On the Indecisiveness of Kelly-Strategyproof Social Choice Functions

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The Gibbard-Satterthwaite theorem has shown that only extremely unattractive single-valued social choice functions (SCFs) are strategyproof when there are more than three alternatives. For set-valued SCFs, or so-called social choice correspondences, the situation is less clear. There are miscellaneous—mostly negative—results using a variety of strategyproofness notions and additional requirements. The simple and intuitive notion of Kelly-strategyproofness has turned out to be particularly compelling because it is weak enough to still allow for positive results. For example, the Pareto rule is strategyproof even when preferences are weak, and a number of appealing SCFs are strategyproof for strict preferences. In this paper, we show that, for weak preferences, *(i)* there are no rank-based SCFs that satisfy Pareto-optimality and strategyproofness, *(ii)* all support-based SCFs (which generalize Fishburn's C2 SCFs) that satisfy Pareto-optimality and strategyproofness return at least one most-preferred alternative of every voter and *(iii)* there are no majority-based SCFs that satisfy non-imposition and strategyproofness.

1 Introduction

Gibbard (1973) and Satterthwaite (1975) have shown that only extremely restricted single-valued social choice functions (SCFs) are immune to strategic manipulation: either the range of the SCF is restricted to only two outcomes or the SCF always returns a most-preferred alternative of the same voter. Perhaps the most controversial assumption of the Gibbard-Satterthwaite theorem is that the SCF can only return a single alternative (see, e.g., Gärdenfors, 1976; Kelly, 1977; Barberà, 1977a; Duggan and Schwartz, 2000; Nehring, 2000; Barberà et al., 2001; Ching and Zhou, 2002; Taylor, 2005). This assumption is at variance with elementary fairness conditions such as anonymity and neutrality. For instance, consider an election with two alternatives and two voters such that each alternative is favored by a different voter. Clearly, both alternatives are equally acceptable, but single-valuedness forces us to pick a single alternative based on the preferences only.

We therefore study the manipulability of *set-valued* SCFs (or so-called social choice correspondences). When SCFs return sets of alternatives, there are various notions of strategyproofness, depending on the circumstances under which one set is considered to be preferred to another. When the underlying notion of strategyproofness is sufficiently strong, the negative consequences of the Gibbard-Satterthwaite theorem remain largely intact (see, e.g., Duggan and Schwartz, 2000; Barberà et al., 2001; Ching and Zhou, 2002; Benoît, 2002; Sato, 2014).[1] In this paper, we are concerned with a rather weak—but natural and intuitive—notion of strategyproofness attributed to Kelly (1977). Some rather attractive SCFs have been shown to be strategyproof for this notion when preferences are strict (Brandt, 2015; Brandt et al., 2016). These include the top cycle, the uncovered set, the minimal covering set, and the essential set. However, when preferences are weak, these results break down and strategyproofness under these assumptions is generally not well understood.

Feldman (1979) has shown that the Pareto rule is strategyproof under these assumptions. Moreover, the omninomination rule and the intersection of the Pareto rule and the omninomination rule are strategyproof as well (Brandt et al., 2020, Remark 1). These results are encouraging because they rule out impossibilities using Pareto-optimality and other weak properties.[2] In the context of strategic abstention (i.e., manipulation by deliberately abstaining from an election), even more positive results can be obtained. Brandl et al. (2019) have shown that all of the above mentioned SCFs that are strategyproof for strict preferences are immune to strategic abstention even when preferences are weak.

A number of negative results were shown for rather restricted classes of SCFs. Kelly (1977) and Barberà (1977a) have shown independently that there is no strategyproof SCF that satisfies quasi-transitive rationalizability. However, this result lacks significance because quasi-transitive rationalizability is almost prohibitive on its own (see, e.g., Mas-Colell and Sonnenschein, 1972).[3] In subsequent work by MacIntyre and Pattanaik (1981) and Bandyopadhyay (1983), quasi-transitive rationalizability has been replaced with weaker conditions such as minimal binariness or quasi-binariness, which are still very demanding and violated by most SCFs. Barberà (1977b) has shown that positively responsive SCFs fail to be strategyproof under mild assumptions. However, positively responsive SCFs are

[1] We refer to Barberà (2010) and Brandt et al. (2020) for a more detailed overview over this extensive stream of research.

[2] For example, Brandt et al. (2020) have shown that Pareto-optimality is incompatible with anonymity and a notion of strategyproofness that is slightly stronger than Kelly's.

[3] This is acknowledged by Kelly (1977) who writes that "one plausible interpretation of such a theorem is that, rather than demonstrating the impossibility of reasonable strategy-proof social choice functions, it is part of a critique of the regularity [rationalizability] conditions."

almost always single-valued and of all commonly considered SCFs only Borda's rule satisfies this criterion. Taylor (2005, Th. 8.1.2) has shown that every SCF that returns the set of all weak Condorcet winners whenever this set is non-empty fails to be strategyproof. This result was strengthened by Brandt (2015), who showed that every SCF that returns a (strict) Condorcet winner whenever one exists fails to be strategyproof. More recently, Brandt et al. (2020) have shown with the help of computers that every Pareto-optimal SCF whose outcome only depends on the pairwise majority margins can be manipulated.

In this paper, we consider three broad classes of SCFs, which cover most SCFs that are commonly considered in the literature. These classes are *rank-based* SCFs (which include all scoring rules), *support-based* SCFs (which generalize Fishburn's C2 SCFs) and *majority-based* SCFs (which are equivalent to Fishburn's C1 SCFs). For the first two classes, we show that Pareto-optimality and strategyproofness imply that *every voter is a nominator*, i.e., the resulting choice sets contain at least one most-preferred alternative of every voter. In the case of ranked-based SCFs, this entails an impossibility whereas for support-based SCFs it demonstrates a high degree of indecisiveness. For majority-based SCFs, we show an even more restrictive result: there are no such SCFs that satisfy both strategyproofness and non-imposition. Similar to the other results, we derive this impossibility from the observation that voters are no nominators for such SCFs. Even though these results are rather negative, they are important to improve our understanding of strategyproof SCFs. Much more positive results are obtained by making minuscule adjustments to the assumptions such as restricting preferences to strict preferences, weakening the underlying notion of strategyproofness, or replacing strategic manipulation with strategic abstention. In these cases, few SCFs such as the top cycle, the uncovered set, the minimal covering set, and the essential set constitute appealing positive examples.

2 The Model

Let $N = \{1, \dots, n\}$ denote a finite set of voters and let $A = \{a, b, \dots\}$ denote a finite set of m alternatives. Moreover, let $[x \dots y]= \{i \in N \colon x \leq i \leq y\}$ denote the subset of voters from x to y and note that $[x \dots y]$ is empty if $x > y$. Every voter $i \in N$ is equipped with a *weak preference relation* $\succsim_i$, i.e., a complete, transitive and reflexive binary relation on A. We denote the strict part of $\succsim_i$ by $\succ_i$, i.e., $x \succ_i y$ if $x \succsim_i y$ and $y \not\succsim_i x$ and the indifference part by $\sim_i$, i.e., if $x \sim_i y$ if $x \succsim_i y$ and $y \succsim_i x$. We compactly represent a preference relation as a comma-separated list, where sets of alternatives express indifferences. For example, $x \succ y \sim z$ is represented by $x, \{y, z\}$. Furthermore, we call a preference relation $\succsim$ strict if it is equal to its strict part $\succ$. The the set of weak preference relations on A is called $\mathcal{R}$. A *preference profile* $R \in \mathcal{R}^n$ is a n-tuple containing the preference relation of every voter $i \in N$. When writing down preference profiles, we indicate the set of voters who submit a specific preference relation directly before the preference. For instance, $[x \dots y]$: a, b, c means that all voters $i \in [x \dots y]$ prefer a to b and b to c. We omit the brackets for singleton sets. Our goal is to determine the set of winners for every preference profile. For this task, we use *social choice functions (SCFs)*, or so-called social choice correspondences, which map from the set of preference profiles to a non-empty set of alternatives, i.e., $f : \mathcal{R}^n \mapsto 2^A \setminus \emptyset$.

The mere mathematical description of SCFs is so general that it allows for rather undesirable functions. We now introduce a number of axioms in order to narrow down the set of SCFs.

The most basic fairness condition is anonymity, which requires that all voters are treated equally: an SCF f is *anonymous* if $f(R) = f(R')$ for all preference profiles R, R' for which there is a permutation $\pi : N \to N$ such that $R_i = R'_{\pi(i)}$ for all $i \in N$.

Perhaps one of the most prominent and least controversial axioms in economic theory is Pareto-optimality, which is based on the notion of Pareto-dominance: an alternative x *Pareto-dominates* another alternative y if $x \succsim_i y$ for all $i \in N$ and there is a voter $j \in N$ with $x \succ_j y$. An alternative is *Pareto-optimal* if it is not Pareto-dominated by any other alternative. This idea leads to the *Pareto rule*, which returns all Pareto-optimal alternatives. An SCF f is Pareto-optimal if it only return Pareto-optimal alternatives for all preference profiles R. An axiom that is closely related to Pareto-optimality is *near unanimity*, as introduced by Benoît (2002). Near unanimity requires that $f(R) = \{x\}$ for all alternatives $x \in A$ and preference profiles R in which $n - 1$ voters uniquely top-rank x. A natural weakening of these axioms is *non-imposition*. This axiom requires from an SCF f that for every alternative $a \in A$, there is a profile R such that $f(R) = \{a\}$.

While the last three axioms make reference to the entire profile of preferences, there also concepts that only refer to the preferences of a single voter. One such concept that is particularly important in our context is that of a nominator. A voter is a *nominator* if $f(R)$ always contains one of his most preferred alternatives. A nominator is a weak dictator in the sense that he can always force an alternative into the choice set by declaring it his uniquely most preferred one.

2.1 Rank-Basedness, Support-Basedness, and Majority-basedness

In this section, we introduce three classes of anonymous SCFs that capture most of the SCFs that are commonly studied in the literature: rank-based SCFs, support-based SCFs, and majority-based SCFs. The basic idea of rank-basedness is that voters assign ranks to the alternatives and that an SCF should only depend on the ranks of the alternatives, but not on which voter assigns which rank to an alternative. In order to formalize this idea, we first need to define the rank of an alternative. In the case of strict preferences, this is straightforward. The rank of alternative x according to $\succsim_i$ is $\bar{r}(\succsim_i, x) = |\{y \in A \colon y \succsim_i x\}|$ (Laslier, 1996). In contrast, there are multiple possibilities how to define the rank in the presence of ties. We will define a very weak notion of ranked-basedness for weak preferences. This will only make our results stronger. To this end, define the *rank tuple* of x with respect to $\succsim_i$ as

$$r(\succsim_i, x) = (\bar{r}(\succ_i, x), \bar{r}(\sim_i, x)) = (|\{y \in A\colon y \succ_i x\}|, |\{y \in A\colon y \sim_i x\}|).$$

The rank tuple contains more information than many other simple generalizations of the rank and therefore, it leads to a more general definition of rank-basedness. Next, we define the *rank vector* of an alternative a which contains the rank tuple of a with respect to every voter in increasing lexicographic order, i.e., $r^*(R, x) = (r(\succsim_{i_1}, x), r(\succsim_{i_2}, x), \dots, r(\succsim_{i_n}, x))$ where $\bar{r}(\succ_{i_j}, x) \leq \bar{r}(\succ_{i_{j+1}}, x)$ and if $\bar{r}(\succ_{i_j}, x) = \bar{r}(\succ_{i_{j+1}}, x)$, then $\bar{r}(\sim_{i_j}, x) \leq \bar{r}(\sim_{i_{j+1}}, x)$. Finally, the *rank matrix* $r^*(R)$ of the preference profile R contains the rank vectors as rows. An SCF f is called *rank-based* if $f(R) = f(R')$ for all preference profiles $R, R' \in \mathcal{R}^n$ with $r^*(R) = r^*(R')$. The class of rank-based SCFs contains many popular SCFs such as all scoring rules or the omninomination rule, which returns all top-ranked alternatives.

A similar line of thought leads to support-basedness, which is based on the pairwise *support* of an alternative x against another one y. Here, support refers to the number of voters who strictly prefer x to y, i.e., $s_{xy}(R) = |\{i \in N\colon x \succ_i y\}|$. Next, we define the *support matrix* $s^*(R) = (s_{xy}(R))_{x,y \in A}$ which contains the supports for all pairs of alternatives. Finally, an SCF f is *support-based* if it yields $f(R) = f(R')$ for all preference profiles $R, R' \in \mathcal{R}^n$ with $s^*(R) = s^*(R')$. Note that support-basedness generalizes Fishburn's C2 to weak preferences (Fishburn, 1977). Hence, many well-known SCFs such as Borda's rule, Kemeny's rule, the Simpson-Kramer rule, Nanson's rule, Schulze's rule and the essential set are support-based.

An important subclass of support-based SCFs are majority-based SCFs, which are merely based on the majority relation R_M of a profile. The majority relation R_M of the profile R is defined as follows $R_M = \{(a, b) \in A^2\colon s_{ab}(R) \geq s_{ba}(R)\}$. Finally, an SCF f is majority-based if $f(R) = f(R')$ for all preference profiles R and R' with $R_M = R'_M$. Fishburn (1977) refers to majority-based SCFs as C1 functions. Many established SCFs such as the top cycle, the uncovered set, the minimal covering set, and the bipartisan set are majority-based.

2.2 Strategyproofness

One of the central problems in social choice theory is manipulation: voters may lie about their true preferences in order to obtain a more preferred outcome. For single-valued SCFs, it is clear what constitutes a more preferred outcome. In the case of set-valued SCFs, there are various ways to define manipulation depending on what is assumed about the voters' preferences over sets of alternatives. Here, we will only make simple and uncontroversial assumption: voter i weakly prefers set X to set Y, denoted by $X \succsim_i Y$, iff $x \succsim y$ for all $x \in X, y \in Y$. Consequently, the strict part of this preference extension is

$$X \succ_i Y \text{ iff } x \succsim_i y \text{ for all } x \in X, y \in Y \text{ and there is } x \in X, y \in Y \text{ with } x \succ_i y.$$

We thus obtain the following definitions of manipulability and strategyproofness. An SCF f is *manipulable* if there are preference profiles R, R' and a voter $i \in N$ such that $\succsim_j = \succsim_j'$ for all $j \in N \setminus \{i\}$ and $f(R') \succ_i f(R)$. Moreover, f is *strategyproof* if it is not manipulable.

These assumptions can, for example, be justified by considering a randomized tie-breaking procedure (a lottery) that is used to select a single alternative from every set of alternatives returned by the SCF. We then have that $X \succ_i Y$ iff all lotteries with support X yield strictly more expected utility than all lotteries with support Y for all utility functions that are ordinally consistent with $\succsim_i$ (see, e.g., Gärdenfors, 1979; Brandt et al., 2020).

3 Results

The unifying theme of our results is that strategyproofness requires a large degree of indecisiveness. In more detail, we show that very voter is a nominator for all ranked-based and support-based SCFs that satisfy Pareto-optimality and strategyproofness. This cannot be true for majority-based SCFs that satisfy strategyproofness and non-imposition, and we therefore derive an impossibility for such SCFs.

In order to prove the claim on rank-based and support-based SCFs, we focus on the contraposition, i.e., we assume that there is such a function f and and a voter $i \in N$ who is no nominator for f. Our first lemma shows that these assumptions imply that f satisfies near unanimity.

Lemma 1. *Let f be an anonymous, Pareto-optimal, and strategyproof SCF that is defined for $m \geq 3$ alternatives and $n \geq 2$ voters. If there is a voter $i \in N$ who is no nominator for f, then f satisfies near unanimity.*

In the interest of space, we only discuss the key ideas for proving this lemma and defer its formal proof to the appendix. First, we assume that there is an SCF f that satisfies all required axioms, and voter i is no nominator for f. This means that there is a profile R such that $f(R)$ does not contain any of voter i's most preferred alternatives. In a first step, we derive a profile R' in which $f(R') = \{a\}$ but a is not among the most preferred alternatives of voter i. This is helpful because strategyproofness becomes much more restrictive when there is only a single winner. Based on the profile R', we derive that $n-1$ voters can ensure that a is the unique winner by submitting it as a uniquely most-preferred alternative. Finally, we show that f satisfies near unanimity by generalizing this observation from a single alternative to all alternatives.

Lemma 1 shows that, under the given assumptions, indecisiveness for a single preference profile of a particularly simple type entails a large degree of indecisiveness for the entire domain of preference profiles. More precisely, if an alternative is not chosen uniquely even if $n-1$ agents uniquely prefer it the most, then all voters are nominators.

Remark 1. A weakened variant of Lemma 1 is also true if the considered SCF f violates anonymity. In this case, an alternative is the unique winner if all voters $j \in N \setminus \{i\}$ prefer it uniquely the most. Thus, requiring the absence of nominators for a strategyproof and Pareto-optimal SCF implies also near unanimity.

Remark 2. Some steps in the proof of Lemma 1 can be generalized from a single voter to a group of voters: if there is a profile R' such that $f(R') = \{a\}$ and there is a group of voters $I \subseteq N$ such that every voter in I prefers b the most and b strictly to a in the profile R', we can construct a profile R'' such that $f(R'') = \{a\}$, all voter in $N \setminus I$ prefer a uniquely the most and every voter in I prefers a uniquely the least. Based on this profile R'', one can show that an alternative is the unique winner for f if all voters in $N \setminus I$ prefer this alternative uniquely the most.

Remark 3. Remarkably, many impossibility results rule out that every voter is a nominator. For instance, Duggan and Schwartz (2000), Benoît (2002), and Sato (2008) invoke axioms prohibiting that every voter is a nominator. Moreover, a crucial step in the computer-generated proofs of Theorem 3.1 by Brandl et al. (2018) and Theorem 1 by Brandt et al. (2020) is to show that there is some voter who is no nominator. Lemma 1 gives intuition about why these assumptions and observations are important.

3.1 Rank-Based SCFs

In this section, we prove that there is no rank-based SCF that satisfies Pareto-optimality and strategyproofness. This result follows from the observation that Pareto-optimality, strategyproofness, and rank-basedness require that every voter is a nominator, but Pareto-optimality and rank-basedness do not allow for such SCFs.

It is possible to show the theorem—as well as all further theorems—by induction proofs where completely indifferent voters and universally bottom-ranked alternatives are used to generalize the statement to arbitrarily many voters and alternatives (see, e.g., Brandl et al., 2018, 2019; Brandt et al., 2020). Instead, we prefer to give universal proofs for any number of voters and alternatives to stress the robustness of the respective constructions. As a consequence, our proofs even hold when restricting the domain of admissible preferences by prohibiting completely indifferent voters or large numbers of Pareto-dominated alternatives.

Theorem 1. *There is no rank-based SCF that satisfies Pareto-optimality and strategyproofness if $m \geq 4$ and $n \geq 3$.*

Proof. Assume for contradiction that there is a rank-based SCF f that satisfies strategyproofness and Pareto-optimality and that is defined for fixed numbers of voters $n \geq 3$ and alternatives $m \geq 4$. We derive a contradiction to this assumption by proving two claims: on the one hand, there is a voter who is no nominator for f. On the other hand, the combination of strategyproofness, Pareto-optimality and rank-basedness requires that every voter is a nominator. These two claims contradict each other and therefore f cannot exist.

Claim 1: Not every voter is a nominator for f
First, we prove that there is a voter who is no nominator for f. Consider therefore the following three profiles in which $X = A \setminus \{a, b, c, d\}$.

R^1:	1: $\{a,b\}, X, \{c,d\}$	2: $\{c,d\}, X, \{a,b\}$	$[3 \dots n]$: $a, \{b,c,d\}, X$
R^2:	1: $\{a,c\}, X, \{b,d\}$	2: $\{b,d\}, X, \{a,c\}$	$[3 \dots n]$: $a, \{b,c,d\}, X$
R^3:	1: $\{a,d\}, X, \{b,c\}$	2: $\{b,c\}, X, \{a,d\}$	$[3 \dots n]$: $a, \{b,c,d\}, X$

It can be easily verified that $r^*(R^1) = r^*(R^2) = r^*(R^3)$ and that a Pareto-dominates b in R^1, c in R^2 and d in R^3. This means that $f(R^1) = f(R^2) = f(R^3) \subseteq \{a\} \cup X$ because of rank-basedness and Pareto-optimality. Consequently, voter 2 is no nominator for f.

Claim 2: Every voter is a nominator for f
Assume for contradiction that a voter is no nominator for f and consider the profiles $R^{k,1}$ and $R^{k,2}$ for $k \in \{1, \dots, n\}$.

$R^{k,1}$:	1: $\{c,d\}, b, X, a$	$[2 \dots k]$: $\{a,b\}, c, d, X$	$[k{+}1 \dots n]$: a, b, c, d, X
$R^{k,2}$:	1: $\{b,d\}, c, X, a$	$[2 \dots k]$: $\{a,b\}, c, d, X$	$[k{+}1 \dots n]$: a, b, c, d, X

We prove by induction on $k \in \{1, \dots, n\}$ that $f(R^{k,1}) = f(R^{k,2}) = \{a\}$. The case $k = n$ yields a contradiction to Pareto-optimality as every voter prefers b to a in R^n.

The base case $k = 1$ follows because $n - 1$ voters prefer a uniquely the most in both R^1 and $\hat{R}^1$. Therefore, Lemma 1 implies that $f(R^{k,1}) = f(R^{k,2}) = \{a\}$. Assume now that the claim is true for some fixed $k \in \{1, \dots, n-1\}$.

By induction and strategyproofness, $f(R^{k+1,1}) \subseteq \{a, b\}$ since otherwise voter $k + 1$ can manipulate by switching back to $R^{k,1}$. Next, we derive the profile $R^{k,3}$ from $R^{k,2}$ by assigning voter $k + 1$ the preference $\{a, c\}, b, d, X$. Formally,

$R^{k,3}$: 1: $\{b,d\}, c, X, a$ $[2 \dots k]$: $\{a,b\}, c, d, X$ $k{+}1$: $\{a,c\}, b, d, X$ $[k{+}2 \dots n]$: a, b, c, d, X

The induction hypothesis entails that $f(R^{k,2}) = \{a\}$ and therefore, strategyproofness implies that $f(R^{k,3}) \subseteq \{a, c\}$; otherwise, voter $k + 1$ could manipulate by switching back to $R^{k,2}$. Next, we apply rank-basedness to conclude that $f(R^{k+1}) = \{a\}$ as $r^*(R^{k+1,1}) = r^*(R^{k,3})$. Finally, $R^{k+1,2}$ evolves from R^{k+1} by having voter 1 change his preferences. As a is the uniquely least preferred alternative of this voter, strategyproofness implies that $f(R^{k+1}, 2) = \{a\}$ as any other outcome benefits voter 1. $\square$

Remark 4. The axioms used in Theorem 1 are independent: the Pareto rule satisfies all axioms except rank-basedness, the trivial SCF which returns always all alternatives only violates Pareto-optimality and Borda's rule only violates strategyproofness.[4] Furthermore, the bounds on n and m are tight if considered simultaneously because the Pareto rule is rank-based if $m \leq 3$ and $n \leq 3$, and if $m \leq 4$ and $n \leq 2$. In contrast, the theorem is also true if $m \geq 5$ and $n = 2$.

Remark 5. Claim 2 in the proof of Theorem 1 even holds for an arbitrary number of voters $n \geq 1$ and an arbitrary number of alternatives $m \geq 4$. This is easily seen as the case $n = 1$ is trivial due to Pareto-optimality and the constructions in the proof can be adapted for $n = 2$. In contrast, Claim 1 requires that $n \geq 3$ or $m \geq 5$.

Remark 6. Note that we assume throughout the proof of Theorem 1 that all voters are indifferent between all alternatives in $X = A \setminus \{a, b, c, d\}$. This assumption is not required and only used for the sake of a simple notation. In fact, the alternatives in X can be ordered arbitrarily and every voter can have different preferences over these alternatives. The two important points for the proof are that the relation between the alternatives in X and the alternatives outside of X is not changed and that the alternatives in X are not reordered during steps that rely on rank-basedness.

Remark 7. Theorem 1 is only an impossibility due to the bad compatibility of rank-basedness and Pareto-optimality. Therefore, we do not interpret this theorem as classical impossibility showing that there are no strategyproof SCFs. Instead, the main consequence of strategyproofness is that every voter is a nominator for a rank-based and strategyproof SCF. This can be easily seen as variants of this claim are even true when replacing Pareto-optimality with weaker axioms.

Remark 8. Theorem 1 is also true if we weaken rank-basedness such that $f(R) = f(R')$ if $r^*(R) = r^*(R')$ and only two voters are allowed to modify their preferences by only renaming two alternatives. This technical condition is significantly less restrictive because rank-basedness allows that multiple voters change their preferences simultaneously.

Remark 9. Theorem 1 also holds if we define rank-basedness based on a more general definition of the rank. The only real restriction on the rank function r is independence of the naming of other alternatives, i.e., $r(\succsim_i, a) = r(\succsim_i', a)$ for all preferences $\succsim_i$, $\succsim_i'$ that only differ in the naming of alternatives in $A \setminus \{a\}$. This observation follows directly from the proof of Theorem 1.

Remark 10. If we require strict preferences, Theorem 1 is no longer true. For instance, the omninomination rule, which returns every alternative that is top-ranked by some voter, satisfies all required axioms for arbitrary numbers of voters and alternatives. Even Claim 2

[4] We define Borda's rule as the SCF that chooses all alternative that maximize $m \cdot n - \sum_{i \in N} \bar{r}(\succ_i, a)$. This definition agrees with the standard notation used in literature on the strict domain and generalizes it to the weak domain.

in the proof no longer holds if preferences are strict as the 2-Plurality rule,[5] which chooses the two alternatives that are first-ranked by the most voters, is rank-based, Pareto-optimal and strategyproof and no voter is a nominator for this rule.

3.2 Support-Based SCFs

It is not possible to replace rank-basedness with support-basedness in Theorem 1 since the Pareto rule is strategyproof, Pareto-optimal, and support-based and always chooses one of the most preferred alternatives of every voter. As consequence of this observation, Claim 1 in the proof of Theorem 1 cannot be true for support-based SCFs. Nevertheless, we can show that Claim 2 in this proof is true for such SCFs, i.e., we show that every voter is a nominator for every support-based SCF that satisfies Pareto-optimality and strategyproofness.

Theorem 2. *In every support-based SCF that satisfies Pareto-optimality and strategyproofness, every voter is a nominator if $m \geq 3$.*

Proof. Let f be a support-based SCF satisfying Pareto-optimality and strategyproofness for fixed numbers of voters $n \geq 1$ and alternatives $m \geq 3$. The theorem follows immediately from Pareto-optimality for $n = 1$ as only the most preferred alternatives of the single voter are Pareto-optimal. Moreover, Lemma 1 proves the theorem for $n = 2$: if a voter is no nominator, a single voter can determine the choice set. However, this means that $f(R) = \{a\}$ and $f(R) = \{b\}$ are simultaneously true if voter 1 prefers a uniquely the most and voter 2 prefers b uniquely the most.

Therefore, we focus on the case $n \geq 3$ and assume for contradiction that a voter is no nominator for f. We derive from this assumption by an induction on $k \in \{1, \dots, n-1\}$ that $n - k$ voters can determine a unique winner by uniquely first-ranking it. This contradicts Lemma 1 if $k = n - 1$ as a single voter can determine the unique winner in this case even if all other voters agree that another alternative is the best one.

The induction basis $k = 1$ follows directly from Lemma 1 as this lemma states that f satisfies near unanimity. Next, we assume that our claim holds for a fixed $k \in \{1, \dots n-2\}$ and prove that also $n-(k+1)$ voters can determine the winner uniquely. We focus for this only on three alternatives a, b, c and on a certain partition of the voters. This is possible as the induction hypothesis allows us to exchange the roles of the alternatives without affecting the proof and support-basedness allows to reorder the voters. Thus, consider the profile $R^{k,1}$ in which $X = A \setminus \{a, b, c\}$.

$$R^{k,1} \qquad [1 \dots k]\colon\ a, X, c, b \qquad k{+}1\colon\ c, X, b, a \qquad [k{+}2 \dots n]\colon\ a, b, X, c$$

[5] For formally defining this rule, we introduce the plurality score $PL(a, R)$ which counts how often an alternative is first-ranked in the profile R. Furthermore, let a denote the alternative with the second highest plurality score. Then, the 2-Plurality rule chooses all alternatives x with $PL(x, R) \geq PL(a, R)$ and $PL(x, R) > 0$.

Our induction hypothesis implies that $f(R^{k,1}) = \{a\}$ as even more than $n-k$ voters prefer a uniquely the most.

Next, we aim to switch the order over a and b for the voters $i \in [k{+}2 \dots n]$. This is achieved by the repeated application application of the following steps explained for voter $k+2$. First, voter $k+2$ changes his preference to $\{a,b\}, c, X$ to derive the profile $R^{k,2}$. Since a subset of $\{a,b\}$ was chosen before this modification, strategyproofness implies that $f(R^{k,2}) \subseteq \{a,b\}$ as otherwise, voter $k+2$ can manipulate by reverting this modification. Next, we use support-basedness to exchange the preferences of voter $k+1$ and $k+2$ over a and b. This leads to the profile $R^{k,3}$ and support-basedness implies that $f(R^{k,3}) = f(R^{k,2}) \subseteq \{a,b\}$. As a subset of the least preferred alternatives of voter $k+1$ is chosen for $R^{k,3}$, strategyproofness implies that this voter cannot make another alternative win by manipulating. Thus, he can switch back to his original preference to derive $R^{k,4}$ and the fact that $f(R^{k,4}) \subseteq \{a,b\}$.

$R^{k,2}$: $[1 \dots k]$: a, X, c, b $\quad$ $k{+}1$: c, X, b, a $\quad$ $k{+}2$: $\{a,b\}, X, c$ $\quad$ $[k{+}3 \dots n]$: a, b, X, c
$R^{k,3}$: $[1 \dots k]$: a, X, c, b $\quad$ $k{+}1$: $c, X, \{a,b\}$ $\quad$ $k{+}2$: b, a, X, c $\quad$ $[k{+}3 \dots n]$: a, b, X, c
$R^{k,4}$: $[1 \dots k]$: a, X, c, b $\quad$ $k{+}1$: c, X, b, a $\quad$ $k{+}2$: b, a, X, c $\quad$ $[k{+}3 \dots n]$: a, b, X, c

It is easy to see that we can repeat the last steps for every voter $i \in [k{+}2 \dots n]$. This process results in the profile $R^{k,5}$ and shows that $f(R^{k,5}) \subseteq \{a,b\}$. Moreover, consider the profile $R^{k,6}$ derived from $R^{k,5}$ by letting voter $k+1$ make b his best alternative. As $n-k$ voters prefer b uniquely the most in $R^{k,5}$, the induction hypothesis entails that $f(R^{k,6}) = \{b\}$. This means that voter $k+1$ can manipulate by switching from $R^{k,5}$ to $R^{k,6}$ if $f(R^{k,5}) = \{a\}$ or $f(R^{k,5}) = \{a,b\}$. Consequently, $f(R^{k,5}) = \{b\}$ is the only valid choice set for $R^{k,5}$.

$R^{k,5}$: $[1 \dots k]$: a, X, c, b $\quad$ $k{+}1$: c, X, b, a $\quad$ $[k{+}2 \dots n]$: b, a, X, c
$R^{k,6}$: $[1 \dots k]$: a, X, c, b $\quad$ $k{+}1$: b, a, X, c $\quad$ $[k{+}2 \dots n]$: b, a, X, c

So far, we have found a profile in which b is chosen when only $n-(k+1)$ voters prefer it uniquely the most. Next, we show that b is always the unique winner if the voters in $[k{+}2 \dots n]$ prefer it uniquely the most. Therefore, consider the profile $R^{k,7}$ which is derived from $R^{k,5}$ by letting the voters $i \in [1 \dots k]$ subsequently change their preference to c, X, a, b. As $f(R^{k,5}) = \{b\}$ and b is the worst alternative for these voters, strategyproofness implies that $f(R^{k,7}) = \{b\}$.

$R^{k,7}$: $[1 \dots k]$: c, X, a, b $\quad$ $k{+}1$: c, X, b, a $\quad$ $[k{+}2 \dots n]$: b, a, X, c

As last step, we change the preferences of voter $k+1$ such that b is his least preferred alternative. For this, we let all voters $i \in [k{+}2 \dots n]$ subsequently change their preference to b, c, X, a. This modification results in the profile $R^{k,8}$ and strategyproofness implies that $f(R^{k,8}) = \{b\}$. Moreover, observe that all alternatives in $A \setminus \{b,c\}$ are Pareto-dominates by c in ^{k,8} and therefore, voter $k+1$ can now make b his least preferred alternative. This step results in the profile $R^{k,9}$ for which Pareto-optimality and strategyproofness imply that $f(R^{k,9}) = \{b\}$.

$$\begin{array}{llll} R^{k,8}\colon & [1\dots k]\colon\ c, X, a, b & k{+}1\colon\ c, X, b, a & [k{+}2\dots n]\colon\ b, c, X, a \\ R^{k,9}\colon & [1\dots k]\colon\ c, X, a, b & k{+}1\colon\ c, X, a, b & [k{+}2\dots n]\colon\ b, c, X, a \end{array}$$

Finally, observe that the voters $i \in [1\dots k{+}1]$ can change their preferences in $R^{k,9}$ arbitrarily without affecting the choice set, and the voters $i \in [k{+}2\dots n]$ can reorder all alternatives in $A \setminus \{b\}$ arbitrarily because of strategyproofness. Thus, b is always the unique winner if all voters $i \in [k+2 : n]$ prefer b uniquely the most. Moreover, interchanging the roles of alternatives and reordering the voters shows that every alternative is chosen if it is uniquely first-ranked by $n-(k+1)$ voters. This completes the induction step and shows that every voter is a nominator for a support-based SCF that satisfies strategyproofness and Pareto-optimality.

□

Remark 11. All axioms used in Theorem 2 are required as the following SCFs show. Every constant SCF satisfies support-basedness and strategyproofness, and violates Pareto-optimality and that every voter is a nominator. The SCF that chooses the lexicographic smallest Pareto-optimal alternative satisfies Pareto-optimality and support-basedness but violates strategyproofness and that every voter is a nominator. For defining an SCF that satisfies Pareto-optimality and strategyproofness but violates support-basedness and that every voter is a nominator, we define a transitive dominance relation by slightly strengthening Pareto-dominance by allowing additionally that an alternative a that is among the most preferred alternatives of $n-1$ voters can dominate another alternative b, even if a single voter prefers b strictly to a. Therefore, we say that an alternative a *dominates* alternative b if a Pareto-dominates b or $n-1$ voters prefer a the most while $s_{ab}(R) \geq 2$ and $s_{ba}(R) \leq 1$. The SCF f^* that chooses all maximal elements with respect to this dominance relation satisfies all required properties. Also the bound on m is tight as the majority rule satisfies all axioms if $m = 2$ but no voter is a nominator for this SCF.

Remark 12. It should be stressed that Theorem 2 is no impossibility result because the Pareto rule satisfies all required axioms and every voter is a nominator for this SCF. Moreover, we have verified using computer-aided techniques that there are also other SCFs that satisfy all required axioms.

Remark 13. If we consider SCFs for variable electorates and strengthen support-basedness to require that $f(R) = f(R')$ for all R, R' with $s^*(R) = s^*(R')$, regardless of the size of the electorates of R and R', then we can characterize the Pareto rule as the only rule that satisfies strategyproofness, support-basedness, neutrality and Pareto-optimality if $3 \leq m \leq 4$. This claim follows by enumerating all profiles on three voters and showing for each profile that, if not all Pareto-optimal alternatives are chosen, there is a profile with the same choice set and an unchosen Pareto-optimal alternative is uniquely first-ranked by a voter. This contradicts Theorem 2 and therefore, the characterization follows for $n = 3$. Moreover, all profiles on more than 3 voters can be decomposed in a profile on 3 voters where a specific alternative is Pareto-optimal and another profile. Therefore, the characterization generalizes to more voters. For more alternatives, this characterization no longer holds.

Remark 14. It has been shown by Brandt et al. (2020) that there is no pairwise, Pareto-optimal and strategyproof SCF if $m \geq 3$ and $n \geq 3$.[6] This result can be obtained as a corollary of Theorem 2 with the help of two preference profiles R^1 and R^2 given below. In these profiles, we let $X = A \setminus \{a, b\}$.

R^1:	1: $\{a,b\}, X$	2: $\{a,b\}, X$	$[3 \dots n]$: a, b, X
R^2:	1: b, a, X	2: a, b, X	$[3 \dots n]$: a, b, X

Every Pareto-optimal SCF f satisfies that $f(R^1) = \{a\}$. Moreover, pairwiseness implies that $f(R^1) = f(R^2)$. As voter 1 prefers b uniquely the most in R^2, it follows that not every voter can be a nominator for every pairwise SCF that satisfies Pareto-optimality. Thus, Theorem 2 implies that these SCFs cannot be strategyproof.

Remark 15. Theorem 2 implies an impossibility for $m \geq 4$ and $n \geq 4$ if we strengthen Pareto-optimality to *SD*-efficiency (aka ordinal efficiency, see Bogomolnaia and Moulin, 2001). This result follows by considering the preference profiles R^1 and R^2 shown in the sequel. In this profile, $X = A \setminus \{a, b, c, d\}$ and all voters $i \in [5 \dots n]$ are assumed to be indifferent between all alternatives.

R^1:	1: a, c, b, d, X	2: a, d, b, c, X	3: b, c, a, d, X	4: b, d, a, c, X
R^2:	1: a, b, c, d, X	2: a, b, d, c, X	3: c, b, a, d, X	4: d, b, a, c, X

Next, consider a support-based and *SD*-efficient SCF f. *SD*-efficiency implies that $f(R^1) \cap X = \emptyset$ and that either $c \notin f(R^1)$ or $d \notin f(R^1)$. Moreover, support-basedness implies that $f(R^1) = f(R^2)$ and therefore, a voter is no nominator as voter 3 or voter 4 does not obtain his uniquely most preferred alternative in R^2. Thus, Theorem 2 implies that f is not strategyproof, which shows the incompatibility of strategyproofness, support-basedness and *SD*-efficiency. Note that there is also no rank-based, strategyproof, and *SD*-efficient SCF if $m \geq 4$ and $n \geq 3$ because of Theorem 1. This leads to the interesting and challenging open question whether there is an anonymous, SD-efficient, and strategyproof SCF.

[6] An SCF f is pairwise if $f(R) = f(R')$ for all preference profiles $R, R' \in \mathcal{R}^n$ with $s_{ab}(R) - s_{ba}(R) = s_{ab}(R') - s_{ba}(R')$ for all $a, b \in A$.

Remark 16. Theorem 2 holds even if we weaken support-basedness to only require that $f(R) = f(R')$ if $s^*(R) = s^*(R')$ and only two voters are allowed to exchange their preferences over two alternatives. This technical restriction is significantly weaker than support-basedness, which allows any number of voters to change their preferences.

Remark 17. If preferences are required to be strict, Theorem 2 no longer holds. Several SCFs including the uncovered set, the minimal covering set, and the essential set are strategyproof, Pareto-optimal and support-based, but no voter is a nominator.

Remark 18. Theorem 1 and Theorem 2 raise the question whether all voters must be nominators for every anonymous, Pareto-optimal, and strategyproof social choice function. This is not the case because the SCF f^*, as defined in Remark 11, is a counter-example.

3.3 Majority-based SCFs

Since majority-based SCFs are a subclass of support-based SCFs, Theorem 2 also holds for this class of functions. Moreover, it is easy to see that the concept of nominators is incompatible with majority-basedness and, therefore, there are no majority-based SCFs that satisfy strategyproofness and Pareto-optimality. As we show in this section, a significantly more restrictive impossibility holds for majority-based SCFs: no such social choice function satisfies both strategyproofness and non-imposition. This means that a majority-based and strategyproof SCF is not even allowed to return a single winner if all voters agree that it is the best choice.

Since this statement does not require Pareto-optimality, Lemma 1 cannot be used. Nevertheless, we can show an even stronger result for majority-based SCFs that satisfy strategyproofness and non-imposition: a group consisting of more than half of the voters can enforce a unique winner by unanimously ranking this alternative first.

Lemma 2. *Let f be a majority-based, non-imposing, and strategyproof SCF that is defined for $m \geq 3$ alternatives and $n \geq 3$ voters. Then, $f(R) = \{x\}$ if more than $n/2$ voters prefer x uniquely the most in R.*

We only discuss the key ideas of the proof of this lemma here and defer its formal proof to the appendix. In the proof, we consider an SCF f that satisfies all required axioms. As f is non-imposing, there is a profile R^1 such that $f(R^1) = \{a\}$. Next, we use the strategyproofness of f to modify the preference of every voter such that they prefer a uniquely the most. This step leads to a profile R^2 and strategyproofness implies that $f(R^2) = \{a\}$. Note that we can reorder all alternatives in $A \setminus \{a\}$ during this step as these preferences do not affect the choice set. Finally, we can use majority-basedness to weaken a in the preference of less than $n/2$ voters. This step results in the profile R^3 and it can easily be verified that $R^3_M = R^2_M$. Consequently, $f(R^3) = \{a\}$ for an arbitrary profile R^3 in which more than $n/2$ voters prefer a uniquely the most.

In the sequel, we use Lemma 2 as replacement of Lemma 1 as it discusses conditions under which a group of voters can enforce a unique winner. In contrast to Lemma 1, this new lemma is not concerned with nominators as it shows that no voter can be a nominator for a majority-based SCF that satisfies strategyproofness and non-imposition. Even more, Lemma 2 entails that multiple voters are simultaneously no nominators for specific profiles. Just as for the previous theorems, this is a key observation in the proof of our next result.

Theorem 3. *There is no majority based SCF that satisfies non-imposition and strategyproofness if $m \geq 3$ and $n \geq 3$.*

Proof. We assume for contradiction that there is a majority-based SCF that is defined for $m \geq 3$ alternatives and $n \geq 3$ voters and that satisfies strategyproofness and non-imposition. Moreover, we define the amount of voters that can make an alternative win uniquely as $l = \lceil \frac{n+1}{2} \rceil$ and let $X = A \setminus \{a, b, c\}$.

As consequence of Lemma 2, we derive that $f(R^1) = \{a\}$, where R^1 is defined as follows.

$$R^1: \quad 1\colon c, X, b, a \qquad [2 \dots l]\colon a, b, c, X \qquad [l{+}1 \dots n]\colon a, c, X, b$$

As first step, we let every voter $i \in [2 \dots l]$ change their preference sequentially to $\{a, b\}, c, X$. This leads to the profile R^2 and strategyproofness implies that $f(R^2) \subseteq \{a, b\}$; otherwise there is a voter who can manipulate by reversing this modification.

$$R^2: \quad 1\colon c, X, b, a \qquad [2 \dots l]\colon \{a, b\}, c, X \qquad [l{+}1 \dots n]\colon a, c, X, b$$

Next, we use the same idea as in the proof of Theorem 2: we let voter 1 and 2 change their preferences over a and b. This results in the profile R^3 and majority-basedness implies that $f(R^3) = f(R^2) \subseteq \{a, b\}$ as this step does not affect the majority relation.

$$R^3: \quad 1\colon c, X, \{a, b\} \qquad 2\colon b, a, c, X \qquad [3 \dots l]\colon \{a, b\}, c, X \qquad [l{+}1 \dots n]\colon a, c, X, b$$

Note that $f(R^3)$ is a subset of the least preferred alternatives of voter 1. Thus, strategyproofness implies that voter 1 cannot make any other alternative but a and b win by lying about his preference as he could manipulate otherwise. As consequence, $f(R^4) \subseteq \{a, b\}$, where R^4 is shown in the sequel.

$$R^4: \quad 1\colon c, X, b, a \qquad 2\colon b, a, c, X \qquad [3 \dots l]\colon \{a, b\}, c, X \qquad [l{+}1 \dots n]\colon a, c, X, b$$

As voter 1 prefers b to a in R^4 after these steps, we can repeat them with every voter in $[3 \dots l]$. This leads to the profile R^5 and the fact that $f(R^5) \subseteq \{a, b\}$.

$$R^5: \quad 1\colon c, X, b, a \qquad [2 \dots l]\colon b, a, c, X \qquad [l{+}1 \dots n]\colon a, c, X, b$$

Moreover, observe that $a \in f(R^5)$ contradicts strategyproofness as voter 1 can manipulate otherwise by making b his best alternative. This step results in the profile R^6 for which Lemma 2 implies that $f(R^6) = \{b\}$. As voter i prefers this set both to $\{a\}$ and to $\{a, b\}$, we can derive from strategyproofness that $f(R^5) = \{b\}$.

$$R^6\text{:} \qquad 1\text{: } b, c, X, a \qquad [2 \dots l]\text{: } b, a, c, X \qquad [l{+}1 \dots n]\text{: } a, c, X, b$$

As last step, we let all voters in $[2 \dots l]$ sequentially replace their preference with b, X, c, a. If at some point, $\{b\}$ is not the choice set anymore, a voter can manipulate by reverting the modification. As consequence, b is still the unique winner after this step. Moreover, we let all voters in $[l{+}1 \dots n]$ subsequently replace their preference with c, X, b, a. If $\{b\}$ is not the choice set after this step, a voter can manipulate as b is the least preferred alternative of these voters. Thus, it follows for the profile R^7 that $f(R^7) = \{b\}$.

$$R^7\text{:} \qquad 1\text{: } c, X, b, a \qquad [2 \dots l]\text{: } b, X, c, a \qquad [l{+}1 \dots n]\text{: } c, X, b, a$$

Finally, we explain why $f(R^7) = \{b\}$ is a contradiction. If n is odd, the contradiction follows directly from Lemma 2: as l voters prefers c uniquely the most, this lemma states that $f(R^7) = \{c\}$. If n is even, the contradiction follows from the symmetry of the profile R^7. For this case, observe that there are $n/2$ voters who submit b, X, c, a and $n/2$ voters who submit c, X, b, a. We can derive the same profile up to renaming the voters if we exchange the roles of c and b in the derivation of R^7. However, this implies that $f(R^7) = \{c\}$. Hence, we see that $f(R^7) = \{b\}$ is no valid choice set and therefore, there is no majority-based SCF that satisfies strategyproofness and non-imposition for $m \geq 3$ and $n \geq 3$. □

Remark 19. The axioms required in Theorem 3 are independent of each other: the Pareto rule only violates majority-basedness, a constant SCF only violates Pareto-optimality, and the top cycle only violates strategyproofness.[7] Moreover, even the bound on m and n are tight as the majority rule satisfies all axioms if $m \leq 2$ and for $n \leq 2$, the Pareto rule satisfies all axioms.

Remark 20. In contrast to our previous results, Theorem 3 and its proof do not directly rely on the notion of nominators. This is possible because Lemma 2 shows that many voters may not obtain any of their best alternatives. Thus, the central idea of Theorem 3 is the same as for the previous results: it is in conflict with strategyproofness if a voter is no nominator.

Remark 21. It is possible to show a stronger variant of Lemma 2: a majority-based SCF that satisfies strategyproofness and non-imposition is always required to return the

[7] The top cycle chooses the smallest set of alternatives X such that $(x, y) \in R_M$ and $(y, x) \notin R_M$ for all $x \in X$, $y \in A \setminus X$.

Condorcet winner as unique winner whenever it exists.[8] Combined with a the incompatibility of Condorcet consistency and strategyproofness shown by Brandt (2015), a variant of Theorem 3 also follows. However, the result by Brandt requires $n \geq 3m$ voters. In contrast, we only require that $n \geq 3$.

4 Conclusion

We have studied whether there are attractive rank-based, support-based and majority-based SCFs that satisfy strategyproofness even when preferences are weak. Our main results are negative. Every rank-based and support-based SCF that satisfies strategyproofness and Pareto-optimality returns at least one most-preferred alternative of every voter. In the case of rank-based SCFs, we also show that at least one voter cannot attain one of his most preferred alternatives and consequently obtain an impossibility. By contrast, there are support-based SCFs that satisfy Pareto-optimality and strategyproofness, for instance the Pareto rule. For majority-based SCFs, we show that strategyproofness and non-imposition implies that even multiple voters may not attain any of their most preferred alternatives. This observation entails the incompatibility of majority-basedness, strategyproofness and non-imposition.

As an immediate implication, these results show that two of the most common approaches for defining SCFs do not lead to reasonable SCFs that satisfy strategyproofness and Pareto-optimality. Our results also have consequences for probabilistic social choice, which studies probabilistic social choice function (PSCFs), which return lotteries over alternatives rather than sets of alternatives. Since the notions of ranked-basedness, support-basesness and majority-basedness are independent of the type of the output of the function and merely define an equivalence relation over preference profiles, they can be straightforwardly extended to PSCFs. Theorem 1 implies that there is no rank-based PSCF that satisfies Pareto-optimality and SD-strategyproofness.[9] Furthermore, Theorem 2 implies that every support-based PSCF that satisfies Pareto-optimality and SD-strategyproofness puts positive probability on at least one most-preferred alternative of every voter, a property that is known as *positive share* in the context of dichotomous preferences (Bogomolnaia et al., 2005). Finally, Theorem 3 implies that there are no majority-based PSCFs that satisfy non-imposition and SD-strategyproofness.

It should be stressed that we derive these results using a weak notion of strategyproofness. In particular, our notion of strategyproofness is weaker than those used by Duggan and Schwartz (2000), Barberà et al. (2001), Ching and Zhou (2002), Rodríguez-Álvarez (2007), and Sato (2008). This is possible because we consider a more general preference domain that also allows for ties.

[8] The Condorcet winner in R is an alternative x such that $s_{xy}(R) > s_{yx}(R)$ for all $y \in A \setminus \{x\}$.

[9] SD-strategyproofness is stronger than our strategyproofness applied to the support of the lotteries returned by a PSCF. See, e.g., Brandl et al., 2018, for a definition of SD-strategyproofness.

Our results are tight in the sense that they turn into a possibility if we remove an axiom, reduce the number of alternatives or voters, weaken the notion of strategyproofness or require strict preferences. For example, the essential set (Dutta and Laslier, 1999; Laslier, 2000) is strategyproof if preferences are strict and satisfies participation for unrestricted preferences (Brandt, 2015; Brandl et al., 2019). Therefore, our results provide important insights on when and why strategyproofness can be attained.

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Appendix: Proofs

Lemma 1. *Let f denote an anonymous, Pareto-optimal and strategyproof SCF that is defined for $m \geq 3$ alternatives and $n \geq 2$ voters. If there is a voter $i \in N$ who is no nominator for f, then f satisfies near unanimity.*

Proof. We first explain some repeatedly used steps. Thus, let f denote a strategyproof SCF, $i \in N$ a voter and R, R' two preference profiles with $\succsim_j = \succsim'_j$ for all $j \in N \setminus \{i\}$. Furthermore, let $X_j^+(R)$ denote the set of voter j's most preferred alternatives in R and $X_j^-(R)$ denote the set of voter j's least preferred alternatives in R. Then, the following two implications follow from strategyproofness:

i) $f(R') \subseteq X_i^-(R)$ if $f(R) \subseteq X_i^-(R)$.

ii) $f(R') \subseteq X_i^+(R')$ if $f(R) \subseteq X_i^+(R')$.

Next, consider a Pareto-optimal and strategyproof SCF and assume that voter i is no nominator for f. Thus, there is a profile R such that $f(R)$ does not contain any of voter i's most preferred alternatives. We derive the lemma by modifying this profile in three steps. Firstly, we deduce a profile R' such that $f(R') \cap X_i^+(R') = \emptyset$ and $f(R') = \{x\}$ for some alternative $x \in f(R)$. Secondly, we infer from this profile that $f(R) = \{x\}$ if $n-1$ voters prefer x uniquely the most. Finally, we generalize this observation from a single alternative to all alternatives. This is reminiscent of the so-called *field expansion lemma* in proofs of Arrow's theorem (see, e.g., Sen, 1986).

Step 1: As a first step, we let every voter $j \in N \setminus \{i\}$ replace his preference in R sequentially such that they prefer the alternatives in $f(R)$ the most. This leads to the preference profile R^1 and it follows from a repeated application of *ii)* that $f(R^1) \subseteq f(R)$. Next, let a denote one of voter i's most preferred alternatives in $f(R)$, i.e., $a \succsim_i b$ for all $b \in f(R)$. We replace the current preference of voter i in R^1 with a preference where a is strictly preferred to all alternatives in $A \setminus (X_i^+(R^1) \cup \{a\})$ and all alternatives in $X_i^+(R^1)$ are strictly preferred to all alternatives in $A \setminus X_i^+(R^1)$. This leads to the preference profile R^2 for which $f(R^2) = \{a\}$. This claim is true as all alternatives in $A \setminus (X_i^+(R^1) \cup \{a\})$ are Pareto-dominated by a and no alternative in $X_i^+(R^1)$ can be chosen; otherwise, voter i can manipulate by switching from R^1 to R^2. Thus, we have derived a profile $R' = R^2$ with $f(R') = \{a\}$ and $f(R') \cap X_i^+(R') = \emptyset$.

Step 2: Given the preference profile R' from the last step, we show that $f(R) = \{a\}$ if $n-1$ voters prefer a uniquely the most. We deduce this result by modifying and analyzing the profile R'. First, we sequentially replace the preference of every voter $j \in N \setminus \{i\}$ in R' with a new preference in which they prefer a uniquely the most and an alternative $b \in X_i^+(R')$ uniquely the second most. This leads to a profile R^3 for which $f(R^3) = \{a\}$ follows from a repeated application of *ii)*. Furthermore, every alternative in $A \setminus \{a, b\}$ is Pareto-dominated by b in R^3. We use this observation to replace voter i's current preference with a preference in which b is his uniquely most preferred alternative and a is his uniquely least preferred alternative. In this new profile R^4, all alternatives in $A \setminus \{a, b\}$ are still Pareto-dominated by b and therefore, $f(R^4) \subseteq \{a, b\}$. Furthermore, if $b \in f(R^4)$, then voter i can manipulate by switching from R^3 to R^4 and consequently, $f(R^4) = \{a\}$. Given this observation, it follows from *i)* that $f(R'') = \{a\}$ for all preference profiles R'' with $\succsim_j'' = \succsim_j^4$ for all $j \in N \setminus \{i\}$ and from *ii)* that a is the unique winner if all voters in $N \setminus \{i\}$ prefer a uniquely the most. Thus, $f(R'') = \{a\}$ for all profiles R'' in which all voters in $N \setminus \{i\}$ prefer a uniquely the most. Finally, anonymity implies that a is chosen if $n-1$ voters agree that it is the uniquely most preferred alternative.

Step 3: It only remains to show that if $n-1$ voters can make a win uniquely by uniquely top-ranking it, then they can make every alternative win uniquely by uniquely top-ranking it. Thus, consider the preference profile R^5 in which $n-1$ voters prefer a uniquely the most, and the remaining voter i prefers c uniquely the most, b uniquely second most and a uniquely the least. It follows from our previous observations that $f(R^5) = \{a\}$. Next, let the voters $j \in N \setminus \{i\}$ change their preferences sequentially such that they prefer a and b the most. This leads to a new preference profile R^6 with $f(R^6) = \{b\}$ because *ii)* implies that $f(R^6) \subseteq \{a, b\}$ and a Pareto-dominates b. Thereafter, we replace the preference of every voter $j \in N \setminus \{i\}$ with a new preference in which he prefers b uniquely the most. This step results in a new preference profile R^7 and the repeated application of *ii)* shows that $f(R^7) = \{b\}$. As voter i does not top-rank b, we can apply the constructions discussed in step 2 to deduce that b is uniquely chosen if $n-1$ voters prefer it uniquely the most. $\square$

For a better understanding of the proof, we provide an example. Therefore, let f denote a anonymous SCF that satisfies Pareto-optimality and strategyproofness. Furthermore, assume that $f(R) = \{a, d\}$ for the profile R shown in the sequel and note that this means that voter 1 is no nominator for f.

$$R: \qquad 1\colon b, \{a, d\}, c \qquad 2\colon d, a, \{b, c\} \qquad 3\colon \{a, c\}, \{b, d\}$$

As a first step, we let both voter 2 and 3 change their preference such that a and d are among there most preferred alternatives, which leads to the profile R^1.

$$R^1: \qquad 1\colon b, \{a, d\}, c \qquad 2\colon \{a, d\}, \{b, c\} \qquad 3\colon \{a, d\}, \{b, c\}$$

As consequence of *ii)*, it follows that $f(R^1) \subseteq \{a, d\}$. Moreover, every alternative that is strictly less preferred than a by voter 1 is Pareto-dominated. We use this observation to break the tie in the preference order of voter 1, which results in R^2.

$$R^2: \qquad 1\colon b, a, d, c \qquad 2\colon \{a, d\}, \{b, c\} \qquad 3\colon \{a, d\}, \{b, c\}$$

Pareto-optimality implies that $f(R^2) \subseteq \{a, b\}$ and strategyproofness requires that if $b \in f(R^1)$, then $c \in f(R^2)$. Consequently, $f(R^2) = \{a\}$. Next, we use again *ii)* to change the preference of voter 2 and 3 such that a is their uniquely most preferred alternative and b is their uniquely second best alternative. This results in the profile R^3 for which $f(R^3) = \{a\}$.

$$R^3: \qquad 1\colon b, a, d, c \qquad 2\colon a, b, \{c, d\} \qquad 3\colon a, b, \{c, d\}$$

Note that b Pareto-dominates every alternative but a in R^3. We use this observation to change the preference of voter 1 while ensuring that a is still the unique winner.

$$R^4: \qquad 1\colon b, d, c, a \qquad 2\colon a, b, \{c, d\} \qquad 3\colon a, b, \{c, d\}$$

Observe that b Pareto-dominates every alternative but a in R^4 and therefore $f(R^4) \subseteq \{a, b\}$. Moreover, b cannot be chosen as otherwise, voter 1 can manipulate by switching from R^3 to R^4. Thus, it follows that $f(R^4) = \{a\}$. Based on this profile, it follows from *i)*, *ii)* and anonymity that a is the unique winner if $n-1$ voters prefer it uniquely the most.

Next, we show why $n-1$ voters can also make another alternative b win uniquely by ranking it first. Therefore, consider the profile R^5 displayed in the sequel.

$$R^5: \qquad 1\colon c, b, d, a \qquad 2\colon a, b, \{c, d\} \qquad 3\colon a, b, \{c, d\}$$

It follows from our previous observations that $f(R^5) = \{a\}$. Next, we let voter 2 and 3 change their preference such that they prefer both a and b uniquely the most, which leads to the profile R^6.

$$R^6 \qquad 1\colon\ c, b, d, a \qquad 2\colon\ \{a, b\}, \{c, d\} \qquad 3\colon\ \{a, b\}, \{c, d\}$$

It holds that $f(R^6) = \{b\}$ as *ii)* implies that $f(R^6) \subseteq \{a, b\}$ and b Pareto-dominates a. Therefore, we let voter 2 and 3 replace their preferences to derive the profile R^7 and *ii)* implies that $f(R^7) = \{b\}$.

$$R^7 \qquad 1\colon\ c, b, d, a \qquad 2\colon\ b, c, \{a, d\} \qquad 3\colon\ b, c, \{a, d\}$$

Finally, note that the profile R^7 is equivalent to R^3 up to renaming alternatives. Therefore, we can use the previously discussed steps to deduce that $f(R) = \{b\}$ if $n-1$ voters prefer b uniquely the most. This concludes the example.

Lemma 2. *Let f be a majority-based, non-imposing and strategyproof SCF that is defined for $m \geq 3$ alternatives and $n \geq 3$ voters. Then, $f(R) = \{x\}$ if more than $n/2$ voters prefer x uniquely the most in R.*

Proof. Let f denote a majority-based SCF that satisfies strategyproofness and non-imposition and that is defined for $m \geq 3$ alternatives and $n \geq 3$ voters. Moreover, let a denote an arbitrary alternative and let R^* denote an arbitrary profile in which more than $n/2$ voters prefer a uniquely the most. We show in the sequel that $f(R^*) = \{a\}$ which proves this lemma. For this, first note that there is a profile R^1 such that $f(R^1) = \{a\}$ as f is non-imposing.

Next, we use strategyproofness to subsequently let every voter $i \in N$ change their preference such that a is his uniquely most preferred alternative and the other alternatives are ordered according to $\succsim_i^*$. This process leads to the profile R^2 for which strategyproofness implies that $f(R^2) = \{a\}$; otherwise, there is a voter such that $\{a\}$ is choice set before he changes his preference and it is not the choice set after he changes the preference. This means that the voter can manipulate by undoing the changes as a is the uniquely best alternative of this voter after the modification.

Finally, observe that R^2 and R^* have the same majority relation. This follows for the alternatives in $A \setminus \{a\}$ as all voters in R^2 order these preferences in the same way as in R^*. Moreover, it holds that $n_{ax}(R^2) > n_{xa}(R^2)$ and $n_{ax}(R^*) > n_{xa}(R^*)$ as in both profiles more than $n/2$ of the voters prefer a uniquely the most. Thus, $R^2_M = R^*_M$ and majority-basedness implies that $f(R^*) = f(R^2) = \{a\}$. $\square$