with a dashed line are not contained in the choice set $S(\cdot)$.

have $b \in S(T_3)$, a contradiction. Hence $a \notin S(T_3)$. Since S satisfies stability and thus $\hat{\alpha}$, we also have $a \notin S(T_1)$, as claimed. See Figure 5 for an illustration.

Now, $\hat{\alpha}$ and regularity of S imply that $S(T_1) = S(T_1|_{A \setminus \{a\}}) = A \setminus \{a\}$. Since S satisfies composition-consistency, we also have that S returns all alternatives except a from the original tournament T , completing our proof. \square

Based on Theorems 8 and 9, we obtain another characterization that does not involve LRS and hence only makes use of properties previously considered in the literature.

Corollary 3. *BP is the unique most discriminating tournament solution that satisfies stability, composition-consistency, monotonicity, and regularity.*

Proof. Suppose that a tournament solution S satisfies stability, composition-consistency, monotonicity, and regularity and is as discriminating as BP . By Theorem 5, S satisfies LRS_{IN} . Since S chooses on average half of the alternatives, it satisfies LRS_{OUT} and hence LRS as well. Theorem 9 then implies that $S = BP$. \square

The only previous characterization of BP that we are aware of was obtained by Laslier (1997, Thm. 6.3.10) and is based on a rather contrived property called *Copeland-dominance*. According to Laslier (1997, p. 153), “this axiomatization of the Bipartisan set does not add much to our knowledge of the concept because it is merely a re-statement of previous propositions.” Corollary 3 essentially shows that, for most discriminating stable tournament solutions, Laslier’s Copeland-dominance is implied by monotonicity and regularity.

We now address the independence of the axioms used in Theorem 9.

Remark 4. LRS is not implied by the other properties. In fact, the trivial tournament solution satisfies stability, composition-consistency, monotonicity, and regularity.

Remark 5. Monotonicity is not implied by the other properties. In fact, the tournament solution that returns $BP(\bar{T})$, where \bar{T} is the tournament in which all edges in T are reversed, satisfies stability, composition-consistency, regularity, and LRS .

The question of whether stability, composition-consistency, and regularity are independent in the presence of the other axioms is quite challenging. We can only provide the following partial answers.

Remark 6. Neither stability nor composition-consistency is implied by *LRS*, monotonicity, and regularity. In fact, there is a tournament solution that satisfies *LRS*, monotonicity, and regularity, but violates stability and composition-consistency (see Appendix A.3).

Remark 7. Neither regularity nor composition-consistency is implied by stability and monotonicity. In fact, there is a tournament solution that satisfies stability and monotonicity, but violates regularity and composition-consistency (see Appendix A.4)

Brandt et al. (2016b) brought up the question whether stability implies regularity (under mild assumptions) because all stable tournament solutions studied prior to this paper were regular.¹³ Remark 7 shows that this does not hold without making assumptions that go beyond monotonicity.

Given the previous remarks, it is possible that composition-consistency and regularity are not required for Theorem 9 and Corollary 3. Indeed, our computer experiments have shown that the only stable and monotonic tournament solution satisfying *LRS* for all tournaments of order up to 7 is *BP*. This may, however, be due to the large number of automorphisms in small tournaments and composition-consistency and regularity could be required for larger tournaments. It is also noteworthy that the proof of Theorem 9 only requires a weak version of composition-consistency, where all components are tournaments in which all alternatives are returned due to automorphisms.

Since stability is implied by Samuelson's weak axiom of revealed preference or, equivalently, by transitive rationalizability, Corollary 3 can be seen as an escape from Arrow's impossibility theorem where the impossibility is turned into a possibility by weakening transitive rationalizability and (significantly) strengthening the remaining conditions (see, also, Brandt and Harrenstein, 2011).

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¹³We checked on a computer that the stable tournament solution $\overset{\circ}{TC}$ (see Brandt et al., 2014, 2016a) satisfies regularity for all tournaments of order 17 or less.

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A. Appendix

A.1. Examples for Remark 2

Brandt and Harrenstein (2011, p. 1729) mention that $\hat{\alpha}$ and $\hat{\gamma}$ are independent from each other in the context of general choice functions. Here, we prove that the same holds even in the context of tournament solutions.

Proposition 1. *There exists a tournament solution that satisfies $\hat{\alpha}$, but not $\hat{\gamma}$.*

Proof. Let S be a stable tournament solution. As mentioned in Section 3, $[S]$ satisfies $\hat{\alpha}$. However, it is easily seen that, unless S is trivial, $[S]$ violates $\hat{\gamma}$. Hence, the statement follows from the existence of non-trivial stable tournament solutions (such as BP). \square

Proposition 2. *There exists a tournament solution that satisfies $\hat{\gamma}$, but not $\hat{\alpha}$.*

Proof. Let S be a stable tournament solution. Define the tournament solution S' such that for each tournament $T = (A, \succ)$,

$$S'(T) = \begin{cases} S(T) & \text{if } |A \setminus S(T)| > 1 \\ A & \text{otherwise.} \end{cases}$$

It can be shown that S' satisfies $\widehat{\gamma}$, but may violate $\widehat{\alpha}$. For the latter, let $S = TC$ and consider a transitive tournament $(\{a, b, c\}, \succ)$ such that $a \succ b$, $b \succ c$, and $a \succ c$. By definition, $S'(\{a, b, c\}) = \{a\}$, but $S'(\{a, b\}) = \{a, b\}$. \square

A.2. Examples for Remark 3

BA satisfies local $\widehat{\alpha}$, but $\widehat{BA} = ME$ violates $\widehat{\alpha}$ (Brandt et al., 2016b).

Similarly, there exists a tournament solution S for which \widehat{S} is well-defined, but \widehat{S} is not stable. For a stable tournament solution S , we have by definition that $S = \widehat{S}$ and hence that \widehat{S} is also stable. The following proposition shows that $\widehat{\alpha}$ does not carry over from S to \widehat{S} even if S is simple and \widehat{S} is well-defined.

Proposition 3. *There exists a simple tournament solution S satisfying $\widehat{\alpha}$ such that \widehat{S} is well-defined but \widehat{S} does not satisfy $\widehat{\alpha}$.*

Proof. Let S be the tournament solution that always chooses all alternatives, with two exceptions:

- If the tournament is of order 2, then S chooses only the Condorcet winner.
- If the tournament is the tournament T_4 given in Figure 6, then S chooses alternatives a, b , and c .

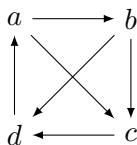


Figure 6: Tournament T_4 .

Clearly, S is simple and satisfies $\widehat{\alpha}$. Since \widehat{S} chooses alternatives a, b , and c from T_4 , but chooses only the Condorcet winner from the transitive tournament of order 3, it does not satisfy $\widehat{\alpha}$.

It remains to show that \widehat{S} is well-defined. One can check that every tournament contains an S -stable set. Suppose for contradiction that some tournament T contains two distinct minimal S -stable sets, which we denote by B and C . Then B and C are also S -stable in $B \cup C$. If B is a singleton, then B is the Condorcet winner in $B \cup C$, which means C cannot be S -stable, a contradiction. Hence both B and C are transitive tournaments of order 3, and $4 \leq |B \cup C| \leq 6$. One can check all the possibilities of $B \cup C$ to conclude that this case is also impossible. \square

For the tournament solution S defined in the proof of Proposition 3, we have that \widehat{S} is not well-defined. Even though no tournament contains two distinct minimal \widehat{S} -stable sets, T_4 does not contain any \widehat{S} -stable set. This example also shows that for a tournament solution S' , \widehat{S}' may fail to be well-defined not because it allows two distinct minimal S' -stable sets in a tournament but because some tournament contains no S' -stable set.

A.3. Example for Remark 6

We show that there exists a tournament solution different from BP that satisfies LRS , monotonicity, regularity, and Condorcet consistency.

To this end, we define a new tournament solution called POS which chooses all alternatives with positive relative degree. More precisely, an alternative is chosen by POS if it dominates strictly more than half of the remaining alternatives, is not chosen if it dominates strictly less than half of the remaining alternatives, and goes to a “tie-break” to determine whether it is chosen if it dominates exactly half of the remaining alternatives.

For tournaments of even size, POS chooses exactly the alternatives that dominate at least (or equivalently, more than) half of the remaining alternatives. Hence we do not need a tie-break for tournaments of even size. The tie-breaking rule for tournaments of odd size $2n + 1$ is as follows: For any (unlabeled) tournament T of order $2n$ and any partition of it into two sets B and C of size n , consider two tournaments T_1 and T_2 of order $2n + 1$. The tournament T_1 contains T and another alternative a that dominates B but is dominated by C , while the tournament T_2 contains T and another alternative a that dominates C but is dominated by B . If T_1 or T_2 is regular, POS chooses a in that tournament and not in the other one. Otherwise, POS arbitrarily chooses a in exactly one of T_1 and T_2 .

Proposition 4. *POS satisfies LRS , monotonicity, and regularity.*

Proof. We need to show that the tie-breaking rule in the definition of POS is well-defined. First, we show that if we perform a local reversal on alternative a , we do not get an isomorphic tournament with alternative a mapped to itself. Indeed, if a were mapped to itself, it would mean that no tournament solution satisfies LRS , which we know is not true since BP satisfies LRS . Secondly, the tournament obtained by performing a local reversal on an alternative in a regular tournament is not regular. Hence we do not obtain a conflict within the tie-breaking rule.

It follows directly from the definition that POS satisfies LRS , monotonicity, and regularity. \square

The tournament T_4 given in Figure 6 shows that POS violates composition-consistency and $\widehat{\alpha}$ (and hence stability).

Interestingly, BP (and all of its coarsenings) always intersect with POS while there exists a tournament for which BA (and all of its refinements such as TEQ and ME) do *not* overlap with POS . This follows from results on the *Copeland value* by Laffond et al. (1993b, 1994).

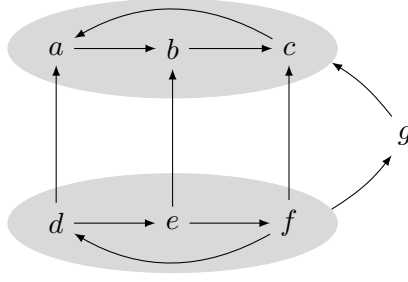


Figure 7: Tournament T_7 . $g \succ \{a, b, c\}$, $\{d, e, f\} \succ g$, and all omitted edges point downwards.

A.4. Example for Remark 7

We construct a tournament solution that satisfies monotonicity and stability, but violates regularity and composition-consistency.

Every tournament solution has to be regular on tournaments of order 5 or less because of non-trivial automorphisms. Consider the tournament T_7 shown in Figure 7. T_7 admits a unique nontrivial automorphism that maps each of the six alternatives in the two 3-cycles to the next alternative in its 3-cycle and maps alternative g to itself.¹⁴

Now, define the simple tournament solution S_7 , which always returns all alternatives unless the tournament is T_7 or it can be modified from T_7 by weakening alternative g . In the latter case, S_7 returns all alternatives except g .

We check that this definition is sound. First, we know that in T_7 , there is no automorphism that maps alternative g to another alternative. When we weaken g , it is the unique alternative with the smallest out-degree, and hence cannot be mapped by an automorphism to another alternative. Now, the alternatives a, b, c form an orbit, and S_7 excludes g whenever it is dominated by d, e, f (and has any dominance relationship to a, b, c). This yields four non-isomorphic tournaments for which S_7 excludes g .

Proposition 5. \widehat{S}_7 satisfies stability and monotonicity.

Proof. First, observe that S_7 trivially satisfies local $\widehat{\alpha}$ because S_7 only excludes an alternative in tournaments of order 7. By virtue of Theorem 3, it therefore suffices to show that \widehat{S}_7 is well-defined.

One can check that every tournament contains an S_7 -stable set. Let T_6 denote the tournament obtained by removing alternative g from T_7 . Suppose for contradiction that there exists a tournament T that contains two distinct minimal S_7 -stable sets, which we denote by B and C . Then B and C are also S_7 -stable in $B \cup C$. Moreover, $T|_B$ must correspond to the tournament T_6 , and each alternative in $C \setminus B$ either has the same dominance relation to B as the alternative g does to T_6 or has a dominance relation that is a weakening of g . The same statement holds for C . We consider the following cases.

Case 1: $10 \leq |B \cup C| \leq 11$. The tournament $T|_B$ has one of its alternatives corresponding to alternatives d, e , and f in Figure 7 outside of $B \cap C$, and this alternative

¹⁴Note that $(T_7)^g$ is the smallest tournament in which BA and UC differ (Brandt et al., 2015).

must dominate all of the alternatives in C . Similarly, there exists an alternative in $C \setminus B$ that dominates all of the alternatives in B . But this implies that some two alternatives dominate each other, a contradiction.

Case 2: $7 \leq |B \cup C| \leq 9$. At least one of the two tournaments $T|_B$ and $T|_C$ must have all of its alternatives corresponding to alternatives d , e , and f in Figure 7 in the intersection $B \cap C$, for otherwise we obtain a contradiction in the same way as in Case 1. Assume without loss of generality that $T|_B$ has its alternatives corresponding to d , e , and f in the intersection. Hence three alternatives in $B \cap C$ that form a cycle dominate the same alternative in C . But this does not occur in T_6 , a contradiction.

It follows from Theorem 4 that \widehat{S}_7 satisfies monotonicity. □

Clearly, \widehat{S}_7 is not regular since it excludes an alternative from the regular tournament T_7 . Moreover, it is not a coarsening of BP since BP selects all of the alternatives in T_7 . Hence we have that stable and monotonic tournament solutions are not necessarily coarsenings of BP .