Consider an urn filled with balls, each labelled with one of several possible collective decisions. Now, draw two balls from the urn, let a random voter pick her more preferred as the collective decision, relabel the losing ball with the collective decision, put both balls back into the urn, and repeat. In order to prevent the permanent disappearance of some types of balls, a randomly drawn ball is labelled with a random collective decision once in a while. We prove that the empirical distribution of collective decisions converges towards the outcome of a celebrated probabilistic voting rule proposed by Peter C. Fishburn (Rev. Econ. Stud., 51(4), 1984). The proposed procedure has analogues in nature recently studied in biology, physics, and chemistry. It is more flexible than traditional voting rules because it does not require a central authority, elicits very little information, and allows agents to arrive, leave, and change their preferences over time.

1 Introduction

The question how to collectively select one of many alternatives based on the preferences of multiple agents has occupied great minds from various disciplines. Its formal study goes back to the Age of Enlightenment, in particular during the French Revolution, and the important contributions by Jean-Charles de Borda and Marie Jean Antoine Nicolas de Caritat, better known as the Marquis de Condorcet. Borda and Condorcet agreed that plurality rule—then and now the most common collective choice procedure—has serious shortcomings. This observation remains a point of consensus among social choice theorists and is largely due to the fact that plurality rule merely asks each voter for her most-preferred alternative (see, e.g., Brams and Fishburn, 2002; Laslier, 2011). When eliciting more fine-grained preferences such as complete rankings

1 For example, plurality rule may select an alternative that an overwhelming majority of voters consider to be the worst of all alternatives.
over all alternatives from the voters, much more attractive choice procedures are available. As a matter of fact, since Arrow’s (1951) seminal work, the standard assumption in social choice theory is that preferences are given in the form of binary relations that satisfy completeness, transitivity, and often anti-symmetry. Despite a number of results which prove critical limitations of choice procedures for more than two alternatives (e.g., Arrow, 1951; Gibbard, 1973; Satterthwaite, 1975), there are many encouraging results (e.g. Young, 1974; Young and Levenglick, 1978; Brams and Fishburn, 1978; Laslier, 2000). In particular, when allowing for randomization between alternatives, some of the traditional limitations can be avoided and there are appealing choice procedures that stand out (Gibbard, 1977; Brandl et al., 2016; Brandl and Brandt, 2020).

The standard framework in social choice theory rests on a number of rigid assumptions that confine its applicability: there is a fixed set of voters, a fixed set of alternatives, and a single point in time when preferences are to be aggregated; all voters are able to rank-order all alternatives; there is a central authority that has access to all preferences, computes the outcome, and convinces voters of the outcome’s correctness, etc. On top of that, computing the outcome of many attractive choice procedures is a demanding task that requires a computer, which can render the process less transparent to voters.\footnote{In some cases, computing the outcome was even shown to be NP-hard, i.e., the computational effort required to compute election winners increases exponentially in the number of alternatives (see, e.g., Bartholdi, III et al., 1989; Brandt et al., 2016).}

In this paper, we propose a simple urn-based procedure that implements a celebrated choice procedure called maximal lotteries (Fishburn, 1984). Our goal is to devise a continuous process in which voters may arrive, leave, and change their preferences over time. Moreover, voters are never asked for their complete preference relations, but rather reveal minimal information about their preferences by choosing between two randomly drawn alternatives from time to time. No central voting authority is required. The process can be executed via a simple physical device: an urn filled with balls that allows for two primitive operations: (i) randomly sampling a ball and (ii) replacing a sampled ball of one kind with a ball of another kind.

More precisely, the process works as follows. Consider an urn filled with balls, each carrying the label of one alternative. The initial distribution of balls in the urn is irrelevant. In each round, a randomly selected voter will draw two balls from the urn at random. Say these two balls are labeled with alternatives 1 and 2, and the voter prefers 1 to 2. She will then change the label of the second ball to 1 and return both balls to the urn. Alternative 1 is declared the winner of this round. Once in a while, with some probability \( r \), which we call \textit{mutation rate}, a randomly drawn ball is labelled with a random alternative.

We show that if the number of balls in the urn is sufficiently large, then the empirical distribution of the winners converges to a lottery close to a maximal lottery almost surely, that is, with probability 1. How far the limiting distribution will be from a maximal lottery depends on \( r \). As \( r \) goes to 0, the limiting distribution converges to a maximal lottery. We can, however, not set \( r \) to 0 as then with probability 1, all alternatives except one will permanently disappear from the urn and the limiting distribution will
be degenerate.

Maximal lotteries are known to satisfy a number of desirable properties that are typically considered in social choice theory. For example, Condorcet winners (i.e., alternatives that defeat every other alternative in a pairwise majority comparison) will be selected with probability 1, and Condorcet losers (i.e., alternative that are defeated in all pairwise majority comparisons) will never be selected. No group of voters benefits by abstaining from an election, removing losing alternatives does not affect maximal lotteries, and each alternative’s probability is unaffected by cloning other alternatives. Maximal lotteries have been axiomatically characterized using Arrow’s independence of irrelevant alternatives and Pareto efficiency (Brandl and Brandt, 2020) as well as population-consistency and composition-consistency (Brandl et al., 2016). The dynamic procedure described above implements maximal lotteries while providing

- myopic strategyproofness (in each round, a randomly selected voter chooses between two alternatives),
- minimal preference elicitation and thus increased privacy protection,
- verifiability realized via a simple physical procedure, and
- increased flexibility in the sense that agents may arrive, leave, and change their preferences over time; similarly, the set of alternatives can be modified while the process is running.

Remarkably, continuous dynamic processes very similar to the discrete process we describe here have recently been studied in quantum physics, population biology, chemical kinetics, and plasma physics to model phenomena such as the condensation of bosons, the coexistence of species, the reactions of molecules, and the scattering of plasmons. In each of these cases, simple interactions between randomly sampled entities can be connected to equilibrium strategies in symmetric zero-sum games. In fact, maximal lotteries are precisely the mixed Nash equilibrium (or maximin) strategies of the symmetric two-player zero-sum game given by the pairwise majority margins of the voters’ preferences. We discuss these relationships, including those to evolutionary game theory, in detail in Section 5

An alternative interpretation of our result can be used to describe the formation of opinions. In this model, there is a population of agents, each of which entertains one of many possible opinions. Agents come together in random pairwise interactions, in which they try to convince each other of their opinion. The probabilities with which one opinion beats another are given as a square matrix and with some small probability, an agent randomly changes her opinion. Our main theorem then shows that, if the population is large enough, the distribution of opinions within the population is close to a maximal lottery of the probability matrix most of the time.
2 The Model

Let \([d] = \{1, \ldots, d\}\) be a set of alternatives and \(\Delta\) the \(d - 1\)-dimensional unit simplex in \(\mathbb{R}^d\), that is, \(\Delta = \{x \in \mathbb{R}_{\geq 0}^d : \sum_{i=1}^d x_i = 1\}\). We refer to elements of \(\Delta\) as lotteries.

Throughout the paper, for a vector \(x \in \mathbb{R}^k\) for some \(k\), \(|x| = \sum_{l=1}^k |x_l|\) denotes its \(L^1\)-norm. For a finite set \(S\), we write \(#S\) for the number of elements of \(S\).

A preference relation \(\succ\) is an asymmetric binary relation over \([d]\). By \(\mathcal{R}\) we denote the set of all preference relations. Let \(V\) be a finite set of voters. A preference profile \(R \in \mathcal{R}^V\) specifies a preference relation for each voter. With each preference profile \(R\), we can associate a matrix \(M_R \in [0,1]^{d \times d}\) that states for each ordered pair of alternatives the fraction of voters who prefer the first to the second. That is, \(M_R(i,j) = \frac{|\{v \in V : i \succ_v j\}|}{|V|}\). This matrix induces a skew-symmetric matrix \(\tilde{M}_R = M_R - M_R^\top\), which we call the matrix of majority margins.\(^3\)

2.1 Maximal Lotteries

A lottery \(p \in \Delta\) is a maximal lottery for a profile \(R\) if \(\tilde{M}_R p \leq 0\). By \(ML(R)\) we denote the set of all lotteries that are maximal for \(R\).

Example 1. Consider, for example, three voters \((V = \{v_1, v_2, v_3\})\), three alternatives \((d = 3)\), and a preference profile \(R\) given by the following table (each column contains the preference ranking of the corresponding voter).

<table>
<thead>
<tr>
<th></th>
<th>(v_1)</th>
<th>(v_2)</th>
<th>(v_3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(v_1)</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>(v_2)</td>
<td>2</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>(v_3)</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

Then,

\[
M_R = \begin{pmatrix}
0 & 2/3 & 2/3 \\
1/3 & 0 & 2/3 \\
1/3 & 1/3 & 0
\end{pmatrix}
\quad \text{and} \quad
\tilde{M}_R = \begin{pmatrix}
0 & 1/3 & 1/3 \\
-1/3 & 0 & 1/3 \\
-1/3 & -1/3 & 0
\end{pmatrix}.
\]

The set of maximal lotteries \(ML(R) = \{(1,0,0)^\top\}\) only contains the degenerate lottery with probability 1 on the first alternative. This alternative is a Condorcet winner; i.e., an alternative that is preferred to every other alternative by some majority of voters.

2.2 Markov Chains

We need some basic concepts for Markov chains. Let \(S\) be a finite set, called the state space. A Markov chain with state space \(S\) is a sequence of random variables \(\{X(n) : n \in \mathbb{N}\}\) with values in \(S\) so that for all \(n \in \mathbb{N}\) and states \(s, s_0, \ldots, s_n \in S\),

\[
\mathbb{P}(X(n+1) = s \mid X(n) = s_n) = \mathbb{P}(X(n+1) = s \mid X(n) = s_n, \ldots, X(0) = s_0)
\]

\(^3\)A matrix \(M\) is skew-symmetric if \(M = -M^\top\).
The defining property of Markov chains is that the probability of transitioning to any state from time \( n \) to time \( n + 1 \) depends only on the state at time \( n \). Conditional on the state at time \( n \), it is independent of the states at times \( 0, \ldots, n - 1 \). The Markov chain is time-homogeneous if the probability on the left-hand side is independent of \( n \). With a time-homogeneous Markov chain, we can associate its *transition probability matrix* \( P \in [0, 1]^{S \times S} \) with

\[
P(s, s') = \mathbb{P} \left( X(n + 1) = s \mid X(n) = s' \right)
\]

for all \( s, s' \in S \). \( P \) is a (row) stochastic matrix, which means that it has non-negative values each of its rows sums to 1. We will frequently write \( X(n, s_0) \) for the random variable \( X(n) \) conditioned on \( X(0) = s_0 \in S \) and call \( s_0 \) the *initial state*. All Markov chains we consider will be time-homogeneous and have a finite state space.

Let \( \{X(n) : n \in \mathbb{N}\} \) be a Markov chain with transition probability matrix \( P \). The period of a state \( s \in S \) is the greatest common divisor of the return times \( \{n \in \mathbb{N} : (P^n)(s,s) > 0\} \). A Markov chain is *aperiodic* if every state has period 1. Aperiodicity requires that the return times of each state are not all multiples of the same prime. A Markov chain is *irreducible* if every state is reached from any other state with positive probability. That is, for any two states \( s, s' \in S \), there is a positive integer \( n \) so that \( (P^n)(s,s') > 0 \). If \( \{X(n) : n \in \mathbb{N}\} \) is irreducible and aperiodic, it has a unique *stationary distribution* \( \pi \in \Delta S \) so that

\[
\pi^\top = \pi^\top P
\]

Hence, \( \pi \) is a left-eigenvector of the transition probability matrix \( P \) for the eigenvalue 1.

### 2.3 The Urn Process

Consider an urn with \( N \in \mathbb{N} \) balls, each labeled with some alternative. Viewing balls with the same label as indistinguishable, we can identify each state of the urn with an element of \( S^{(N)} = \{s \in \mathbb{N}^d : \sum_{i=1}^d s_i = N\} \). Fix a *mutation rate* \( r \in [0, 1] \).

We are interested in a Markov chain with state space \( S^{(N)} \), which can be informally described as follows. First, we flip a coin that has probability \( 1 - r \) of landing heads. If the coin shows heads, we choose one voter \( v \in V \) uniformly at random and ask the voter to draw two balls from the urn. Say these two balls are labeled with alternatives 1 and 2. If \( 1 \succ_v 2 \), the label of the second ball is changed to 1. Likewise, if \( 2 \succ_v 1 \), the first ball is relabeled with label 2. If both balls carry the same label, the labels remain unchanged. If the coin shows tails, we draw a single ball from the urn, relabel it with an alternative chosen uniformly at random, and put it back into the urn.

In order to formally capture this process, we define a transition probability matrix \( P^{(N,r)} \) that specifies for every pair of states the probability that the distribution of the urn transitions from the first to the second. Denote by \( e_i \) the \( i \)th unit vector in \( \mathbb{N}^d \). For
that one of the balls of the second type is replaced with one of the first type is
\[ \text{if } i \neq j \]
\[ \{1 - r\} \sum_{k=1}^{d} \frac{\binom{s_k}{k}}{\binom{s}{k}} + \frac{r}{d} \text{ if } i = j \]
be the probability of transitioning from \( s \) to \( s' \). For the remaining pairs of states \( s, s' \in S^{(N)} \), let \( P^{(N,r)}(s, s') = 0 \). Then, \( P^{(N,r)} \) has non-negative values and its rows sum to 1 so that it is a valid transition probability matrix. For an initial state \( s_0 \in S^{(N)} \), we consider a Markov chain \( \{X^{(N,r)}(n, s_0): n \in \mathbb{N}\} \) with transition probability matrix \( P^{(N,r)} \). The distribution of \( X^{(N,r)}(n, s_0) \) over \( S^{(N)} \) is given by the row of \( (P^{(N,r)})^n \) with index \( s_0 \). If \( r > 0 \), this Markov chain is irreducible and aperiodic. It corresponds to the urn process described above when the initial state of the urn is \( s_0 \).

Continuing our previous example, consider an urn containing \( N = 5 \) balls. Then, the transition probability matrix \( P^{(N,r)} \) is an \( \frac{3+5-1}{5} = 21 \)-dimensional square matrix. Let the mutation rate be \( r = 0.1 \) and the initial state \( s_0 = (1, 2, 2)^T \). Then, the probability that one of the balls of the second type is replaced with one of the first type is
\[ P^{(5,0.1)}(s_0, (2, 1, 2)^T) = 0.9 \cdot \frac{2}{10} \cdot \frac{2}{3} + 0.1 \cdot \frac{1}{3} \cdot \frac{2}{5} = \frac{2}{15} \approx 0.133. \]

## 3 The Result

We show that for any initial state \( s_0 \in S^{(N)} \), the distribution of alternatives in the state \( X^{(N,r)}(k, s_0) \) is close to a maximal lottery for all but a small fraction of rounds \( k \) for some \( r > 0 \) provided the number of balls \( N \) is large enough. More precisely, take any \( \delta, \tau > 0 \). Then we can find \( N_0 \in \mathbb{N} \) and \( r > 0 \) so that for every \( N \geq N_0 \) and every initial state \( s_0 \in S^{(N)} \), the following event has probability 1: the lower density of the \( k \in \mathbb{N} \) for which \( \frac{1}{N} X^{(N,r)}(k, s_0) \) is no more than \( \delta \) away from \( ML(R) \) is at least \( 1 - \tau \).\(^5\)

**Theorem 1.** Let \( \delta, \tau > 0 \). Then, there are \( N_0 \in \mathbb{N} \) and \( r > 0 \) such that for all \( N \geq N_0 \) and \( s_0 \in S^{(N)} \),
\[ \mathbb{P} \left( \lim_{n \to \infty} \frac{1}{n} \# \left\{ k \in [n] : \frac{1}{N} X^{(N,r)}(k, s_0) \in B_\delta(ML(R)) \right\} \geq 1 - \tau \right) = 1 \]

We outline the main steps in the proof of Theorem 1. Consider the process \( X^{(N,r)}(k, s_0) = \frac{1}{N} X^{(N,r)}(k, s_0) \) obtained by scaling to the discrete unit simplex \( \Delta^{(N)} = \{ p \in \Delta : Np \in \mathbb{Z}^d \} \). For any \( p \in \Delta^{(N)} \), we consider the expected value of \( x^{(N,r)}(k + 1, p) - x^{(N,r)}(k, p) \), which, conditional on \( x^{(N,r)}(k, p) \), is independent of \( k \) since \( x^{(N,r)} \) is time-homogeneous. These expected values induce a continuous function \( f^{(N,r)}: \Delta \to \mathbb{R}^d \) by interpolating linearly. For \( r > 0 \), \( f^{(N,r)} \) has a unique zero \( p^{(N,r)} \), which is close to the set of maximal lotteries \( ML(R) \) if \( r \) is small.

\(^4\)The lower density of a set \( A \subset \mathbb{N} \) is \( \lim_{n \to \infty} \frac{1}{n} \#(A \cap [n]) \).

\(^5\)For a set \( S \subset \mathbb{R}^d \) and \( \delta > 0 \), \( B_\delta(S) = \{ x \in \mathbb{R}^d : \inf \{ |x - y| : y \in S \} < \delta \} \) denotes the set of points with distance less than \( \delta \) to some element of \( S \).
Consider now the following differential equation with \( p \in \Delta, \ t \in \mathbb{R}_+, \) and \( y(\cdot, p): \mathbb{R}_+ \to \Delta. \)

\[
\frac{d}{dt} y(t, p) = f^{(N,r)}(y(t, p)) \\
y(0, p) = p
\]  

A solution to (1) is a deterministic dynamic process that can be interpreted as the stochastic process we consider with a continuum of balls. We show that the unique solution \( y^{(N,r)}(\cdot, p) \) of (1) converges to \( p^{(N,r)} \) for any initial state \( p \in \Delta \) as \( t \) goes to infinity and the convergence is uniform in \( p \).

To relate the discrete-time process \( x^{(N,r)} \) to the continuous-time process \( y^{(N,r)} \), we extend the former to the real time axis by letting \( \bar{x}^{(N,r)}(t, p) = x^{(N,r)}(k, p) \) for \( t \in \left[\frac{k-1}{N}, \frac{k}{N}\right) \). Given any \( T > 0 \), one can show that with probability close to 1, \( \bar{x}^{(N,r)} \) approximately satisfies the integral equation corresponding to (1) for \( t \) between 0 and \( T \) and uniformly in \( p \in \Delta(N) \) if \( N \) is large. Using Grönwall’s inequality, we show that with probability close to 1, \( \bar{x}^{(N,r)}(t, p) \) and \( y^{(N,r)}(t, p) \) are close to each other for all \( t \) from 0 to \( T \). However, for \( t \) larger than \( T \), they may (and with probability 1 will) be arbitrarily far apart.

To deal with this, we partition the time axis into consecutive intervals of length \( T \) and synchronize the continuous process with the discrete process at the beginning of each interval. More precisely, take any \( \delta, \tau > 0 \). Since \( y^{(N,r)}(t, p) \) converges to \( p^{(N,r)} \) as \( t \) goes to infinity uniformly in \( p \), we can find \( T > 0 \) such that \( y^{(N,r)}(t, p) \) is no more than \( \frac{\delta}{2} \) away from \( p^{(N,r)} \) for all but possibly a \( 1 - \frac{\tau}{2} \) fraction of the interval \([0, T]\) for all \( p \). Moreover, we can choose \( N \) large enough so that with probability at least \( 1 - \frac{\tau}{2} \), the distance between \( \bar{x}^{(N,r)} \) and \( y^{(N,r)} \) is less than \( \frac{\delta}{2} \) for all \( t \) in an interval of length \( T \) provided both processes start at the same point at the beginning of the interval. We chop up the time axis into intervals \([0, T], [T, 2T], \ldots\). On the interval \([(k - 1)T, kT]\), we compare \( \bar{x}^{(N,r)}(t, p) = \bar{x}^{(N,r)}(t - (k - 1)T, \bar{x}_{k-1}) \) to \( y^{(N,r)}(t - (k - 1)T, \bar{x}_{k-1}) \), where \( \bar{x}_{k-1} = x^{(N,r)}((k-1)T, p) \). That is, we reset \( y^{(N,r)} \) to the position of \( \bar{x}^{(N,r)} \) at the beginning of the interval. In those intervals where the distance between both processes is never more than \( \frac{\delta}{2} \), \( \bar{x}^{(N,r)} \) is no more than \( \delta/2 + \delta = \delta \) away from \( p^{(N,r)} \) for all but a \( \frac{\tau}{2} \) fraction of the interval. By the choice of \( N \), those intervals make up at least a \( \frac{\tau}{2} \) fraction of all intervals in expectation. Summing over all intervals, this is enough to conclude that \( \bar{x}^{(N,r)} \) is no more than \( \delta \) away from \( p^{(N,r)} \) at least a \( 1 - \tau \) fraction of the time. Since \( p^{(N,r)} \) is close to \( ML(R) \) when \( r \) is small, we can get the same conclusion with \( ML(R) \) in the place of \( p^{(N,r)} \). Translating this statement back to \( X^{(N,r)} \) gives Theorem 1.

A straightforward corollary of Theorem 1 is that the time average of the \( x^{(N,r)}(k, s_0) \) is almost surely close to \( ML(R) \) for some small \( r \) provided that \( N \) is large enough. We define the time-averages of the \( x^{(N,r)}(k, s_0) \).

\[
z^{(N,r)}(n, s_0) = \frac{1}{n} \cdot \sum_{k=0}^{n-1} x^{(N,r)}(k, s_0).
\]

By the ergodic theorem for Markov chains, \( z^{(N,r)} \) converges almost surely to the stationary distribution \( \pi \in \Delta(N) \) of the Markov chain \( \{x^{(N,r)}(n, s_0): n \in \mathbb{N}\} \). Theorem 1
Figure 1: Simulation of the urn process for the preference profile and corresponding majority margin matrix given in Example 1. Here, the first alternative is a Condorcet winner and the distribution in the urn mostly contains balls of the first type most of the time.

shows that $x^{(N,r)}$ is almost surely in a ball of radius $\delta$ around $ML(R)$ for all but a $\tau$-fraction of rounds. Hence, by choosing $\tau$ sufficiently small, we get that almost surely, the limit $\lim_{n \to \infty} z^{(N,r)}(n, s_0)$ exists and lies in a ball of radius $2\delta$ around $ML(R)$. In particular, $z^{(N,r)}(n, s_0)$ is almost surely in $B_{\delta}(ML(R))$ for all but a finite number of $n$.

**Corollary 1.** Let $\delta > 0$. Then, there are $N_0 \in \mathbb{N}$ and $r > 0$ such that for all $N \geq N_0$ and $s_0 \in S^{(N)}$,

$$
\Pr \left( \lim_{n \to \infty} z^{(N,r)}(n, s_0) \in B_{\delta}(ML(R)) \right) = 1
$$

Figure 1 shows a simulation of the urn process for the preference profile and corresponding majority margin matrix given in Example 1. Here, the first alternative is a Condorcet winner and the distribution in the urn mostly contains balls of the first type most of the time.
We now give two other examples, for which the unique maximal lottery is not degenerate.

**Example 2.** Consider three voters \( V = \{v_1, v_2, v_3\} \), three alternatives \( d = 3 \), and the following preference profile \( R \), known as the Condorcet cycle or Condorcet paradox.

\[
\begin{array}{ccc}
  v_1 & v_2 & v_3 \\
  1 & 2 & 3 \\
  2 & 3 & 1 \\
  3 & 1 & 2 \\
\end{array}
\]

Then,

\[
M_R = \begin{pmatrix}
  0 & \frac{2}{3} & \frac{1}{3} \\
  \frac{1}{3} & 0 & \frac{2}{3} \\
  \frac{2}{3} & \frac{1}{3} & 0 \\
\end{pmatrix}
\]

and \( \tilde{M}_R = \begin{pmatrix}
  0 & \frac{1}{3} & -\frac{1}{3} \\
  -\frac{1}{3} & 0 & \frac{1}{3} \\
  \frac{1}{3} & -\frac{1}{3} & 0 \\
\end{pmatrix} \).

The set of maximal lotteries \( ML(R) = \{(\frac{1}{3},\frac{1}{3},\frac{1}{3})\} \) consists of the uniform lottery over the three alternatives. A simulation of an urn process for this profile is given in Figure 2.

**Example 3.** There exists a profile \( R \) with 12 voters and 4 alternatives that induces the matrix of majority margins below.

\[
\tilde{M}_R = \begin{pmatrix}
  0 & \frac{1}{3} & -\frac{1}{9} & \frac{1}{3} \\
  -\frac{1}{3} & 0 & \frac{2}{9} & \frac{1}{3} \\
  \frac{1}{9} & -\frac{2}{9} & 0 & \frac{1}{3} \\
  -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 \\
\end{pmatrix}
\]

The set of maximal lotteries \( ML(R) = \{(\frac{1}{3},\frac{1}{9},\frac{1}{2},0)\} \) consists of a single lottery, which is supported on the first three alternatives. A simulation of an urn process for this profile is given in Figure 3.

4 The Case of a Condorcet Winner

We give an elementary proof of Theorem 1 for profiles that admit a Condorcet winner. In that case, we can analyze directly the stationary distribution \( \pi \in \Delta(\Delta(N)) \) of the Markov chain induced by the urn process (which exists and is unique if \( r > 0 \)). This allows us to give a concrete lower bound on the number of balls \( N \) required so that \( x^{(N,r)} \) is no more than \( \delta > 0 \) away from the maximal lottery (the lottery with probability 1 on the Condorcet winner) for at least a \( 1 - \tau \) fraction of rounds as a function of \( \delta \) and \( \tau \).

Let \( M = M_R \) be the majority matrix of a profile \( R \) with Condorcet winner \( i \in [d] \).

Hence, \( M_{ij} > \frac{1}{2} \) for all \( j \in [d] - \{i\} \). Let \( \alpha = \min\{M_{ij} : j \in [d] - \{i\}\} - \frac{1}{2} \). We slice up \( \Delta(N) \) into the level sets of \( i \). For \( k \in \{0, \ldots, N\} \), let \( S_k = \{p \in \Delta(N) : p_i = k/N\} \) be the states corresponding to distributions with \( k \) of the \( N \) balls of type \( i \). Then \( \sigma_k = \sum_{p \in S_k} \pi(p) \) is the limit probability that the urn is in a state in \( S_k \) as the number of
Figure 2: Simulation of the urn process for the profile given in Example 2 using an urn with $N = 5000$ balls for 500000 rounds and mutation rate $r = 0.04$. The green line indicates the distribution of balls starting from the degenerate distribution in which all balls are labelled with Alternative 2. The orange line depicts the time-average considered in Corollary 1.

rounds goes to infinity. We want to show that if $r$ is sufficiently small and $N$ sufficiently large, $\pi$ has most of the probability on states in $S_k$ with $k$ close to $N$.

For 4 alternatives, one can illustrate the ensuing argument as follows. The set of states $\Delta^{(N)}$ corresponds to rooms in a tetrahedral-shaped pyramid. The rooms on the $k$th floor correspond to $S_k$, so that the tip of the pyramid is the state where all balls are of type $i$. The urn process is a random walk through the pyramid, moving from one room to an adjacent one (which could be on the same floor, the floor below, or the floor above). With the exception of few floors close to the tip, the probability of going up is always larger than the probability of going down. It is then intuitively clear that if the pyramid is large enough, one should expect to find the random walk close to the tip of the pyramid most of the time.

Recall that $P^{(N,r)}(p,q)$ is the probability of transitioning from state $p$ to state $q$. Since $\pi$ is a stationary distribution, we have $\pi^\top P^{(N,r)} = \pi^\top$. Consider any partition of $\Delta^{(N)}$ into two sets. For the stationary distribution, the probability of transitioning from the
Figure 3: Simulation of the urn process for the profile in Example 3 on an urn with \( N = 5000 \) balls for \( 10^6 \) rounds and mutation rate \( r = 0.004 \). The dashed lines indicate the probabilities of the alternatives in the unique maximal lottery \((\frac{1}{3}, \frac{1}{6}, \frac{1}{2}, 0)\). The dotted line is the distance between the distribution in the urn and the maximal lottery.
first set of the second is equal to the probability of transitioning from the second set to
the first since the probabilities of both sets of conserved. Apply this to the sets \( \bigcup_{l=0}^{k-1} S_l \) and \( \bigcup_{l=k}^{N} S_l \) for \( k \in [N] \) and notice that the only transitions between the two sets with
positive probability are from \( S_{k-1} \) to \( S_k \) and vice versa. We get
\[
\sum_{p \in S_{k-1}} \pi(p) \sum_{q \in S_k} P^{(N,r)}(p,q) = \sum_{p \in S_k} \pi(p) \sum_{q \in S_{k-1}} P^{(N,r)}(p,q). \tag{2}
\]
That is, the probability of being in a state in \( S_{k-1} \) and transitioning to a state in \( S_k \)
equals the probability of being in a state in \( S_k \) and transitioning to a state in \( S_{k-1} \).

Now observe that for \( p \in S_k, k \in \{0, \ldots, N-1\} \), we have
\[
\sum_{q \in S_{k+1}} P^{(N,r)}(p,q) \geq (1-r) \frac{k(N-k)}{\binom{N}{2}} \left( \frac{1}{2} + \frac{\alpha}{d} \right) + \frac{r(N-k)}{d} =: u_k
\]
where the left hand side is the probability of replacing a ball of type other than \( i \) by one
of type \( i \) in state \( p \in S_k \) (moving up one floor in the pyramid). Similarly, we find that
for \( p \in S_k, k \in \{1, \ldots, N\} \), we have
\[
\sum_{q \in S_{k-1}} P^{(N,r)}(p,q) \leq (1-r) \frac{k(N-k)}{\binom{N}{2}} \left( \frac{1}{2} - \frac{\alpha}{d} \right) + \frac{r}{d} \frac{k-N}{N} =: d_k
\]
for the probability of replacing a ball of type \( i \) by one of type other than \( i \) in state \( p \in S_k \)
(moving down one floor in the pyramid). Plugging this into (2), we get
\[
\sigma_{k-1} u_{k-1} \leq \sigma_k d_k. \tag{3}
\]
All terms in (3) are strictly positive if \( r > 0 \).

Let \( N \) be so that \( \frac{r}{N \alpha} \geq \frac{1}{2} \) (we choose \( r > 0 \) later). Then,
\[
u_k \geq (1-r) \frac{k(N-k)}{\binom{N}{2}} \left( \frac{1}{2} + \frac{\alpha}{d} \right) + (1-r) \frac{N-k}{\binom{N}{2}} \geq (1-r) \frac{k+1}{\binom{N}{2}} \left( \frac{1}{2} + \frac{\alpha}{d} \right)
\]
where the last inequality uses \( 1 \geq \frac{1}{2} + \alpha \). Similarly, we find that for \( r \leq \frac{1}{2} \) and \( k \leq N \left( 1 - \frac{\alpha}{2} \right) \),
\[
d_k \leq (1-r) \frac{k(N-k)}{\binom{N}{2}} \frac{1-\alpha}{2}
\]
Hence, with this bound on \( k \), we have
\[
\frac{d_k}{u_{k-1}} \leq \frac{1-\alpha}{2 \left( \frac{1}{2} + \frac{\alpha}{d} \right)} = \frac{1-\alpha}{1+2\alpha} =: \beta.
\]
Thus, by (3), \( \sigma_{k-1} \leq \beta < 1 \). We have shown that the cumulative probability \( \sigma_k \) of the
states \( S_k \) decreases at least as fast the geometric series with parameter \( \beta \) from some \( k \)
(close to \( N \)) downwards.
The maximal lottery for $R$ is the degenerate lottery with probability 1 on $i$. For given $\delta, \tau > 0$, we are aiming for a lower bound on $N$ so that the probability on states with at least $1 - \delta$ fraction of balls of type $i$ in the stationary distribution $\pi$ is at least $1 - \tau$. That is,

$$\sum_{k=\lceil N(1-\delta) \rceil}^{N} \sigma_k \geq 1 - \tau.$$ 

First observe that

$$\sum_{k \geq k_0} \beta^k = \beta^{k_0} \frac{1}{1 - \beta} \leq \tau$$

(4)

for $k_0 \geq \frac{\log(\tau(1 - \beta))}{\log \beta}$. For our bound, $N$ needs to be large enough so that there are at least $k_0$ integers in the interval $\{(1 - \delta)N, \ldots, (1 - \frac{\tau}{\alpha})N\}$. The probability on states in $S_k$ with $k < (1 - \delta)N$ will then be below $\tau$ by (4) and the choice of $k_0$ (since the bound on $d_k$ assumes that $k \leq N(1 - \frac{\tau}{\alpha})$). Choosing $r \leq \frac{\alpha \delta}{2}$ and

$$N \geq \frac{k_0}{\delta - \frac{\tau}{\alpha}} \geq \frac{1}{\delta} \left\lceil \log \left( \frac{\tau(1 - \beta)}{\log \beta} \right) \right\rceil$$

achieves this.

In Example 1, there are three alternatives and three voters. Alternative 1 is a Condorcet winner and for every other alternative, at least two out of three voters prefer alternative 1 over it ($\alpha = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}, \beta = \frac{5}{8}$). Suppose we want that at least 90% of the balls in the urn are of type 1 in at least 90% of rounds ($\delta = 0.1, \tau = 0.1$). Choosing $r = \frac{\alpha \delta}{2} = \frac{1}{120}$, we need $N \geq 140$ balls in the urn.

**Remark 1.** The calculations above suggest that when a Condorcet winner exists, a reasonable choice for simulations is $N \geq -\frac{1}{\delta} \log(\tau)$ and $\frac{1}{N} \leq r \leq \delta$.

**Remark 2.** If $r$ is too small compared to $N$, it will in general not be the case that the distribution in the urn is close to a maximal lottery for most rounds. For any long enough time interval, the distribution in the urn will for all $r$ with high probability degenerate within the interval, that is, it will only contain balls of one type. If $r$ is very small, it will stay in a degenerate state for a long time (compared to the chosen interval) with high probability. When the process leaves the degenerate state, the same will repeat itself (possibly with a different degenerate state), so that the process spends most rounds in a degenerate state.

## 5 Discussion of Related Work

Laslier and Laslier (2017) consider a model similar to ours in which the number of balls in the urn increases. Two balls are drawn at random and a fixed comparison matrix specifies which alternative wins against which alternative (this could be seen as a single
voter with possibly intransitive preferences in our model). Rather then replacing the losing ball, a new ball of the same type as the winning ball is added to the urn. They show that the distribution in the urn does not converge unless one alternative beats all alternatives (which corresponds to the Condorcet winner case). However, the fraction of alternatives not contained in the support of the maximal lottery of the comparison matrix goes to zero.

They then consider a modified process, in which three balls are drawn from the urn. Whenever one of three balls beats both other balls, a new ball of the same type is added to the urn. Otherwise, one of the three types is chosen at random and a ball of that type added. Their main result is that, for this modified process, the distribution in the urn converges towards the (unique) maximal lottery of the comparison matrix. Laslier and Laslier do not consider the empirical distribution of winners or the time-average of the distribution in the urn. For the process with two drawn balls, it can be shown that not even the time-average converges. Since the number of balls in the urn increases, convergence is generally very slow.

Equilibrium dynamics

When interpreting the majority margin matrix as a symmetric two-player zero-sum game and maximal lotteries as equilibrium strategies, our result can be phrased as a result about a dynamic process that converges towards equilibrium play. Equilibrium dynamics have been extensively studied in game theory and, in particular for zero-sum games, a number of simple and attractive processes have been proposed. The earliest of these is fictitious play (Brown, 1951). More recently, the multiplicative weights update algorithm (e.g., Freund and Schapire, 1999; Arora et al., 2012) and regret matching (Hart and Mas-Colell, 2000, 2013) have been celebrated in game theory, optimization, and machine learning. When translating the multiplicative weights update algorithm to our setting, one obtains a dynamic urn process, in which voters need to compare a drawn ball to all possible alternatives and adjust the distribution in the urn accordingly. It does not suffice to replace a single ball and the total number of balls does not remain constant.

Continuous models in the natural sciences

Knebel et al. (2015) study a process very similar to ours (evolving through pairwise interactions and mutations) in the context of quantum physics. Here, balls in the urn model correspond to bosons and alternatives to quantum states. The distribution of quantum states determines which states are condensates and, thus, observed macroscopically. Since the number of particles in such systems is typically large, they focus on a process with a continuum of particles as described in Section 3 (which is deterministic). They show that the time-average of this process converges to an equilibrium strategy (i.e., a maximal lottery) of the zero-sum game induced by the transition probabilities between quantum states. All states with probability zero in the equilibrium strategy are depleted; the fractions of the remaining states are bounded away from 0 for all times. However, they neglect mutations for the continuous process, which may cause the process to cycle around the equilibrium strategy without converging to it. Part of our proof of Theorem 1 is showing that the
continuous process with mutations does converge (and not only its time-average). Knebel et al. (2015, Supplementary Note 1) argue that the discrete process with mutations is well-approximated by the continuous process if the number of particles is large and mutations become vanishingly unlikely. Hence, they conclude that the time-average of the discrete process converges to an equilibrium strategy, which is in the spirit of Corollary 1. Our understanding is that their arguments are heuristic and not intended to provide a rigorous derivation of this result. For further connections between physical phenomena and evolutionary game theory, the reader is referred to theoretical physicists. Frey (2010) and Knebel et al. (2013) give further examples of how techniques from evolutionary game theory are used to analyze dynamic processes in theoretical physics.

Our process is also related to the replicator equation in population dynamics and evolutionary game theory (see, e.g., Taylor and Jonker, 1978; Schuster and Sigmund, 1983; Hofbauer and Sigmund, 1998). In its basic form, it states that the change in the relative frequency of a species equals the relative fitness of the species (that is, its fitness relative to the entire population) minus the change in the size of the entire population. When the fitness depends linearly on the relative frequencies of the species and the population size is constant, this is the differential equation (1) with \( r = 0 \).

Grilli et al. (2017) consider dynamical processes in population biology that can explain the stable coexistence of multiple species. It is based on the replicator equation for interactions of triples of individuals: in each round, a randomly chosen individual dies; it is replaced by the winner of a comparison between three randomly selected individuals, where the winner is determined as in Laslier and Laslier’s process with three drawn balls based on a dominance matrix between species. Hence, the number of individuals remains constant. They show that with a continuum of individuals, this process converges to an equilibrium strategy of the zero-sum game corresponding to the dominance matrix. For a finite number of individuals, permanent coexistence of multiple species is a probability zero event. However, they argue that interactions of thee or more individuals can prolong coexistence compared to pairwise interactions. Further connections between symmetric zero-sum games and the coexistence of species have been drawn by Allesina and Levine (2011) and Levine et al. (2017).

A common solution concept for population dynamics is evolutionary stability introduced by Maynard Smith and Price (1973). A distribution of species is evolutionary stable if there is some neighborhood of it that contains no state with higher relative fitness. Foster and Young (1990) argue that evolutionary stability is not an appropriate solution concept when stochastic events (such as random mutations or chance events in nature) affect the population. They propose the stochastically stable set, which is the smallest set of states such that for every neighborhood of it, with probability 1 the state is in that neighborhood all but a small fraction of the time. Theorem 1 shows that the stochastically stable set is contained in a small neighborhood of the set of maximal lotteries. As the mutation rate \( r \) goes to 0, it converges to the set of maximal lotteries. Foster and Young (1990) show that the stochastically stable set is always non-empty and consists of those states that minimize a potential function. In much of the work on the replicator equation, the distribution of species changes continuously with time.
Acknowledgments

This material is based on work supported by the Deutsche Forschungsgemeinschaft under grants BR 2312/11-1 and BR 5969/1-1. The authors thank Stefano Allesina, Vincent Conitzer, Erwin Frey, Philipp Geiger, and Johannes Knebel for stimulating discussions.

References


J. Bartholdi, III, C. A. Tovey, and M. A. Trick. Voting schemes for which it can be difficult to tell who won the election. Social Choice and Welfare, 6(3):157–165, 1989.


