Extending Tournament Solutions

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Abstract

An important subclass of social choice functions, so-called *majoritarian (or C1) functions*, only take into account the pairwise majority relation between alternatives. In the absence of majority ties—e.g., when there is an odd number of agents with linear preferences—the majority relation is antisymmetric and complete and can thus conveniently be represented by a tournament. Tournaments have a rich mathematical theory and many formal results for majoritarian functions assume that the majority relation constitutes a tournament. Moreover, most majoritarian functions have only been defined for tournaments and allow for a variety of generalizations to unrestricted preference profiles, none of which can be seen as the unequivocal extension of the original function. In this paper, we argue that restricting attention to tournaments is justified by the existence of a *conservative extension*, which inherits most of the commonly considered properties from its underlying tournament solution.

1 Introduction

Perhaps one of the most natural ways to aggregate binary preferences from individual agents to a group of agents is *simple majority rule*, which prescribes that one alternative is socially preferred to another whenever a majority of agents prefers the former to the latter. Majority rule has an intuitive appeal to democratic principles, is easy to understand and—most importantly—satisfies some attractive formal properties (May 1952). Moreover, almost all common social choice functions coincide with majority rule in the two-alternative case. It would therefore seem that the existence of a majority of individuals preferring alternative a to alternative b signifies something fundamental and generic about the group's preferences over a and b.

A *majoritarian (or C1) function* is a social choice function that is based on the majority relation only. When dealing with majoritarian functions, it is often assumed that there are no majority ties. For example, this can be guaranteed by insisting on an odd number of voters with linear preferences. Under this assumption, a preference profile gives rise to a *tournament* and a majoritarian function is equivalent to a *tournament solution*, i.e., a function that associates with every complete and antisymmetric directed graph a subset of the vertices of the graph. Examples of well-studied tournament solutions are the Copeland set, the top cycle, the uncovered set, or the Slater set (see, e.g., Laslier 1997).

Recent years have witnessed an increased interest in the axiomatic and algorithmic aspects of tournament solutions from the AI community (Brandt and Fischer 2008; Faliszewski et al. 2009; Brandt et al. 2010b; Brandt, Brill, and Seedig 2011; Brandt et al. 2010a; Aziz et al. 2012; Brandt, Conitzer, and Endriss 2013) as well as from the theoretical computer science community (Woeginger 2003; Alon 2006; Baumeister et al. 2013).

It is natural to ask how a given majoritarian function can be generalized to the class of preference profiles that may admit majority ties. Mathematically speaking, we are looking for ways to apply a tournament solution to a complete, but not necessarily antisymmetric, directed graph—a so-called *weak tournament*. For many tournament solutions, generalizations or extensions to weak tournaments have been proposed. Often, it turns out that there are several sensible ways to generalize a tournament solution and it is unclear whether there exists a unique "correct" generalization. Even for something as elementary as the Copeland set or the top cycle, there is a variety of extensions that are regularly considered in the literature. A natural criterion for evaluating the different proposals is whether the extension satisfies appropriate generalizations of the axiomatic properties that the original tournament solution satisfies.

In this paper, we propose a generic way to extend any tournament solution to the class of weak tournaments. This so-called *conservative extension* of a tournament solution S returns all alternatives that are chosen by S in *some* orientation of the weak tournament at hand. We show that many of the most common axiomatic properties of tournament solutions are "inherited" from S to its conservative extension (see Table 1 for an overview). We argue that these results provide a justification for restricting attention to tournaments when studying majoritarian functions.

The conservative extension also leads to interesting *computational* problems that are strongly related to the possible winner problem (Lang et al. 2012). In fact, computing the conservative extension of a tournament solution is equivalent to solving its possible winner problem when pairwise comparisons are only partially specified. Of course, there is an exponential number of orientations of a weak tournament in general. Early results, however, indicate that for many well-known tournament solutions, the corresponding conservative extensions can be computed efficiently by ex-

Property inherited by $[S]$	Result
monotonicity	Prop. 1
independence of irrelevant alternatives	Prop. 2
set monotonicity	Prop. 3
$\widehat{\alpha}$	Prop. 4
stability ($\widehat{\alpha} \wedge \widehat{\gamma}$)	Prop. 5
$\widehat{\alpha}_{\subseteq}$ and $\widehat{\alpha}_{\supseteq}$	Prop. 16
$\widehat{\alpha} \wedge \gamma_{\subseteq}$ and $\widehat{\alpha} \wedge \widehat{\gamma}_{\supseteq}$	Prop. 17
composition-consistency	Prop. 6
weak composition-consistency	Prop. 20
weak regularity	Prop. 21

Table 1: Properties that $[S]$ inherits from S

ploiting individual peculiarities of these concepts.

The paper is organized as follows. After introducing the necessary notation in Section 2, we define the conservative extension in Section 3 and show that it inherits many desirable properties in Section 4. Furthermore, we compare the conservative extension to other generalizations that have been proposed in the literature (Section 5) and study its computational complexity (Section 6) for a number of common tournament solutions. Due to the space constraint, several proofs are delegated to the appendix.

2 Preliminaries

Let U be a universe of alternatives. For notational convenience we assume that $\mathbb{N} \subseteq U$. Every nonempty finite subset of U is called a *feasible set*. For a binary relation $\succsim$ on U and alternatives $a, b \in U$, we usually write $a \succsim b$ instead of the more cumbersome $(a, b) \in \succsim$. A *weak tournament* is a pair $W = (A, \succsim)$, where A is a feasible set and $\succsim$ is a complete binary relation on U, i.e., for all $a, b \in U$, we have $a \succsim b$ or $b \succsim a$ (or both).[1] Intuitively, $a \succsim b$ signifies that alternative a is (weakly) preferred to b. Note that completeness implies reflexivity, i.e., $a \succsim a$ for all $a \in U$. We write $a \succ b$ if $a \succsim b$ and not $b \succsim a$, and $a \sim b$ if both $a \succsim b$ and $b \succsim a$. We denote the class of all weak tournaments by $\mathcal{W}$.

The relation $\succsim$ is often referred to as the *dominance relation*. One of the best-known concepts defined in terms of the dominance relation is that of a Condorcet winner. Alternative a is a *Condorcet winner* in a weak tournament $W = (A, \succsim)$ if $a \succ b$ for all alternatives $b \in A \setminus \{a\}$.

A *tournament* is a weak tournament $(A, \succsim)$ whose dominance relation $\succsim$ is also antisymmetric, i.e., for all *distinct* $a, b \in A$, we have that $a \succsim b$ and $b \succsim a$ imply $a = b$.[2] For a tournament $T = (A, \succsim)$ and distinct alternatives $a, b \in A$, $a \succsim b$ if and only if $a \succ b$. We therefore often write $T = (A, \succ)$ instead of $T = (A, \succsim)$. We denote the class of all tournaments by $\mathcal{T}$. Obviously, $\mathcal{T} \subseteq \mathcal{W}$.

For a pair of weak tournaments $W = (A, \succsim)$ and $W' = (A', \succsim')$, we say that W is *contained* in W', and write $W \subseteq W'$, if $A = A'$ and $a \succsim b$ implies $a \succsim' b$ for all $a, b \in A$. We will often deal with the set of all tournaments that are contained in a given weak tournament W.

Definition 1. *For a weak tournament $W \in \mathcal{W}$, the set of* orientations *of W is given by $[W] = \{T \in \mathcal{T} : T \subseteq W\}$.*

Every orientation of a weak tournament $W = (A, \succsim)$ can be obtained from W by eliminating, for all distinct alternatives a and b such that $a \sim b$, one of (a, b) and (b, a) from $\succsim$.

The relation $\succsim$ can be raised to sets of alternatives and we write $A \succsim B$ to signify that $a \succsim b$ for all $a \in A$ and all $b \in B$. For a weak tournament $W = (A, \succsim)$ and a feasible set $B \subseteq A$, we will sometimes consider the *restriction* $W|_B = (B, \succsim)$ of W to B.

A *tournament solution* is a function S that maps each tournament $T = (A, \succ)$ to a nonempty subset $S(T)$ of its alternatives A called the *choice set*. It is generally assumed that choice sets only depend on $\succ|_A$ and that tournament solutions cannot distinguish between isomorphic tournaments.

Two examples of well-known tournament solutions are the top cycle and the Copeland set. The *top cycle* $TC(T)$ of a tournament $T = (A, \succ)$ is defined as the smallest set $B \subseteq A$ such that $B \succ A \setminus B$. The *Copeland set* $CO(T)$ consists of all alternatives whose dominion is of maximal size, i.e., $CO(T) = \arg\max_{a \in A} |\{b \in A \setminus \{a\} : a \succ b\}|$.

3 The Conservative Extension

In order to render tournament solutions applicable to general preference profiles, we need to generalize them to weak tournaments. A *generalized tournament solution* is a function S that maps each weak tournament $W = (A, \succsim)$ to a nonempty subset $S(W)$ of its alternatives A. A generalized tournament solution S is called an *extension* of tournament solution S' if $S(W) = S'(W)$ whenever W is a tournament. For several tournament solutions, extensions have been proposed in the literature (e.g., Dutta and Laslier 1999; Peris and Subiza 1999). Of course, there are many ways to extend any given tournament solution, and there is no definite obvious way of judging whether one proposal is better than another one.

We are interested in a *generic* way to extend any tournament solution to the class of weak tournaments. In particular, our goal is to extend tournament solutions in such a way that common axiomatic properties are "inherited" from a tournament solution to its extension. This task is not trivial, as even the arguably most cautious approach has its problems. Let the *trivial extension* of a tournament solution S be defined as the generalized tournament solution that always selects the whole feasible set A whenever the weak tournament $W = (A, \succsim)$ is *not* a tournament. It is easy to see that the trivial extension does not satisfy Condorcet-consistency, i.e., the requirement that a Condorcet winner should be uniquely selected whenever it exists. Indeed, for the weak tournament

[1] This definition slightly diverges from the common graph-theoretic definition where $\succsim$ is defined on A rather than on U. However, it facilitates the definition of tournament solutions and their properties.

[2] Defining tournaments with a reflexive dominance relation is non-standard. The reason we define tournaments in such a way is to ensure that every tournament is a weak tournament. Whether the dominance relation of a tournament is reflexive or not does not make a difference for any of our results.

$(\{a,b,c\},\succsim)$ with $a \succ \{b,c\}$ and $b \sim c$, the trivial extension of any tournament solution selects $\{a,b,c\}$. The trivial extension also fails to inherit composition-consistency, which will be defined in Section 4.4.

We therefore propose to extend tournament solutions in a slightly more sophisticated way. The *conservative extension* of a tournament solution S returns all alternatives that are chosen by S in *some* orientation of the weak tournament at hand.

Definition 2. *Let S be a tournament solution. The conservative extension $[S]$ of S is the generalized tournament solution that maps a weak tournament $W \in \mathcal{W}$ to*

$$[S](W) = \bigcup_{T\in[W]} S(T).$$

This definition is reminiscent of the parallel-universes tie-breaking approach in social choice theory (Conitzer, Rognlie, and Xia 2009; Brill and Fischer 2012) and corresponds to selecting the set of all *possible* winners of W when ties are interpreted as missing edges (Lang et al. 2012; Aziz et al. 2012).

For example, the weak tournament depicted in Figure 1 has four orientations. It can be checked that $\{CO(T) : T \in [W]\} = \{\{a\},\{a,b\},\{a,c\}\}$ and $\{TC(T) : T \in [W]\} = \{\{a\},\{a,b,c,d\}\}$. Therefore, $[CO](W) = \{a,b,c\}$ and $[TC](W) = \{a,b,c,d\}$.

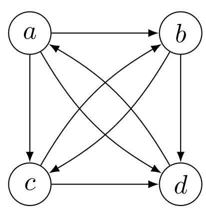

Figure 1: Weak tournament $W = (\{a,b,c,d\},\succsim)$ with $[CO](W) = \{a,b,c\}$ and $[TC](W) = \{a,b,c,d\}$. An edge from vertex x to vertex y represents $x \succsim y$.

4 Inheritance of Properties

The literature on (generalized) tournament solutions has identified a number of desirable properties for these concepts. In this section, we study which properties are inherited when a tournament solution is generalized via the conservative extension. After stating a useful lemma, we consider four classes of properties: *dominance-based properties* (Section 4.2) that deal with changes in the dominance relation, *choice-theoretic properties* that deal with varying feasible sets (Section 4.3), *composition-consistency* (Section 4.4), and *regularity* (Section 4.5).

4.1 A General Lemma

Many properties express the invariance of alternatives being chosen (or alternatives not being chosen) under certain type of transformation of the weak tournament. That is, they have the form that if an alternative a is chosen (not chosen) from some weak tournament W, then a is also chosen (not chosen) from $f(W)$, where f is an operation that transforms weak tournaments in a particular way.

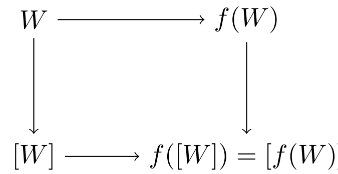

Figure 2: Orientation-consistency

Formally, a *tournament operation* is a mapping f from the class of all weak tournaments to itself. A tournament operation f is *orientation-consistent* if applying the operation to any orientation of a weak tournament W results in a tournament that is an orientation of $f(W)$.

Definition 3. *A tournament operation f is* orientation-consistent *if for all weak tournaments W and all $T \in [W]$,*

$$f([W]) = [f(W)],$$

where $f([W]) = \{f(T) : T \in [W]\}$. Furthermore, a class F of tournament operations is orientation-consistent *if each operation in F is orientation-consistent.*

In other words, f is orientation-consistent if the diagram in Figure 2 commutes. Observe that a necessary condition for f to be orientation-consistent is that $f(\mathcal{T}) \subseteq \mathcal{T}$.

Let F be a class of tournament operations and $\mathcal{C}$ a subclass of weak tournaments. We then say that a generalized tournament solution S is *inclusion-invariant under F on $\mathcal{C}$* if, for all weak tournaments W in $\mathcal{C}$,

$$a \in S(W) \quad \text{implies} \quad a \in S(f(W)) \text{ for all } f \in F.$$

Similarly, we say that S is *exclusion-invariant under F on $\mathcal{C}$* if, for all weak tournaments W in $\mathcal{C}$,

$$a \notin S(W) \quad \text{implies} \quad a \notin S(f(W)) \text{ for all } f \in F.$$

We are now in a position to formulate a very useful lemma.

Lemma 1. *Let F be an orientation-consistent class of tournament operations and S a tournament solution. Then,*

(i) *if S is inclusion-invariant under F on $\mathcal{T}$, so is $[S]$ on $\mathcal{W}$,*

(ii) *if S is exclusion-invariant under F on $\mathcal{T}$, so is $[S]$ on $\mathcal{W}$.*

Proof. We give the proof for (i); the proof for (ii) runs along analogous lines. Assume that S is inclusion-invariant under F on $\mathcal{T}$, i.e., for all $T \in \mathcal{T}$ and all alternatives a,

$$a \in S(T) \text{ implies } a \in S(f(T)).$$

Consider an arbitrary weak tournament W in $\mathcal{W}$ and an alternative $a \in [S](W)$. By definition of $[S]$,

$$a \in S(T) \text{ for some } T \in [W]. \qquad (*)$$

We show that $a \in [S](f(W))$ for all $f \in F$. For a contradiction assume $a \notin [S](f(W))$ for some $f \in F$. Then, $a \notin S(T)$ for all $T \in [f(W)]$. By orientation-consistency of f, then also $a \notin S(T)$ for all $T \in f([W])$. Recall that $f([W]) = \{f(T) : T \in [W]\}$. Hence, $a \notin S(f(T))$ for all $T \in [W]$. Our initial assumption then finally yields $a \notin S(T)$ for all $T \in [W]$, which contradicts $(*)$. $\square$

4.2 Dominance-Based Properties

We first look at three properties that deal with changes in the dominance relation, namely monotonicity, independence of unchosen alternatives, and set-monotonicity.

A tournament solution is monotonic if a chosen alternative remains in the choice set when it is strengthened against some other alternative, while leaving everything else unchanged. Here, *strengthening* a versus b refers to replacing $b \succ a$ with $a \succ b$. In weak tournaments, we can also strengthen a against b by replacing $a \sim b$ with $a \succ b$.[3] In order to formalize monotonicity, let $W = (A, \succsim)$ be a weak tournament and define $W_{a\succ b} = (A, \succsim')$, where

$$\succsim' \;=\; \succsim \setminus \{(b,a)\} \cup \{(a,b)\}.$$

Definition 4. *A generalized tournament solution S is* monotonic *if for all $W = (A, \succsim)$ and $b \in A$,*

$$a \in S(W) \quad \textit{implies} \quad a \in S(W_{a\succ b}).$$

It is easy to see that monotonicity can be phrased as an inclusion-invariance condition. Invoking Lemma 1, we then obtain the following result.

Proposition 1. *If a tournament solution S is monotonic on $\mathcal{T}$, so is $[S]$ on $\mathcal{W}$.*

Proof sketch. Let $f_{a\succ b}$ be the tournament operation that maps each weak tournament W to $W_{a\succ b}$ and define

$$F_a^{MON}(W) = \{f_{a\succ b} : b \in A \setminus \{a\}\}.$$

It can then easily be appreciated that a generalized tournament solution is monotonic if and only if it is inclusion-invariant under F_a^{MON}. Observe that F_a^{MON} is orientation-consistent. Lemma 1 then gives the result. □

Independence of unchosen alternatives (IUA) prescribes that the choice set is invariant under any changes in the dominance relation among unchosen alternatives.

Definition 5. *A generalized tournament solution S is* independent of unchosen alternatives (IUA) *if for all $W = (A, \succsim)$ and $a, b \in A \setminus S(A)$,*

$$S(W) = S(W_{a\succ b}).$$

Reasoning along similar lines as for monotonicity, we find that IUA is inherited from S to $[S]$.

Proposition 2. *If a tournament solution S is independent of unchosen alternatives on $\mathcal{T}$, so is $[S]$ on $\mathcal{W}$.*

Proof sketch. Let $F^{IUA}(W) = \{f_{a\succ b} : a, b \in A \setminus S(W)\}$ and observe that a generalized tournament solution satisfies IUA if and only if it is both inclusion-invariant and exclusion-invariant under F^{IUA}. To appreciate this observe that inclusion-invariance under F^{IUA} implies that $S(W) \subseteq S(W_{a\succ b})$ and exclusion-invariance under F^{IUA} implies $S(W) \supseteq S(W_{a\succ b})$, the other direction being straightforward. Moreover, $F^{IUA}(W)$ is orientation-consistent. An application of Lemma 1 then yields the result. □

[3] A subtler way to strengthen a against b consists in replacing $b \succ a$ with $a \sim b$. Although this case is not covered by our definition of monotonicity, it can be shown that $[S]$ satisfies this additional property as long as S is monotonic.

Set-monotonicity is a strengthening of both monotonicity and IUA and is the defining property in a characterization of group-strategyproof social choice functions (Brandt 2011). A tournament solution is *set-monotonic* if the choice set remains the same whenever some alternative is strengthened against some unchosen alternative.

Definition 6. *A generalized tournament solution S is* set-monotonic *if for all $W = (A, \succsim)$, $a \in A$, and $b \in A \setminus S(A)$,*

$$S(W) = S(W_{a\succ b}).$$

Set-monotonicity can be characterized in terms of inclusion-invariance and exclusion-invariance with respect to an orientation-consistent class of tournament operations given by $F^{SMON}(W) = \{f_{a\succ b} : a \in U, b \in A \setminus S(W)\}$. Thus, Lemma 1 also yields the following result.

Proposition 3. *If a tournament solution S is set-monotonic on $\mathcal{T}$, so is $[S]$ on $\mathcal{W}$.*

4.3 Choice-Theoretic Properties

We now turn to a class of properties that relate choices from different feasible sets to each other. For all of these properties, the dominance relation $\succsim$ is fixed. We can therefore simplify notation and write $S(A)$ for $S((A, \succsim))$.

The central property in this section is stability (Brandt and Harrenstein 2011), which requires that a set is chosen from two different sets of alternatives if and only if it is chosen from the union of these sets.

Definition 7. *A generalized tournament solution S is* stable *if for all feasible sets A, B and $X \subseteq A \cap B$,*

$$X = S(A) = S(B) \quad \textit{if and only if} \quad X = S(A \cup B).$$

Stability can be factorized into conditions $\widehat{\alpha}$ and $\widehat{\gamma}$ by considering each implication in the above equivalence separately.[4]

$$X = S(A \cup B) \text{ implies } X = S(A) = S(B) \qquad (\widehat{\alpha})$$
$$X = S(A) = S(B) \text{ implies } X = S(A \cup B) \qquad (\widehat{\gamma})$$

Lemma 1 is not directly applicable to the choice-theoretic properties $\widehat{\alpha}$ and $\widehat{\gamma}$. For $\widehat{\alpha}$, however, we can utilize the following characterization: S satisfies $\widehat{\alpha}$ if and only if

$$S(A) \subseteq B \subseteq A \text{ implies } S(A) = S(B)$$

for all feasible sets A, B. This characterization allows us to reformulate $\widehat{\alpha}$ as the conjunction of two invariance properties. For a feasible set B, we let f_B denote the tournament operation that maps a weak tournament $W = (A, \succsim)$ with $B \subseteq A$ to its restriction to B, i.e., $f_B(W) = W|_B$. Furthermore, define $F^{\widehat{\alpha}} = \{f_B : S(A) \subseteq B \subseteq A\}$. Observe that a generalized tournament solution S satisfies $\widehat{\alpha}$ if and only if S is both inclusion-invariant under $F^{\widehat{\alpha}}$ and exclusion-invariant under $F^{\widehat{\alpha}}$. Since $F^{\widehat{\alpha}}$ is orientation-consistent, we can apply Lemma 1.

[4] Property $\widehat{\alpha}$ is also known as Chernoff's *postulate* 5^* (Chernoff 1954), the *strong superset property* (Bordes 1979), or *outcast* (Aizerman and Aleskerov 1995) (see Monjardet (2008) for a more thorough discussion of the origins of this condition).

Proposition 4. *If a tournament solution S satisfies $\widehat{\alpha}$ on $\mathcal{T}$, so does $[S]$ on $\mathcal{W}$.*

For $\widehat{\gamma}$, no characterization similar in spirit to the reformulation of $\widehat{\alpha}$ above is known. In fact, we were not able to prove that $\widehat{\gamma}$ is inherited from a tournament solution S to its conservative extension $[S]$. However, it is inherited if S also satisfies $\widehat{\alpha}$.

Proposition 5. *Let S be a tournament solution that satisfies $\widehat{\alpha}$. If S satisfies $\widehat{\gamma}$ on $\mathcal{T}$, so does $[S]$ on $\mathcal{W}$.*

Since stability is equivalent to the conjunction of $\widehat{\alpha}$ and $\widehat{\gamma}$, the following statement follows as an immediate consequence of Propositions 4 and 5.

Corollary 1. *If S is stable on $\mathcal{T}$, so is $[S]$ on $\mathcal{W}$.*

Interestingly, requiring $\widehat{\alpha}$ so that $\widehat{\gamma}$ is inherited is less restrictive than it might seem because all common tournament solution satisfy $\widehat{\alpha}$ *if and only if* they satisfy $\widehat{\gamma}$.[5] In general, however, it is the case that $\widehat{\alpha}$ and $\widehat{\gamma}$ are independent from each other, even though this requires the construction of rather artificial tournament solutions.[6]

In Appendix B, we show that several weakenings of $\widehat{\alpha}$ and $\widehat{\gamma}$ are also inherited from S to $[S]$.

4.4 Composition-Consistency

We finally consider a structural property that deals with sets of similar alternatives. A component of a tournament is a subset of alternatives that bear the same dominance relationship to all alternatives not in the set. A decomposition is a partition of the alternatives into components. A decomposition induces a summary tournament with the components as alternatives. A tournament solution is then said to be *composition-consistent* if it selects the best alternatives from the components it selects from the summary tournament.

In order to extend the definition of composition-consistency to weak tournaments, we need to generalize the concept of a component. By a *component* of a weak tournament $W = (A, \succsim)$ we understand a feasible set $X \subseteq A$ such that X is a singleton or for all $y \in A \setminus X$, either $X \succ y$ or $y \succ X$. We have the following lemma.

Lemma 2. *Let $W = (A, \succsim)$ be a weak tournament and $X \subseteq A$. Then, X is a component of W if and only if X is a component of every orientation $T \in [W]$.*

Given the definition of a component, decompositions and summaries of weak tournaments, as well as composition-consistency of generalized tournament solutions, are then defined analogously to the case of tournaments (see Appendix C.1 for details). We find that composition-consistency is inherited to the conservative extension.

Proposition 6. *If a tournament solution S is composition-consistent on $\mathcal{T}$, so is $[S]$ on $\mathcal{W}$.*

The literature on tournaments also distinguishes the concept of *weak composition-consistency* (see, e.g., Laslier 1997). We find that, rendering it applicable to weak tournaments in a way much similar as above, weak composition-consistency is also inherited by $[S]$ from S. For definitions and proofs we refer the interested reader to Appendix C.2.

The notion of a component of a weak tournament defined here is rather strong and the associated concept of composition-consistency correspondingly weak. A natural stronger notion of composition-consistency could be based on a weaker concept of component. Thus, for weak tournaments $W = (A, \succsim)$, a component could be defined as a subset $X \subseteq A$ such that for all $y \in A\backslash X$, either $X \succ y, y \succ X$, or $X \sim y$. Observe that for such components Lemma 2 does no longer hold. Moreover, it can easily be seen that the conservative extension $[S]$ of *no* Condorcet-consistent tournament solution S satisfies the associated concept of composition-consistency (see Appendix C.3).

4.5 Regularity

A tournament solution is regular if it selects all alternatives from regular tournaments, i.e., tournaments in which the indegree and outdegree of every alternative are equal. Regularity extends naturally to weak tournaments, but we find that it is not generally inherited from S to $[S]$. A weaker, but likewise conservative, extension of the notion of regularity, which we call *weak regularity*, requires a generalized solution concept to choose all alternatives from regular weak tournaments *of odd order* only. Weak regularity is inherited from S to $[S]$. For details, see Appendix D.

5 Comparison to Other Generalizations

For many tournament solutions, generalizations or extensions to weak tournaments have been proposed in the literature. In this section, we compare these extensions to the conservative extension for a number of well-known tournament solutions (for definitions, see Laslier 1997). For generalized tournament solutions S and S', we write $S' \subset S$ if $S \neq S'$ and $S'(W) \subseteq S(W)$ for all weak tournaments W. In this case, we say that S' is a *refinement* of S.

Copeland Set The Copeland set CO gives rise to a whole class of extensions that is parameterized by a number α between 0 and 1. The generalized tournament solution CO^α selects all alternatives that maximize the variant of the Copeland score in which each tie contributes α points to an alternative's score (see, e.g., Faliszewski et al. 2009). Henriet (1985) axiomatically characterized $CO^{\frac{1}{2}}$, arguably the most natural variant in this class. While it is easy to check that $[CO] \not\subset CO^\alpha$ for all $\alpha \in [0, 1]$, the inclusion of CO^α in $[CO]$ turns out to depend on the value of α.

Proposition 7. *$CO^\alpha \subset [CO]$ if and only if $\frac{1}{2} \leq \alpha \leq 1$.*

Top Cycle Schwartz (1972; 1986) defined two generalizations of the top cycle TC (see also Sen 1986 and Brandt, Fischer, and Harrenstein 2009). *GETCHA* (or the *Smith set*) contains the maximal elements of the transitive closure of $\succsim$ whereas *GOCHA* (or the *Schwartz set*) contains the maximal elements of the transitive closure of $\succ$. The conservative extension $[TC]$ coincides with GETCHA.

Proposition 8. *GOCHA $\subset$ GETCHA $= [TC]$.*

[5] For example, this statement holds for all tournament solutions considered in Section 5: TC, BP, and MC satisfy both $\widehat{\alpha}$ and $\widehat{\gamma}$, and CO, UC, BA, and TEQ satisfy neither $\widehat{\alpha}$ nor $\widehat{\gamma}$.

[6] See Appendix B for examples.

Bipartisan Set Dutta and Laslier (1999) generalized the bipartisan set BP to the *essential set* ES, which is given by the set of all alternatives that are contained in the support of *some* Nash equilibrium of the underlying weak tournament game. The essential set does not coincide with the conservative extension $[BP]$. Whether $ES \subset [BP]$ holds is an open problem.

Uncovered Set Duggan (2013) surveyed several extensions of the covering relation to weak tournaments. Any such relation induces a generalization of the *uncovered set*. The so-called *deep covering* and *McKelvey covering* relations are particularly interesting extensions. Duggan (2013) showed that for all other generalizations of the covering relation he considered, the corresponding uncovered set is a refinement of the deep uncovered set UC_D. Another interesting property of UC_D is that it coincides with the conservative extension of UC.

Proposition 9. $UC_D = [UC]$.

It follows that all other UC generalizations considered by Duggan (2013) are refinements of $[UC]$.

Minimal Covering Set The generalization of MC is only well-defined for the McKelvey covering relation and the deep covering relation. The corresponding generalized tournament solutions are known to satisfy stability. We have constructed a weak tournament in which $[MC]$ is strictly contained in both the McKelvey minimal covering set MC_M and the deep minimal covering set MC_D. There are also weak tournaments in which MC_M is strictly contained in $[MC]$.

Proposition 10. *$[MC] \subset MC_D$, $[MC] \not\subset MC_M$, and $MC_M \not\subset [MC]$.*

Corollary 1 implies that $[MC]$ satisfies the very demanding stability property. Hence, we have found a new sensible generalization of MC which is a refinement of MC_D and sometimes yields strictly smaller choice sets than MC_M.

Banks Set Banks and Bordes (1988) discussed four different generalizations of the Banks set BA to weak tournaments, denoted by BA_1, BA_2, BA_3, and BA_4. Each of those generalizations is a refinement of the conservative extension $[BA]$.

Proposition 11. *$BA_m \subset [BA]$ for all $m \in \{1, 2, 3, 4\}$.*

Tournament Equilibrium Set Finally, Schwartz (1990) suggested six ways to extend the *tournament equilibrium set* TEQ—and the notion of retentiveness in general—to weak tournaments. However, all of those variants can easily be shown to lead to disjoint minimal retentive sets even in very small tournaments, and none of the variants coincides with $[TEQ]$.

6 Computational Complexity

When a tournament solution S is generalized via the conservative extension to $[S]$, it is natural to ask whether the choice set of $[S]$ can be computed efficiently. Since the number of orientations of a weak tournament can be exponential in the size of the weak tournament, tractability of the winner determination problem of S is a necessary, but not a sufficient, condition for the tractability of $[S]$.

Proposition 12. *There is a tournament solution S such that the winner determination problem is in P for S, and NP-complete for $[S]$.*

In light of Proposition 12, it is interesting to check for each tractable tournament solution S, whether the choice set of $[S]$ can be computed efficiently. This question is mathematically equivalent to the problem of computing the set of *possible winners for a partially specified tournament*. The latter problem has been studied for the Copeland set CO, the top cycle TC, and the uncovered set UC.

Proposition 13 (Cook et al. 1998)**.** *Computing $[CO]$ is in P.*

Proposition 14 (Lang et al. 2012)**.** *Computing $[TC]$ is in P.*

Proposition 15 (Aziz et al. 2012)**.** *Computing $[UC]$ is in P.*

While the proof of Proposition 13 consists in a polynomial-time reduction to maximum network flow, $[TC]$ and $[UC]$ can be computed by greedy algorithms. It is a very interesting open problem whether the conservative extensions of more elaborate tournament solutions such as the minimal covering set or the bipartisan set can be computed efficiently.

If computing winners is NP-complete for a tournament solution, the same is true for its conservative extension.

Lemma 3. *If winner determination for S is NP-complete, then winner determination for $[S]$ is NP-complete.*

Proof. Hardness of computing $[S]$ immediately follows from hardness of computing S, because $[S]$ and S agree whenever the weak tournament is in fact a tournament. For membership in NP, we can guess an orientation T of the weak tournament and then verify that the designated alternative is contained in $S(T)$. □

Since the winner determination problem is NP-complete for the Banks set BA (Woeginger 2003), we have an immediate corollary.

Corollary 2. *Computing $[BA]$ is NP-complete.*

7 Conclusion

We have shown that the conservative extension inherits many desirable properties from its underlying tournament solution (see Table 1). In general, the conservative extension $[S]$ of tournament solution S is rather large and there might be more discriminating extensions of S that still satisfy its characterizing properties. However, the conservative extension may serve as "proof of concept" to show that generalizing a tournament solution in a meaningful way is possible. Whether there are more discriminating solutions that are equally attractive is a different issue that needs to be settled for each tournament solution at hand.

Besides its axiomatic properties, the conservative extension is also interesting from a computational point of view. Particularly intriguing is the question whether the conservative extension of the minimal covering set or the bipartisan set can be computed efficiently.

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A Omitted Proofs

Proposition 5. *Let S be a tournament solution that satisfies $\widehat{\alpha}$. If a tournament solution S satisfies $\widehat{\gamma}$ on $\mathcal{T}$, so does $[S]$ on $\mathcal{W}$.*

Proof. Let S be a tournament solution satisfying $\widehat{\alpha}$ and $\widehat{\gamma}$ and let $W = (A \cup B, \succsim)$ be a weak tournament such that $[S](A) = [S](B) = X \subseteq A \cap B$. We need to show that $[S](A \cup B) = X$.

By definition of $[S]$, we have

$$[S](A) = \bigcup_{T_A \in [W|_A]} S(T_A),$$
$$[S](B) = \bigcup_{T_B \in [W|_B]} S(T_B), \text{ and}$$
$$[S](A \cup B) = \bigcup_{T \in [W]} S(T).$$

We will show that for all $T \in [W]$, $S(T|_A) = S(T|_B) = S(T)$. The statement then follows from the trivial observation that every orientation of $W|_A$ can be obtained as a restriction of an orientation of W, i.e., for all $T_A \in [W|_A]$ there is a $T \in [W]$ such that

$$T|_A = T_A.$$

Now consider an arbitrary $T \in [W]$. Obviously, $T|_A \in [W|_A]$ and $T|_B \in [W|_B]$. By assumption, we have $S(T|_A) \subseteq A \cap B$ and $S(T|_B) \subseteq A \cap B$. Applying $\widehat{\alpha}$ to A and $A \cap B$ yields

$$S(T|_A) = S(T|_{A \cap B}),$$

and applying $\widehat{\alpha}$ to B and $A \cap B$ yields

$$S(T|_B) = S(T|_{A \cap B}).$$

Therefore, $S(T|_A) = S(T|_B)$. Since S satisfies $\widehat{\gamma}$ on $\mathcal{T}$, this yields $S(T) = S(T|_A) = S(T|_B)$. □

Lemma 2. *Let $W = (A, \succsim)$ be a weak tournament and $X \subseteq A$. Then, X is a component of W if and only if X is a component of every orientation $T \in [W]$.*

Proof. For the "only if"-direction, assume that X is a component of W and, for contradiction, that there is some orientation $T = (A, \succ')$ of W for which X is not a component. Then, there are $x, x' \in X$ and $y \in A \setminus X$ such that $x \succ' y$ and $y \succ' x'$. With X being a component of W, both $x \succ y$ and $x' \succ y$ or both $y \succ x$ and $y \succ x'$. Moreover, this has to hold in every orientation of W and a contradiction follows.

For the "if"-direction, let X be a subset of A that is not a component of W. Then, in particular, X is not a singleton. Moreover, there are $x, x' \in X$ and $y \in A \setminus X$ such that one of the following cases obtains: both $x \succ y$ and $y \succ x$, both $x \succ y$ and $x' \sim y$, or both $x \sim y$ and $x' \sim y$. In each of these cases there is an orientation $T = (A, \succ')$ of W such that both $x \succ' y$ and $y \succ' x'$. □

Proposition 6. *If a tournament solution S is composition-consistent on $\mathcal{T}$, so is $[S]$ on $\mathcal{W}$.*

Proof. Assume S is composition-consistent. Consider an arbitrary weak tournament $W = (A, \succsim)$ along with a decomposition $\{X_1, \ldots, X_k\}$ of W. Let $\tilde{W}$ be such that $W = \prod(\tilde{W}, W|_{X_1}, \ldots, W|_{X_k})$. We have to show that, for all $a \in A$,

$$a \in [S](W) \text{ iff } a \in [S](W|_{X_i}) \text{ for some } i \in [S](\tilde{W}).$$

First, assume that $a \in [S](W)$. Then, there is some orientation $T \in [W]$ such that $a \in S(T)$. By virtue of Lemma 2, $\{X_1, \ldots, X_k\}$ is also a decomposition of T. Therefore, $T = \prod(\tilde{T}, T|_{X_1}, \ldots, T|_{X_k})$, where $\tilde{T} \in [\tilde{W}]$ and $T|_{X_i} \in [W|_{X_i}]$ for all i with $1 \leq i \leq k$. Having assumed that S is composition-consistent, $a \in S(T|_{X_i})$ for some $i \in S(\tilde{T})$. As $\tilde{T} \in [\tilde{W}]$, it follows that $a \in [S](W|_{X_i})$ for some $i \in [S](\tilde{W})$.

For the opposite direction, assume $a \in [S](W|_{X_i})$ for some $i \in [S](\tilde{W})$. Then there are orientations $\tilde{T} \in [\tilde{W}]$ and $T|_{X_i} \in [W|_{X_i}]$ such that $i \in S(\tilde{T})$ and $a \in S(T|_{X_i})$. Let $T'|_{X_j} \in [W|_{X_j}]$ for all j distinct from i and define $T'' = \prod(\tilde{T}, T'_{X_1}, \ldots, T_{X_i}, \ldots, T'_{X_k})$. Observe that T'' is an orientation of W. By Lemma 2, moreover, $\{X_1, \ldots, X_k\}$ is also a decomposition of T' and by composition-consistency of S we obtain $a \in S[T'']$. Finally, with T'' being an orientation of W, we may conclude that $a \in [S](W)$. □

Proposition 7. *$CO^\alpha \subset [CO]$ if and only if $\frac{1}{2} \leq \alpha \leq 1$.*

Proof. For notational convenience, define the *tie graph* of a weak tournament $(A, \succsim)$ as the undirected graph (A, E) with $\{a, b\} \in E$ if and only if $a \sim b$. Furthermore, $t(a)$ denotes the degree of alternative a in the tie graph, i.e., the number of ties involving a, and $d^+(a)$ denotes the cardinality of $\{b \in A : a \succ b\}$.

Let $0 \leq \alpha < \frac{1}{2}$. We will construct a weak tournament W_α such that $CO^\alpha(W_\alpha) \nsubseteq [CO](W_\alpha)$. Define

$$k = \left\lceil \frac{2 - 2\alpha}{1 - 2\alpha} \right\rceil.$$

The weak tournament $W_\alpha = (A, \succsim)$ has alternatives $A = \{a_i : 1 \leq i \leq k\} \cup \{x\} \cup \{b_j : 1 \leq j \leq k-1\}$. For all $i \leq k$, $a_i \succ x$, and for all $j \leq k-1$, $x \succ b_j$. Finally, $u \sim v$ for all pairs $(u, v) \in (A \setminus \{x\}) \times (A \setminus \{x\})$.

Let $s_\alpha(a)$ denote CO^α score of alternative $a \in A$, i.e., $s_\alpha(a) = d^+(a) + t(a) \cdot \alpha$. We have $s_\alpha(a_i) = 1 + (2k-2)\alpha$ for all $i \leq k$, $s_\alpha(x) = k - 1$, and $s_\alpha(b_j) = (2k-2)\alpha$ for all $j \leq k-1$. The definition of k yields that $s_\alpha(x) \geq s_\alpha(a_i) > s_\alpha(b_j)$. Therefore, $x \in CO^\alpha(W_\alpha)$.

We will now show that $x \notin [CO](W_\alpha)$. Since x has no ties, we already know that its Copeland score is $k-1$ in any orientation of W_α. Let $T \in [W_\alpha]$ be such an orientation and let $\hat{T}$ be the restriction of T to $A \setminus \{x\}$. Since $\hat{T}$ has $2k-1$ alternatives, the average Copeland score in $\hat{T}$ is $k-1$. We distinguish two cases. If all alternatives in $A \setminus \{x\}$ have Copeland score $k-1$ in $\hat{T}$, then the Copeland score of alternative a_1 in T is k. If, on the other hand, not all alternatives in $A \setminus \{x\}$ have Copeland score $k-1$ in $\hat{T}$, then there exists an alternative $c \in A \setminus \{x\}$ that has a Copeland score of at least k in $\hat{T}$. The Copeland score of c in tournament T is therefore greater or equal to k. In both cases, we have found an alternative whose Copeland score in T is strictly greater than the Copeland score of x. It follows that $x \notin CO(T)$ for any orientation $T \in [W_\alpha]$ and, consequently, $x \notin [CO](W_\alpha)$.

Now let $\frac{1}{2} \leq \alpha \leq 1$. Consider a weak tournament G and an alternative $x \in CO^\alpha(G)$. We will show that $x \in [CO](G)$ by constructing an orientation $T \in [G]$ with $x \in CO(T)$.

Call an alternative y *active* if $t(y) > 0$, and *inactive* otherwise. As a first step, we make x inactive by letting x dominate all alternatives to which it was tied. Let s^* be Copeland score of x after this step and observe that all other alternatives have a CO^α score of at most s^*.

We then iteratively eliminate all remaining ties via the procedure described below. Throughout the procedure, the CO^α score of x will always remain maximal among the CO^α scores of all alternatives.

While there are still active alternatives, we iteratively do one of the following two operations:

(i) if there is an active node y whose current CO^α score is less then or equal to $s^* - (1-\alpha)$, choose an arbitrary alternative z with $y \sim z$ and replace the tie with $y \succ z$.

(ii) if all active nodes have a current CO^α score strictly greater than $s^* - (1-\alpha)$, find a cycle in the tie graph and orient the cycle in one direction.

It is left to be shown that both operations maintain the invariant that all alternatives have a CO^α score of less than or equal to s^*. For the second operation, we also have to argue that there always exists a cycle in the tie graph.

As for the first operation, observe that turning a tie $a \sim b$ into $a \succ b$ increases the CO^α score of a by $1-\alpha$ and decreases the score of b by α. The first operation therefore maintains the invariant.

As for the second operation, the existence of a cycle in the tie graph is guaranteed by the fact that every active alternative has at least two neighbors in the tie graph. Indeed, an active alternative y with $t(y) = 1$ has a CO^α score of $d^+(y) + \alpha$, and since $d^+(y)$ is a natural number, it is impossible that

$$s^* - (1-\alpha) < d^+(y) + \alpha \leq s^*.$$

Furthermore, orienting the cycle (arbitrarily in one of the two possible directions) decreases the CO^α score of all involved alternatives by $2\alpha - 1 \geq 0$. □

Proposition 8. *GOCHA* $\subset$ *GETCHA* $= [TC]$.

Proof. GOCHA $\subset$ GETCHA was shown by Schwartz (1972; 1986). We show that GETCHA $= [TC]$. For a weak tournament $W = (A, \succsim)$, let $D^*_\succsim(a)$ denote the set of alternatives that can be reached by a via a $\succsim$-path.

For the inclusion GETCHA $\subseteq [TC]$, consider a weak tournament $W = (A, \succsim)$ and let $a \in$ GETCHA(W). By definition of GETCHA, $D^*_\succsim(a) = A$. We can construct an orientation $T_a \in [W]$ by iteratively substituting ties $x \sim y$ with $x \in D^*_\succsim(a)$ and $y \notin D^*_\succsim(a)$ with $x \succ y$. In T_a, alternative a can reach every other alternative via a $\succ$-path. Thus, $x \in TC(T_a) \subseteq [TC](W)$.

For the inclusion $[TC] \subseteq$ GETCHA, consider a weak tournament $W = (A, \succsim)$ and an arbitrary orientation $T \in [W]$. We show that $X = TC(T) \subseteq$ GETCHA(W). Assume for contradiction that there exists $x \in X \setminus$ GETCHA(W). Minimality of X implies that GETCHA(W) cannot be a strict subset of X. Therefore, there exists $y \in$ GETCHA$(W) \setminus X$. Since $X = TC(T)$, $x \succ y$. But this contradicts the assumption that $x \notin$ GETCHA(W). □

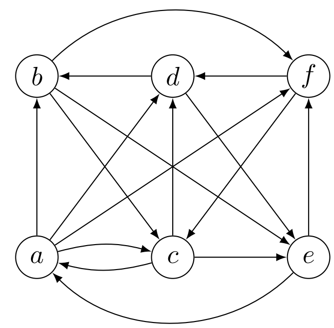


Figure 3: A weak tournament W on $A = \{a,b,c,d,e,f\}$ with $MC_M(W) = MC_D(W) = A$ and $[MC](W) = A \setminus \{f\}$.

Proposition 9. $UC_D = [UC]$.

Proof. In a tournament $T = (A, \succ)$, an alternative $y \in A$ is said to be *covered in* T if there exists an alternative $x \in A \setminus \{y\}$ such that (1) $x \succ y$ and (2) $z \succ x$ implies $z \succ y$ for all $z \in A \setminus \{x, y\}$. The uncovered set $UC(T)$ of T consists of all alternatives in A that are not covered in T.

In a weak tournament $W = (A, \succsim)$, an alternative $y \in A$ is said to be *deeply covered in* W if there exists an alternative $x \in A \setminus \{y\}$ such that (1) $x \succ y$ and (2) $z \succsim x$ implies $z \succ y$ for all $z \in A \setminus \{x, y\}$. The deep uncovered set $UC_D(W)$ of W consists of all alternatives in A that are not deeply covered in W.

Let $W = (A, \succsim)$ be a weak tournament. The identity of $UC_D(W)$ and $[UC](W)$ follows from the fact that an alternative $a \in A$ is deeply covered in W if and only if a is covered in T for all orientations $T \in [W]$. □

Proposition 10. $[MC] \subset MC_D$, $[MC] \not\subseteq MC_M$, *and* $MC_M \not\subseteq [MC]$.

Proof. In a tournament $T = (A, \succ)$ the *minimal covering set* $MC(T)$ is defined as the unique smallest set $B \subseteq A$ such that $x \notin UC(T|_{B \cup \{x\}})$ for all $x \in A \setminus B$. Moreover, in a weak tournament $W = (A, \succsim)$, an alternative $y \in A$ is said to be *McKelvey-covered in* W if there exists an alternative $x \in A \setminus \{y\}$ such that (1) $x \succ y$, (2) $z \succ x$ implies $z \succ y$ for all $z \in A$, and (3) $y \succ z$ implies $x \succ z$. The *McKelvey uncovered set* $UC_M(W)$ of W consists of all alternatives

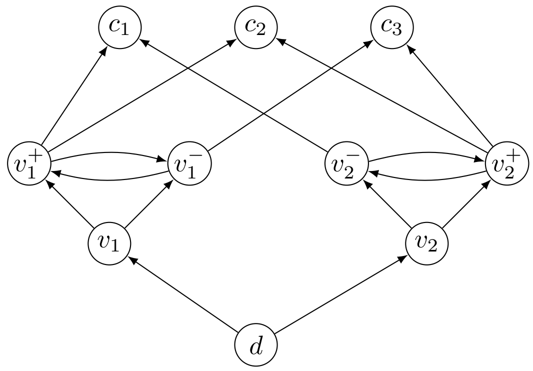


Figure 4: The weak tournament W_φ for the formula $\varphi = (v_1 \vee \overline{v_2}) \wedge (v_1 \vee v_2) \wedge (\overline{v_1} \vee v_2)$. All omitted edges point downwards or to the right.

in A that are not McKelvey-covered in W. In a weak tournament $W = (A \succsim)$, the *minimal deep covering set* $MC_D(W)$ and the *minimal McKelvey covering set* $MC_M(W)$ are then defined as the the (unique) smallest sets $B \subseteq A$ such that $x \notin UC_D(W|_{B\cup\{x\}})$ and $x \notin UC_M(W|_{B\cup\{x\}})$ for all $x \in A \setminus B$, respectively.

In tournaments, McKelvey-covering coincides with deep covering and is simply referred to as *covering*. Observe that if x deeply covers y in a weak tournament W, then x covers y in all tournaments $T \in [W]$.

We first show that $[MC] \subseteq MC_D$. Consider a weak tournament W and let $X = MC_D(W)$. By definition, X is externally stable w.r.t. deep covering, i.e., for all $y \in A \setminus X$ there exists $x \in A$ such that x deeply covers $y \in X \cup y$. Let T be an orientation of W. The above observation implies that X is externally stable in T. Since $MC(T)$ is contained in all externally stable sets, $MC(T) \subseteq X$.

In order to show that $[MC] \neq MC_D$ and $MC_M \not\subset [MC]$, consider the weak tournament W in Figure 3. It can be checked that both the McKelvey minimal covering set and the deep minimal covering set contain all alternatives, i.e., $MC_M(W) = MC_D(W) = \{a,b,c,d,e,f\}$. There are two orientations of W. Let T_1 be the orientation with $a \succ c$ and let T_2 be the orientation with $c \succ a$. Since $MC(T_1) = \{a,b,c,d,e\}$ and $MC(T_2) = \{a,b,c\}$, we have $[MC](W) = \{a,b,c,d,e\} \cup \{a,b,c\} = \{a,b,c,d,e\}$. In particular, $MC_D(W) \neq [MC](W)$ and $MC_M(W) \not\subseteq [MC](W)$.

In order to show that $[MC] \not\subset MC_M$, consider the tournament W' on $\{a_1,a_2,a_3,a_4,a_5,b\}$ such that $a_1 \succ a_2 \succ a_3 \succ a_4 \succ a_5 \succ a_1$, $a_i \succ b$ for $i \in \{1,2\}$, $b \succ a_3$, and $x \sim y$ for all other pairs. It can be checked that $MC_M(W') = \{a_1,a_2,a_3,a_4,a_5\}$ and that $b \in [MC](W')$. □

Proposition 12. *There is a tournament solution S such that the winner determination problem is in P for S, and NP-complete for $[S]$.*

Proof sketch. Let $T = (A,\succ)$ be a tournament. A *T-path of length k* is a sequence $(a_0,a_1,\dots,a_k)$ of alternatives such that $a_{i-1} \succ a_i$ for all $i \in \{1,\dots,k\}$. Define $P(T)$ as the set of all alternatives $a \in A$ that can reach every alternative $b \in A \setminus \{a\}$ by (1) a T-path (a,b) of length 1, (2) a T-path (a,c,b) of length 2, or (3) a T-path (a,c_1,c_2,c_3,b) of length 4 such that $\{c_2,c_3\} \succ a$ and $c_1 \succ c_3$. The tournament solution S maps a tournament $T = (A,\succ)$ to

$$S(T) = \begin{cases} P(T) & \text{if } P(T) \neq \emptyset \\ A & \text{otherwise.} \end{cases}$$

It can be shown that the winner determination problem for S is in P, whereas computing winners for $[S]$ is NP-hard. The latter is shown by a reduction from SAT, in which a Boolean formula φ is transformed into a weak tournament $W_\varphi = (A_\varphi, \succsim_\varphi)$. Every orientation of W_φ then corresponds to an assignment of the variables in φ, and a designated node $d \in A_\varphi$ is contained in $[S](W_\varphi)$ if and only if φ is satisfiable. Figure 4 depicts W_φ for an example formula. □

Proposition 11. $BA_m \subset [BA]$ *for* $m \in \{1,2,3,4\}$.

Proof. We start by defining the four generalizations of the Banks set that were proposed by Banks and Bordes (1988). All generalizations are based on an extension of the definition of a trajectory or maximal transitive subset. Let $\vec{a} = (a_1,\dots,a_k)$ be a sequence of alternatives of some weak tournament $W = (A,\succsim)$. Then, following Banks and Bordes (1988), we say

$\vec{a}$ is *transitive*$_1$ if $a_i \succ a_j$ for all $1 \le i < j \le k$, and
$\vec{a}$ is *transitive*$_2$ if $a_i \succsim a_j$ for all $1 \le i < j \le k$

Furthermore, $\vec{a}$ is *transitive*$_3$ whenever $\vec{a}$ is transitive$_2$ and $a_i \succ a_j$ for some $1 \le i < j \le k$. Finally, $\vec{a}$ is *transitive*$_4$ whenever $\vec{a}$ is transitive$_2$ and $a_i \succ a_{i+1}$ for all $1 \le i < k$. For $m \in \{1,2,3,4\}$, we say that $\vec{a}$ is *maximal transitive*$_m$ in W if $(a,a_1,\dots,a_k)$ is transitive$_m$ for no $a \in A$ and define BA_m such that, for all weak tournaments W,

$$BA_m(W) = \{a_1 : (a_1,\dots,a_k) \text{ is maximal transitive}_m\}.$$

Banks and Bordes (1988) showed that each of their generalizations BA_m always selects a nonempty subset of alternatives. Moreover, on tournaments each each BA_m coincides with the Banks set BA.

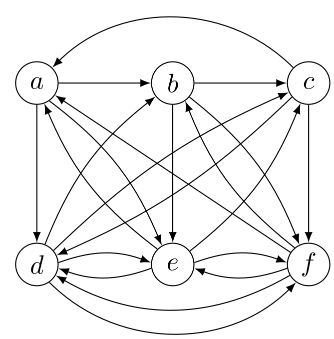


Figure 5: A weak tournament W with $BA_1(W) = \{a,b,c\}$ and $BA_2(W) = \{d,e,f\}$.

We now show that each of the generalizations BA_m is a refinement of $[BA]$.

First, let $m \in \{1,2,3,4\}$ and $W = (A, \succsim)$ a weak tournament. Assume that $a \in BA_m(W)$, i.e., $a = a_1$ for some $\vec{a} = (a_1, \dots, a_k)$ that is maximal transitive$_m$ in W. Observe that an orientation $T = (A, \succ')$ of W exists such that

(i) $a_i \succ' a_j$, for all a_i, a_j with $1 \le i < j \le k$, and

(ii) $a_i \sim x$ implies $a_i \succ' x$, for all a_i with $1 \le i \le k$ and $x \in A \setminus \{a_1, \dots, a_k\}$.

Also observe that there is no $x \in A \setminus \{a_1, \dots, a_k\}$ with $x \succ' a_i$ for all $1 \le i \le k$. Otherwise, also $x \succ a_i$ for all $1 \le i \le k$ and $\vec{a}$ would not be maximal transitive$_m$ in W. It thus follows that $\vec{a}$ is maximal transitive in T and that $a \in BA(T)$. As $T \in [W]$, we may conclude that $a \in [BA](W)$.

Second, Banks and Bordes (1988) demonstrate in their paper that for each $m \in \{2,3,4\}$, there is a weak tournament $W = (A, \succsim)$ with $BA_1(W) \cap BA_m(W) = \emptyset$ (see Figure 5 for the case $m = 2$). As none of the generalizations of the Banks set ever yields the empty set, there are $a, b \in A$ such that $a \in BA_1(W) \setminus BA_m(W)$ and $b \in BA_m(W) \setminus BA_1(W)$. Since both $BA_1(W) \subseteq [BA](W)$ and $BA_m(W) \subseteq [BA](W)$, it follows that $b \in [BA](W)$ whereas $b \notin BA_1(W)$ and $a \in [BA](W)$ although $a \notin BA_m(W)$. That is, $BA_m(W) \subset [BA](W)$ for each $m \in \{1,2,3,4\}$, as desired. $\square$

B More Choice-Theoretic Properties

In this section, we introduce a new perspective on the properties $\widehat{\alpha}$ and $\widehat{\gamma}$ by splitting them up further. For A, B feasible sets, $X \subseteq A \cap B$, and S a generalized tournament solution, we define the properties $\widehat{\alpha}_\subseteq$, $\widehat{\alpha}_\supseteq$, $\widehat{\gamma}_\subseteq$, and $\widehat{\gamma}_\supseteq$ as follows.

$$S(A \cup B) = X \text{ implies } S(A) \subseteq X \text{ and } S(B) \subseteq X \quad (\widehat{\alpha}_\subseteq)$$
$$S(A \cup B) = X \text{ implies } S(A) \supseteq X \text{ and } S(B) \supseteq X \quad (\widehat{\alpha}_\supseteq)$$
$$X = S(A) = S(B) \text{ implies } X \subseteq S(A \cup B) \quad (\widehat{\gamma}_\subseteq)$$
$$X = S(A) = S(B) \text{ implies } X \supseteq S(A \cup B) \quad (\widehat{\gamma}_\supseteq)$$

Property $\widehat{\alpha}_\subseteq$ is also known as the *weak superset property* or the *Aïzerman property* in the literature.

Obviously, we have $\widehat{\alpha} \Leftrightarrow \widehat{\alpha}_\subseteq \wedge \widehat{\alpha}_\supseteq$ and $\widehat{\gamma} \Leftrightarrow \widehat{\gamma}_\subseteq \wedge \widehat{\gamma}_\supseteq$. Furthermore note that $\widehat{\alpha}_\subseteq$ implies *idempotency*, i.e., $S(S(W)) = S(W)$ for all weak tournaments W. Figure 6 shows the implications between the properties considered in this section.

Similar to the case of $\widehat{\alpha}$, the properties $\widehat{\alpha}_\subseteq$ and $\widehat{\alpha}_\supseteq$ can be formulated in a more accessible way.

Lemma 4. *Let S be a generalized tournament solution.*

(i) *S satisfies $\widehat{\alpha}_\subseteq$ if and only if, for all feasible sets A, B,*

$$S(A) \subseteq B \subseteq A \textit{ implies } S(A) \subseteq S(B).$$

(ii) *S satisfies $\widehat{\alpha}_\supseteq$ if and only if, for all feasible sets A, B,*

$$S(A) \subseteq B \subseteq A \textit{ implies } S(A) \supseteq S(B).$$

Using these characterizations, both properties can be reformulated as invariance properties with respect to $F^{\widehat{\alpha}}$.

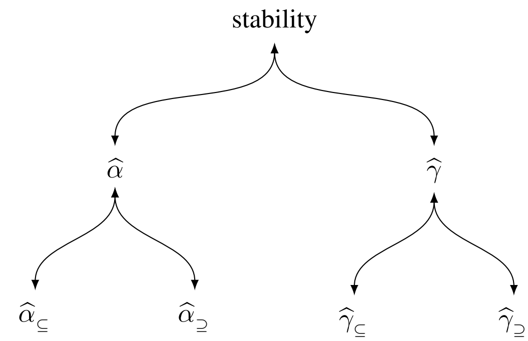


Figure 6: Implications of stability properties

Lemma 5. *Let S be a generalized tournament solution.*

(i) *S satisfies $\widehat{\alpha}_\subseteq$ if and only if S is inclusion-invariant under $F^{\widehat{\alpha}}$.*

(ii) *S satisfies $\widehat{\alpha}_\supseteq$ if and only if S is exclusion-invariant under $F^{\widehat{\alpha}}$.*

Proposition 4 can thus be extended to also cover $\widehat{\alpha}_\subseteq$ and $\widehat{\alpha}_\supseteq$.

Proposition 16. *Let $\phi \in \{\widehat{\alpha}_\subseteq, \widehat{\alpha}_\supseteq\}$. If a tournament solution S satisfies property ϕ on $\mathcal{T}$, so does $[S]$ on $\mathcal{W}$.*

Similarly, Proposition 5 extends to $\widehat{\gamma}_\subseteq$ and $\widehat{\gamma}_\supseteq$.

Proposition 17. *Let S be a tournament solution that satisfies $\widehat{\alpha}$ and let $\phi \in \{\widehat{\gamma}_\subseteq, \widehat{\gamma}_\supseteq\}$. If S satisfies property ϕ on $\mathcal{T}$, so does $[S]$ on $\mathcal{W}$.*

Finally, we prove that the properties $\widehat{\alpha}$ and $\widehat{\gamma}$ are independent

Proposition 18. *There exists a tournament solution that satisfies $\widehat{\alpha}$, but not $\widehat{\gamma}$.*

Proof sketch. A *Condorcet loser* in a tournament $T = (A, \succ)$ with $|A| > 1$ is an alternative $a \in A$ with $b \succ a$ for all $b \in A \setminus \{a\}$. Every tournament can have at most one Condorcet loser. Let S be the tournament solution that selects all alternatives that survive the process of iteratively deleting Condorcet losers until a tournament without a Condorcet loser is reached.

It can be shown that S satisfies $\widehat{\alpha}$, but violates $\widehat{\gamma}$. For the latter, consider a tournament $(\{a,b,c,d\}, \succ)$ such that $a \succ \{b,c,d\}$, $b \succ c$, $c \succ d$, and $d \succ b$. By definition $S(\{a,b\}) = \{a\}$ and $S(\{a,c,d\}) = \{a\}$, but $S(a,b,c,d) = \{a,b,c,d\}$. $\square$

Proposition 19. *There exists a tournament solution that satisfies $\widehat{\gamma}$, but not $\widehat{\alpha}$.*

Proof sketch. Let S' be a stable tournament solution. Define the tournament solution S such that for each tournament $T = (A, \succ)$,

$$S(T) = \begin{cases} S'(T) & \text{if } |A \setminus S'(T)| > 1 \\ A & \text{otherwise.} \end{cases}$$

It can be shown that S satisfies $\widehat{\gamma}$, but violates $\widehat{\alpha}$. For the latter, let S' be the top cycle and consider a transitive tournament $(\{a,b,c\}, \succ)$ such that $a \succ b$, $b \succ c$, and $a \succ c$. By definition, $S(\{a,b,c\}) = \{a\}$, but $S(\{a,b\}) = \{a,b\}$. $\square$

C Composition-Consistency

This section contains definitions and results for different variants of composition-consistency.

C.1 Generalizing Composition-Consistency

A *decomposition* of a weak tournament $W = (A, \succsim)$ we define as a partition $\{X_1, \dots, X_k\}$ of A such that each X_i is a component of W. Moreover, let $W_1 = (B_1, \succsim_1), \dots, W_k = (B_k, \succsim_k)$, and $\tilde{W} = (\{1, \dots, k\}, \tilde{\succsim})$ be weak tournaments with $B_1, \dots, B_k$ pairwise disjoint. Then, define the *product* $\prod(\tilde{W}, W_1, \dots, W_k)$ of $W_1, \dots, W_k$ with respect to $\tilde{W}$ as the weak tournament $(A, \succsim')$ such that $A = \bigcup_{i=1}^{k} B_i$ and, for all $b_1 \in B_i, b_2 \in B_j$,

$$b_1 \succsim' b_2 \text{ if and only if } i = j \text{ and } b_1 \succsim_i b_2, \text{ or } i \neq j \text{ and } i \tilde{\succsim} j.$$

We are now in a position to define composition-consistency for weak tournaments.

Definition 8. *A generalized tournament solution S is* composition-consistent (on $\mathcal{W}$) *if for all weak tournaments W, decompositions $\{X_1, \dots, X_k\}$ of W, and $\tilde{W} = (\{1, \dots, k\}, \tilde{\succsim})$ such that $W = \prod(\tilde{W}, W|_{X_1}, \dots, W|_{X_k})$,*

$$S(T) = \bigcup_{i \in S(\tilde{W})} S(W|_{X_i}).$$

Let $W = \prod(\tilde{W}, W_1, \dots, W_k)$ where $W_i = (B_i, \succsim_i)$ for all i with $1 \leq i \leq k$. Observe that then $\{B_1, \dots, B_k\}$ is a decomposition of W whenever $\tilde{W}$ is a tournament. If, moreover, no B_i is a singleton, the implication also holds in the opposite direction.

C.2 Weak Composition-Consistency

Where A is feasible set, we denote by $\mathcal{W}(A)$ the set of weak tournaments with A as the set of alternatives. For Y a component of a weak tournament $W = (A, \succsim)$ and $W' \in \mathcal{W}(Y)$ a weak tournament on Y, let $W^Y_{W'} = (A, \succsim'')$ denote the weak tournament that is like W except that the subtournament $W|_Y$ induced by component Y is replaced by W'. Formally, for weak tournaments W with components X and Y such that $W = \prod(\tilde{W}, W|_X, W|_Y)$, and $W' \in \mathcal{W}_Y$, we have $W^Y_{W'}$ denote $\prod(\tilde{W}, W|_X, W')$. We are now in a position to give the definition of *weak composition-consistency* for weak tournaments.

Definition 9. *A generalized tournament solution S is* weakly composition-consistent (on $\mathcal{W}$) *if for all weak tournaments W, component Y of W, and $W' \in \mathcal{W}(Y)$,*

(i) *$S(W) \setminus Y = S(W^Y_{W'}) \setminus Y$, and*
(ii) *$S(W) \cap Y \neq \emptyset$ implies $S(W^Y_{W'}) \cap Y \neq \emptyset$.*

It can easily be verified that this definition conservatively extends the regular definition of weak composition-consistency for tournament solutions (Laslier 1997), which we will refer to as *weak composition-consistency on $\mathcal{T}$*.

Proposition 20. *If a tournament solution S is weakly composition-consistent on $\mathcal{T}$, so is $[S]$ on $\mathcal{W}$.*

Proof. Let S be a tournament solution that is weakly composition-consistent on $\mathcal{T}$ and $W = (A, \succsim)$ a weak tournament with components X and Y such that $W = \prod(\tilde{W}, W|_X, W|_Y)$. Furthermore, let $W' \in \mathcal{W}(Y)$. We prove that

(i) $[S](W) \setminus Y = [S](W^Y_{W'}) \setminus Y$, and
(ii) $[S](W) \cap Y \neq \emptyset$ implies $[S](W^Y_{W'}) \cap Y \neq \emptyset$.

First observe that, by virtue of Lemma 2, for every orientation $T \in [W]$ we have that $\{X, Y\}$ is a decomposition of T as well, i.e., $T = \prod(\tilde{T}, T|_X, T|_Y)$ for some $\tilde{T} \in \mathcal{T}(\{1, 2\})$. Moreover, it holds that $\tilde{T} \in [\tilde{W}]$, $T|_X \in [W|_X]$, and $T|_Y \in [W|_Y]$. Also observe that, for each $T' \in [W']$, we may assume that $T' \in \mathcal{T}(Y)$ and $T^Y_{T'} \in [W^Y_{W'}]$.

For (i), having assumed S to be weakly composition-consistent on $\mathcal{T}$, the following equivalences hold.

$$\begin{aligned}
& a \in [S](W) \setminus Y \\
\text{iff } & a \in S(T) \setminus Y \quad \text{for some } T \in [W] \\
\text{iff } & a \in S(T^Y_{T'}) \setminus Y \quad \text{for some } T \in [W] \text{ and some } T' \in [W'] \\
\text{iff } & a \in [S](W^Y_{W'}) \setminus Y.
\end{aligned}$$

For (ii), assume $[S](W) \cap Y \neq \emptyset$. Then, there is some orientation $T \in [W]$ such that $S(T) \cap Y \neq \emptyset$. Let $T \in [W']$. Then also $T \in \mathcal{T}(Y)$. Having assumed that S satisfies weak composition-consistency on $\mathcal{T}$, we obtain $S(T^Y_{T'}) \cap Y \neq \emptyset$. As $T^Y_{T'} \in [W^Y_{W'}]$, we conclude that $[S](W^Y_{W'}) \cap Y \neq \emptyset$. ☐

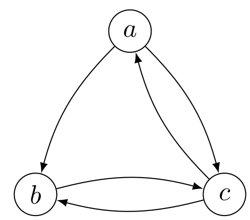


Figure 7: Weak tournament $W = (\{a, b, c\}, \succsim)$ showing that stronger concepts of composition-consistency are not inherited by $[S]$ if S is Condorcet-consistent.

C.3 Other Notions of Composition-Consistency

A natural stronger notion of composition-consistency could be based on a weaker concept of component. Thus, for weak tournaments $W = (A, \succsim)$, a component could be defined as a subset $X \subseteq A$ such that for all $y \in A \setminus X$, either $X \succ y$, $y \succ X$, or $X \sim y$. Observe that for such components Lemma 2 does no longer hold. Moreover, it can easily be seen that the conservative extension $[S]$ of *no* Condorcet-consistent tournament solution S satisfies the associated concept of composition-consistency. To appreciate this, let S be Condorcet-consistent and consider the weak tournament $W = (A, \succsim)$ with $A = \{a, b, c\}$ and $a \succ b$, $a \sim c$, and $b \sim c$ (see Figure 7). Observe that W can be oriented such that $b \succ c$ and $c \succ a$, resulting in a cyclical tournament from which every tournament solution chooses $\{a, b, c\}$. Hence, $[S](W) = \{a, b, c\}$. However, $\{\{a, b\}, \{c\}\}$ would be a decomposition under the alternate definition and, by Condorcet-consistency, alternative b would not be chosen by S from $T|_{\{a,b\}}$ for any orientation $T \in [W]$. Accordingly, if $[S]$ had been composition-consistent in the new sense, $b \notin [S](W)$, a contradiction.

D Regularity and Weak Regularity

For a weak tournament $W = (A, \succsim)$ and alternative $a \in A$, we let $d^+_W(a)$ and $d^-_W(a)$ denote the *outdegree* of a, i.e., cardinality of the dominion, and the *indegree* of a, i.e., the cardinality of the set of dominators of a, i.e.,

$$d^+_W(a) = |\{x \in A : a \succsim x\}|, \text{ and}$$
$$d^-_W(a) = |\{x \in A : x \succsim a\}|.$$

Moreover, $t_W(a)$ denotes the number of alternatives a ties with, i.e.,

$$t_W(a) = |\{x \in A \setminus \{a\} : a \sim x\}|.$$

We omit the subscript when W is clear from the context.

A tournament $T = (A, \succ)$ is said to be *regular* whenever the indegree equals the outdegree for each alternatives $a \in A$. This definition can immediately be extended to weak tournaments. Thus, we say that a weak tournament $W = (A, \succ)$ is *regular* if $d^+(a) = d^-(a)$ for all $a \in A$.[7]

It can easily be appreciated that a tournament being regular implies its order to be odd. This, however, is not generally the case for weak tournaments, i.e., regular weak tournaments of even order exist. We also have the following lemma, which implies that, by orienting edges in a regular weak tournament of even order all ties can be eliminated without impairing regularity.

Lemma 6. *Let $W = (A, \succsim)$ be a weak tournament such that $|A|$ is odd and $d^+_W(a) = d^-_W(a)$ for all alternatives $a \in A$. Then, there is a regular orientation $T \in [W]$.*

Proof. We proceed by induction on $\sum_{a\in A} t_W(a)$. If $\sum_{a\in A} t_W(a) = 0$, then W is already a tournament and we are done immediately. So assume that $\sum_{a\in A} t_W(a) > 0$. Having assumed that $t_W(a)$ is even for all $a \in A$, there is some alternative a with $t_W(a) \geq 2$. By the same token, it follows that there is a cycle $a_1, \ldots, a_m$ with $a = a_1 = a_m$ in the tie graph $G = (A, E)$, in which $\{x, y\} \in E$ if and only if $x \sim y$. Now, let $W' = (A, \succsim')$ be the weak tournament in which the cycle $a_1, \ldots, a_m$ is oriented in one direction, i.e.,

$$\succsim' \;=\; \succsim \setminus \{(a_{i+1}, a_i) : 1 \leq i < m\}.$$

Then, for every $x \in \{a_1, \ldots, a_m\}$ we have

$$d^+_{W'}(x) = d^+_W(x) - 1 \quad\text{and}\quad d^-_{W'}(x) = d^-_W(x) - 1.$$

Moreover, $t_{W'}(x) = t_W(x) - 2$. For all $y \notin \{a_1, \ldots, a_m\}$, it holds that $d^+_{W'}(y) = d^+_W(y)$ and $d^-_{W'}(y) = d^-_W(y)$, as well as $t_{W'}(y) = t_W(y)$. Since now, obviously,

$$\sum_{a\in A} t_{W'}(a) < \sum_{a\in A} t_W(a),$$

the statement follows by the induction hypothesis. □

A generalized tournament solution S is then said to be *regular* if $S(W) = A$ for every regular weak tournament $W = (A, \succsim)$. The order of regular tournaments necessarily being odd, regularity of a tournament solution S as such does not impose any restriction on its behavior on tournaments of even order and, *ipso facto*, neither on the orientations of a weak tournament of even order. From this perspective, an arguably more natural extension of the concept of regularity to weak tournaments takes into account the parity of weak tournaments. Thus, we say a generalized tournament solution S is *weakly regular* if $S(W) = A$ for every regular weak tournament $W = (A, \succsim)$ such that $|A|$ is odd.

Observe that regularity implies weak regularity. On tournaments, the notions coincide.

Weak regularity is inherited by the conservative extension.

Proposition 21. *If a tournament solution S is weakly regular on $\mathcal{T}$, so is $[S]$ on $\mathcal{W}$.*

Proof. Let S be a weakly regular tournament solution and consider an arbitrary regular weak tournament $W = (A, \succsim)$ such that $|A|$ is odd. By weak regularity of W, for every alternative a, we have that $t(a)$ is even as well as that $d^-_W(a) = d^+_W(a)$. By Lemma 6, there is a regular orientation $T \in [W]$. Accordingly, $S(T) = A$. It follows that $[S](W) = A$, as desired. □

However, regularity of a tournament solution S does not in general extend to regularity of $[S]$ on weak tournaments.

Proposition 22. *There exists a regular tournament solution S such that $[S]$ is not regular on weak tournaments.*

Proof. A sequence $(a_1, \ldots, a_k)$ of alternatives is a *trajectory* if $i < j$ implies $a_i \succ a_j$. Let S be the tournament solution that selects the $\succ$-maximal elements of the trajectories that are of maximal length. Note that S is not regular: for the regular tournament T depicted in Figure 8(a), $S(T) = \{a, c, f\}$. Now define S^* as the tournament solution that is exactly like S apart from it choosing all alternatives from every regular tournament, i.e., for all tournaments $T = (A, \succ)$,

$$S^*(T) = \begin{cases} A & \text{if } T \text{ is regular,} \\ S(T) & \text{otherwise.} \end{cases}$$

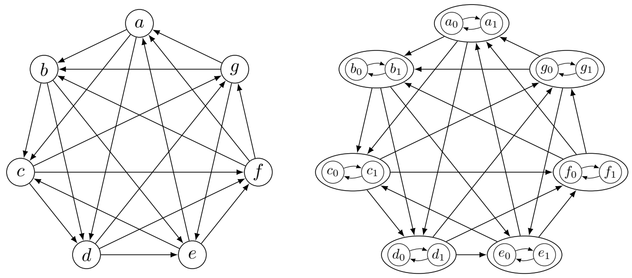
By definition, S^* is regular.

Now consider the weak tournament $W = (A', \succsim)$ depicted in Figure 8(b). An easy check reveals that W is

[7] An alternate definition regularity would result if one requires regular weak tournaments $W = (A, \succ)$ to be such that for all $a, b \in A$,

$$d^+(a) = d^-(a) = d^+(b) = d^-(b).$$

This definition gives rise to a stronger notion of a regular weak tournament and a weaker notion of a regular and a weakly regular generalized tournament solution. For these notions the results in this section also hold.



(a) tournament T (b) weak tournament W

Figure 8: The tournament T as depicted in (a) is regular but not vertex-homogeneous. For instance, there is no automorphism mapping alternative a to alternative b. The weak tournament W depicted in (b), results from T by 'replacing' *every* alternative x by a subtournament X on alternatives x_0 and x_1 such that $x_0 \sim x_1$.

regular: observe that W results from the regular tournament T in Figure 8(a) by 'replacing' *every* alternative x by a subtournament X on alternatives x_0 and x_1 such that $x_0 \sim x_1$. It can also easily be verified that for every orientation $T' \in [W]$ we have that $S^*(T') \subseteq \{a_0, a_1, c_0, c_1, f_0, f_1\}$. (For an example see the orientation in Figure 9, from which S^* selects $\{a_0, c_0, f_0\}$). Accordingly, $[S^*](W) \subseteq \{a_0, a_1, c_0, c_1, f_0, f_1\}$. It follows that $[S^*]$ is not regular.[8] □

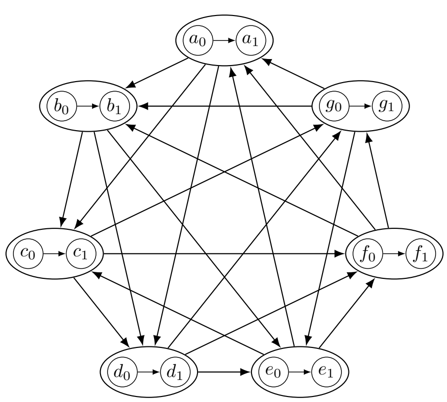


Figure 9: An orientation of the weak tournament W

[8] A similar argument, involving a more complicated weak tournament can be given for the tournament solution BA_{reg} defined such that, for all tournaments $T = (A, \succ)$,

$$BA_{reg}(T) = \begin{cases} A & \text{if } T \text{ is regular,} \\ BA(T) & \text{otherwise.} \end{cases}$$