Fishburn’s Maximal Lotteries

A randomized rule that is immune to splitting electorates, cloning alternatives, abstention, and crude manipulation

Felix Brandt

(based on joint papers with H. Aziz, F. Brandl, M. Brill, J. Hofbauer, H. G. Seedig)

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Fishburn’s Maximal Lotteries

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Preliminaries

- $n$ voters with transitive and complete preference relations
  - transitivity not required for results
- majority margin: $g(x,y)$
  - number of voters who prefer $x$ to $y$ minus the number of voters who prefer $y$ to $x.$
- (weak) Condorcet winner: $x$ such that $g(x,y) \geq 0$ for all $y.$
- Condorcet winners may fail to exist
Maximal Lotteries

- Kreweras (1965) and Fishburn (1984)

- Extend $g$ to lotteries: $g(p,q) = \sum_{x,y} p(x) \cdot q(y) \cdot g(x,y)$
  - expected majority margin

- $p$ is a maximal lottery if $g(p,q) \geq 0$ for all $q \in \Delta(A)$.
  - randomized Condorcet winner
  - always exists due to Minimax Theorem (v. Neumann, 1928)

- Maximal lotteries are “almost always” unique.
  - always unique for odd number of voters with strict preferences (Laffond et al., 1997)
  - generalized uniqueness conditions by Le Breton (2005)
Examples

- Two alternatives

- \( g \) can be interpreted as a symmetric zero-sum game.
  - Maximal lotteries are mixed minimax strategies.

\[
\begin{array}{ccc}
2 & 2 & 1 \\
\hline
a & b & c \\
b & c & a \\
c & a & b \\
\end{array}
\quad
\begin{array}{ccc}
a & 0 & -1 \\
b & -1 & 0 \\
c & 1 & -3 \\
\end{array}
\]

- The unique maximal lottery is \( \frac{2}{5} a + \frac{1}{5} b + \frac{1}{5} c \).
Population Consistency ✅

Whenever two disjoint electorates agree on a lottery, this lottery should also be chosen by the union of both electorates.

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½ \(a + \frac{1}{2}b\)

- also known as “reinforcement” (Moulin, 1988)
- famously used for the characterization of scoring rules and Kemeny
Agenda Consistency

A lottery should be chosen from two agendas iff it is also chosen in the union of both agendas.

\[
\begin{array}{ccc}
1 & 1 & 1 \\
& a & b & a & b \\
d & c & b & c \\
b & d & c & a \\
c & a & b & a \\
R & R\vert_A & R\vert_B \\
\frac{1}{2} a + \frac{1}{2} b & \frac{1}{2} a + \frac{1}{2} b & \frac{1}{2} a + \frac{1}{2} b \\
\end{array}
\]

- Sen (1971)’s \( \alpha \) (contraction) and \( \gamma \) (expansion)
- at the heart of numerous impossibilities (e.g., Blair et al., 1976; Sen, 1977; Kelly, 1978; Schwartz, 1986)
Composition Consistency

Composed preference profiles are treated component-wise. In particular, alternatives are not affected by the cloning of other alternatives.

\[
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
a & a & b & b' \\
b' & b & b' & b \\
b & b' & a & a \\
\end{array}
\quad \begin{array}{cc}
2 & 2 \\
a & b \\
b & a \\
\end{array}
\quad \begin{array}{cc}
2 & 2 \\
b' & b \\
b & b' \\
\end{array}
\]

\[
\begin{align*}
R & : \frac{1}{2} a + \frac{1}{4} b + \frac{1}{4} b' \\
R\big|_A & : \frac{1}{2} a + \frac{1}{2} b \\
R\big|_B & : \frac{1}{2} b + \frac{1}{2} b' \\
\end{align*}
\]

- Laffond, Laslier, and Le Breton (1996)
Theorem (Brandl, B., Seedig; 2015): A randomized rule satisfies population consistency and composition consistency iff it returns all maximal lotteries.
No agent can obtain more expected utility by misreporting his preferences.

- maximal lotteries are not strategyproof with respect to stochastic dominance
  - \( q \) will always yield more expected utility than \( p \)

\[
p = \frac{1}{3}a + \frac{1}{3}b + \frac{1}{3}c \\
q = \frac{3}{5}a + \frac{1}{5}b + \frac{1}{5}c
\]
However, maximal lotteries are **group-strategyproof** with respect to the “sure thing” lottery extension.

- loosely based on Savage’s sure-thing principle
- ignore alternatives that receive the same probability in \( p \) and \( q \)
- all remaining alternatives in the support of \( q \) should be preferred to all remaining alternatives in the support of \( p \).
- Almost all randomized versions of classic rules fail to satisfy even this weak notion of strategyproofness
  - e.g., Borda, Copeland, STV, Kemeny, Dodgson
Participation

No agent can obtain more expected utility by abstaining from an election.

\[
\begin{array}{ccc}
\frac{1}{3}a + \frac{1}{3}b + \frac{1}{3}c & b &
\end{array}
\]

- cannot be satisfied by *resolute* Condorcet extensions (Moulin, 1988)
- satisfied by maximal lotteries with respect to stochastic dominance
Pareto Efficiency

The expected utility of a voter can only be increased by decreasing the expected utility of another.

- maximal lotteries are efficient with respect to stochastic dominance
- violated by random serial dictatorship: there can even be lotteries that give strictly more expected utility to all voters!
- maximal lotteries are social-welfare-maximizing lotteries for canonical skew-symmetric bilinear (SSB) utility functions.
References