Donor Coordination: Collective Distribution of Individual Contributions

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We propose a novel mechanism design setting in which each agent contributes some amount of a divisible resource (such as money or time) to a common pool. The agents then collectively decide how to efficiently distribute the resources on a fixed number of public goods called projects. An important application of this setting is donor coordination, which allows philanthropists with different goals to find mutually accepted causes. In general, we find that no efficient mechanism can guarantee that each agent only needs to distribute her individual contribution on her most-preferred projects and that no efficient mechanism can incentivize agents to actually participate in the mechanism. On the other hand, for the important case of dichotomous preferences, these impossibilities disappear, and we show that the *Nash product rule* satisfies all of the above-mentioned properties. However, the Nash product can be strategically manipulated, and we settle a long-standing open question of Bogomolnaia, Moulin, and Stong (2005) by proving that no strategyproof and efficient mechanism can guarantee that at least one approved project of each agent receives a positive amount of the resource. An interesting alternative to the Nash product rule is the *conditional utilitarian rule*, which satisfies strategyproofness and a natural weakening of efficiency.

1 INTRODUCTION

Philanthropists have different aims and values that determine how they would like to spend their money. As a consequence, the decisions about what to do with their money are usually made independently from each other, even though there may be opportunities for efficiency improvements through coordination and compromise. In the simplest example, there are two donors: one likes charities a and b and one likes charities b and c. Without coordination, their mutual interest in charity b goes unnoticed and there can be significant efficiency losses by spending money on charities a and c rather than on b.

Donor coordination could improve donation decisions on small scales (such as by those involved in effective altruism) or on large scales (such as foundations). The Open Philanthropy Project, which distributes the multi-billion dollar endowment of the foundation Good Ventures, has written extensively about the need for methods to coordinate granting decisions, and has called on academics to study this problem [23]. In their case, several teams are working on identifying promising recipients in several focus areas (such as global health, scientific research, U.S. policy), and they require methods to apportion their annual spending (of about 100 million U.S. dollars) among recipients while identifying synergies between the causes. These opportunities also exist for individual donors, who usually do not explicitly look for compromise potential with other donors' plans.

We propose to study the problem of donor coordination from a mechanism design perspective. We envision a central board to which donors submit information about their philanthropic goals and the size of their planned donation. The board then recommends to each donor a specific apportionment of her donation to charities, which takes into account others' inputs. In another interpretation of our formal model, agents transfer their resources to a common pool, and then "vote" over the division of the total resources in the pool among different projects.[1]

The proposed mechanism design setting could be applied on different scales and in a variety of situations. In the most basic terms, our model describes the collective distribution of individual contributions: each individual owns some amount of resources (such as money or time) which can be used for several purposes. The individuals also have preferences over how their and others' resources should be used. Thus, there may be benefits from aggregating those preferences. Apart from donor coordination, this model could, for example, be used for allotting time to research projects in academia or helping a group of investors to coordinate the allocation of their capital.

In our formal model, there are m projects which may be charities, cause areas, or joint activities. Each of n individual agents contributes an amount C_i of resources to be spend on these projects, giving rise to a total endowment $C = \sum_{i=1}^{n} C_i$. The mechanism then returns how the total endowment should be divided among the m projects based on the agents' preferences. In principle, the agents' preferences over all possible distributions could account for various effects such as complementarities, substitutabilities, decreasing marginal returns, and budget caps. However, we find that, even for the relatively simple case of linear utility functions, basic desirable properties are already incompatible with each other.

Since we assume that agents own the resources, they cannot be forced to participate in the mechanism. Thus, we need to ensure that it is weakly better for individuals to contribute their resources (and preferences) to the mechanism, rather than spending them independently. Mechanisms guaranteeing this are *individually rational.* Further, we assume that agents are not bound to the recommendations of the mechanism, and hence the outcome of the mechanism needs to be

[1] Amazon smile, for example, allows customers to let Amazon donate 0.5% of the price of the customer's purchases to a charitable organization of their choice (from a list of charities provided by Amazon). If customers were allowed to approve more than a single charity (as in the example above), efficiency could be easily increased.

implementable: the mechanism cannot force agents to spend their money on projects that they think are suboptimal. The main goal of these mechanisms as coordination devices is to exploit efficiency gains. Hence, the output of a mechanism should be Pareto efficient, i.e., no other distribution should be preferred by all agents. Unfortunately, we show that efficiency is incompatible with either individual rationality or implementability, even for linear utility functions.

We therefore focus on the important special case of *dichotomous utility functions*, where agents either approve or disapprove a project. The utility gained from a distribution is equal to the amount of resource spent on approved projects. Approval ballots of this form are a popular input format in many settings. For dichotomous preferences, we find that the impossibilities vanish. In particular, the rule that maximizes *Nash welfare* (the product of agents' utilities) satisfies efficiency, individual rationality, and implementability. One can check implementability by analyzing the first-order conditions of maximizing Nash welfare. In contrast, the proof that Nash satisfies individual rationality is difficult, and is based on a careful estimate of the derivative of an agent's utility at the optimum solution, depending on the amount of their contribution C_i. We also analyze four other mechanisms based on varying notions of maximizing utilitarian and egalitarian welfare. None of them satisfies our three desiderata. However, the *conditional utilitarian rule* (which maximizes utilitarian welfare subject to the constraint that the returned distribution be implementable) satisfies implementability, individual rationality, and a weaker form of efficiency, which only requires that the returned distribution should not be Pareto dominated by a distribution which itself is implementable. We briefly discuss some monotonicity conditions and show that under the Nash product rule, a project may receive less resources if it becomes more popular while this is impossible under the conditional utilitarian rule.

In our model, agents report two kinds of information to the mechanism: the size of their contribution, and their preferences. Individual rationality implies that strategic agents do not have an incentive to underreport the size of their contribution. However, there may be incentives to misreport the preferences. While the conditional utilitarian rule has the attractive feature of being strategyproof, none of the efficient mechanisms avoid incentives for manipulation. For linear utilities, a classic result by Hylland [19] shows that only dictatorial mechanisms can be efficient and strategyproof at the same time. We establish a collection of strong impossibility theorems that apply even to the case of dichotomous utilities. We show that any efficient mechanism that assigns a positive amount of resources to at least one approved project of each agent (a property much weaker than implementability) can be manipulated. These manipulations take the form of "free-riding": agents pretend not to approve popular projects, which on fairness grounds induces the mechanism to redirect funds to other projects that they approve. Our impossibilities confirm a long-standing conjecture by Bogomolnaia, Moulin, and Stong [6]. The proof of the strongest result is extremely complicated and was constructed with the help of a SAT solver; it reasons about manipulations between hundreds of type profiles.

2 RELATED WORK

In general, our model belongs to the area of collective decision making where the set of alternatives is some subset of the Euclidian space, modeling divisible public goods or lotteries over indivisible public goods [see, e.g., 20]. Two concrete applications in this context that have recently gained a lot of attention by computer scientists are that of *participatory budgeting* [e.g., 4, 15, 16] and *probabilistic social choice* [e.g., 2, 8, 9].

The former is concerned with a fixed budget that needs to be allocated to projects and the latter extends the traditional social choice model by allowing for randomizations between alternatives. While both models are mathematically equivalent, the literature has focused on slightly different axioms due to the different interpretations of the mechanisms's output (allocations vs. lotteries).

The key difference between both of the above settings and our model is that in our our model, the individual contributions to the endowment (or the accumulated probability mass) are *owned* by the agents. This enables us to define the axioms of implementability and individual rationality, which—to the best of our knowledge—have not been considered in previous work. The Nash product rule has featured prominently in both streams of research [2, 11, 12, 15, 16]. The conditional utilitarian rule was first implicitly used by Duddy [14] and studied in more detail by Aziz, Bogomolnaia, and Moulin [2].

Some of our results can be viewed as results in probabilistic social choice by letting $C_i = 1/n$ for all i. From this angle, individual rationality is a strengthening of *strict participation* (aka *very strong participation*) for concrete utility functions [2, 9]. Hence, Theorem 4 strengthens Aziz, Bogomolnaia, and Moulin [2]'s proof showing that the Nash product rule satisfies strict participation (their Theorem 3). The proof of their result is a relatively simple contradiction argument, while we need a precise estimate of the change induced by an extra voter.

This is not the first paper that considers donations to charities from a mechanism design standpoint. Conitzer and Sandholm [13] have proposed a bidding language to formulate matching offers over multiple charities and Buterin, Hitzig, and Weyl [10] recently extended ideas from quadratic voting to allow a large funder to subsidize donations by others in an efficient way.

3 PRELIMINARIES

Let A be a finite set of m *projects* (e.g., charities or joint activities) and N a finite set of n *agents*. For each $i \in N$, agent i's *contribution* is $C_i \in \mathbb{R}_{>0}$. Contributions may, for example, be time shares or monetary contributions. When all individual contributions are equal, we refer to this as *uniform contributions*.

A *distribution* δ is a function that describes how some value V (e.g., an individual contribution or the entire endowment) is distributed among the projects, i.e., $\delta : A \to \mathbb{R}_{\geq 0}$ such that $\sum_{x\in A} \delta(x) = V$. For convenience, we will write distributions as linear combinations of projects, i.e., $a + 2b$ stands for distribution δ with $\delta(a) = 1$ and $\delta(b) = 2$. The set of all distributions of value V is denoted by $\Delta(V)$. Every distribution of the endowment $\delta \in \Delta(C)$ can be divided into n individual distributions $\delta_i \in \Delta(C_i)$ such that $\delta = \sum_{i\in N} \delta_i$. Clearly, the division of a distribution into individual distributions is not unique.

Apart from her contribution, each agent possesses a utility function $u_i : \Delta(C) \to \mathbb{R}$, which describes how much utility an agent derives from a distribution. Let agent i's type be $\theta_i = (u_i, C_i)$ and the set of all types Θ. A *mechanism* f is a function that maps a profile $\theta = (\theta_i)_{i\in N}$ to a distribution $\delta \in \Delta(C)$.

4 LINEAR UTILITY FUNCTIONS

In general, utility functions can capture many effects such as complementarities, substitutabilities, decreasing marginal returns, and budget caps. In this section, we observe that, even for linear utility functions, basic desirable properties of distribution mechanisms turn out to be incompatible with each other.

We assume that linear utilities are normalized such that the utility assigned to most-preferred projects is 1 and that assigned to least-preferred projects is 0 (unless an agent is completely indifferent, in which case all utilities are 1). With slight abuse of notation, linear utilities are given by some function $u_i : A \to [0, 1]$, which maps projects to utilities such that

$$u_i(\delta) = \sum_{x\in A} \delta(x) \cdot u_i(x)$$

for any distribution δ.

The aim of distribution mechanisms is to exploit synergies in the agent's preferences. It should thus return distributions that are not Pareto dominated.

Definition 1 (Efficiency)**.** Let $\theta \in \Theta^n$ be a type profile. A distribution $\delta' \in \Delta(C)$ Pareto *dominates* another distribution $\delta \in \Delta(C)$ if $u_i(\delta') \geq u_i(\delta)$ for all $i \in N$ and $u_i(\delta') > u_i(\delta)$ for some $i \in N$. A distribution $\delta \in \Delta(C)$ is *efficient* if no distribution dominates it.

Every distribution $\delta \in \Delta(C)$ can be decomposed into individual distributions $\delta_i \in \Delta(C_i)$. Depending on the concrete application, the mechanism may not be able to directly control the use of the agents' contributions (for example, when a donor coordination service does not actually collect money from its participants). Then, the mechanism's output is better understood as a recommendation to the agents about how they should use their resources. In such a case, the output δ of the mechanism should be implementable, that is, decomposable into individual distributions such that no agent is asked to spend resources on projects that she considers suboptimal.

Definition 2 (Implementability)**.** Let $\theta \in \Theta^n$ be a type profile. A distribution $\delta \in \Delta(C)$ is *implementable* if it can be divided into individual distributions $(\delta_i)_{i \in N}$ with $\delta_i \in \Delta(C_i)$ for all $i \in N$ and $\delta = \sum_{i \in N} \delta_i$ such that $u_i(\delta_i) = C_i$ for all $i \in N$.

Clearly, if a distribution δ is implementable then $u_i(\delta) \geq C_i$. Hence, implementability implies a condition known elsewhere as *individual fair share* [see, e.g., 6]. We say that a mechanism f is efficient (resp. implementable) if $f(\theta)$ is efficient (resp. implementable) for all $\theta \in \Theta^n$.

It is easily seen that implementability clashes with efficiency. Consider the type profile shown in Table 1. Implementability implies that $\delta_1(a) = 1$ and $\delta_2(b) = 1$. Hence, $u_1(\delta) = u_2(\delta) = 1$. However,

	$u_i(a)$	$u_i(b)$	$u_i(x)$	C_i
Agent 1	1	0	0.9	1
Agent 2	0	1	0.9	1

Table 1. Type profile showing the incompatibility of implementability and efficiency.

δ is dominated by δ' with $\delta'(x) = 2$ since $u_1(\delta') = u_2(\delta') = 1.8$. Intuitively, it would be socially efficient to spend the endowment on the "compromise project" x, but given the non-cooperative environment suggested by Definition 2, both agents will use their resources on their "pet project" which they slightly prefer to x.

One can also define implementability in game-theoretic terms: imagine the agents are players in a non-cooperative normal-form game. Each player can choose a distribution $\delta_i \in \Delta(C_i)$, i.e., how to spend their money. The outcome of the game is a distribution $\delta = \sum_{i \in N} \delta_i$ of the endowment, which induces utilities for the players. Then, a distribution is implementable if and only if it can be obtained as a Nash equilibrium of this game. The game induced by the profile in Table 1 is essentially a prisoner's dilemma, where spending on x is "cooperate", and spending on a or b is "defect".

Agents cannot be forced to participate in the mechanism. They could choose to keep their resources, and use them according to their own will. Conservatively, we will assume that agents have good outside options, and that they can use their resources optimally: they can achieve utility C_i with their resources outside the mechanism. This is certainly realistic in the case of donor coordination: agents may decide to donate to their favorite charities directly. Further, even if not participating, agents still benefit from the projects funded by the participating agents. However, there is a downside to not participating: the mechanism only considers the utility functions of those who contribute to the mechanism. Thus, agents who do not participate forego the possibility of affecting the contributions of other agents. Mechanisms under which it is always weakly better for an agent to participate rather than to act independently are called individually rational.

Definition 3 (Individual rationality)**.** A mechanism f is individually rational if for each $\theta \in \Theta^n$, $u_i(f(\theta)) \geq u_i(f(\theta_{-i})) + C_i$.

In other words, no agent is better off by spending her contribution alone while letting the others coordinate the expenditure of their contributions using the mechanism.

Individual rationality is a rather demanding condition, because it assumes that agents have a maximally attractive outside option. If we assume that agents' resources are worthless unless contributed to the mechanism, so that $u_i(f(\theta)) \geq u_i(f(\theta_{-i}))$, the resulting condition is weak enough to be satisfiable together with efficiency and other conditions (for example by welfare-maximizing mechanisms). However, full individual rationality is not compatible with efficiency.

Theorem 1. No individually rational mechanism satisfies efficiency when $m, n \geq 4$.

	$u_i(a)$	$u_i(b)$	$u_i(c)$	$u_i(x)$	C_i
Agent 1	1	0	0	$0.5 + \varepsilon$	1
Agent 2	0	1	0	$0.5 + \varepsilon$	1
Agent 3	0	0	0	1	1
Agent 4	0	0	1	$0.5 + \varepsilon$	1

Table 2. Type profile with $0 < \varepsilon < \frac{1}{6}$ for the proof of Theorem 1.

Proof. Assume for contradiction that there exists a mechanism f satisfying individual rationality and efficiency. We first determine a sequence of agents to be used in the proof. For $\theta = (\theta_1, \theta_2, \theta_3, \theta_4)$ like in Table 2, the distribution $\delta = f(\theta)$ should only allocate resources to at most one of a, b, and c. Otherwise, if there is any subset $\{y, z\} \subset \{a, b, c\}$, $y \neq z$ with $\delta(y) > 0$ and $\delta(z) > 0$, the distribution

$$(\delta(y) - \kappa)\, y + (\delta(z) - \kappa)\, z + (\delta(x) + 2\kappa)\, x$$

with $\kappa = \min(\delta(y), \delta(z))$ is strictly preferred by all four agents. Thus, without loss of generality, we can assume an ordering of the agents such that $\delta(a) \geq 0$ and $\delta(b) = \delta(c) = 0$.

Observe that for θ the total endowment C equals 4. We will now show the contradiction by arguing that ultimately more than the total endowment would have to be spent on project x alone already.

Let $\theta' = (\theta_1, \theta_2)$ and $\delta' = f(\theta')$. As above, by efficiency we can assume that without loss of generality $\delta'(b) = 0$. Otherwise, if $\delta'(a) > 0$ and $\delta'(b) > 0$, the distribution

$$(\delta'(a) - \kappa')\, a + (\delta'(b) - \kappa')\, b + (\delta'(x) + 2\kappa')\, x$$

with $\kappa' = \min(\delta'(a), \delta'(b))$ is strictly preferred by both agents with a utility improvement of $2\kappa'\varepsilon > 0$.

By individual rationality, Agent 2 must get at least the utility as if both would act uncoordinatedly: $u_2(\delta') \geq u_2(a) + 1 = 1$ and with $\delta'(b) = 0$, we get $\delta'(x) \geq \frac{1}{u_2(x)} = \frac{2}{1+2\varepsilon}$.

If now Agent 3 joins, we get $\theta'' = (\theta_1, \theta_2, \theta_3)$ and $\delta'' = f(\theta'')$. Individual rationality for Agent 3 requires that $u_3(\delta'') \geq u_3(\delta') + 1 \geq \frac{2}{1+2\varepsilon} + 1 = \frac{3+2\varepsilon}{1+2\varepsilon}$. However, Agent 3 gets positive utility only from project x, thus we have $\delta''(x) \geq \frac{3+2\varepsilon}{1+2\varepsilon}$.

If we finally add the last agent to get the full type profile $\theta = (\theta_1, \theta_2, \theta_3, \theta_4)$ with $\delta = f(\theta)$ again, we already know, e.g., that $\delta(c) = 0$.

A final application of individual rationality yields $u_4(\delta) \geq u_4(\delta'') + 1 \geq \frac{3+2\varepsilon}{1+2\varepsilon} \cdot \frac{1+2\varepsilon}{2} + 1 = \frac{5+2\varepsilon}{2}$. Due to $\delta(c) = 0$, Agent 4 can only get positive utility from project x, thus $\delta(x) \geq \frac{5+2\varepsilon}{2} / \frac{1+2\varepsilon}{2} = \frac{5+2\varepsilon}{1+2\varepsilon} > 4$ for $0 < \varepsilon < \frac{1}{6}$, which contradicts the fact that the total endowment is 4. □

The reason for this incompatibility is structurally similar to the one for implementability: efficiency requires spending resources on "compromise project" x, but individual rationality can only be satisfied if "pet projects" a, b, and c are funded.

5 DICHOTOMOUS UTILITY FUNCTIONS

On the domain of all linear utility functions, we have found that both implementability and individual rationality are incompatible with efficiency. Aiming for more positive results, we now consider the important subdomain of *dichotomous* utility functions, in which agents only assign values 0 and 1 to the individual projects [see, e.g., 2, 5, 6]. In many applications, this is a reasonable simplifying assumption: agents only distinguish between approved and disapproved projects. For example, a donor may identify a number of charities that she is interested in funding, without further discriminating between these charities or between the charities she is uninterested in. Notably, this simplified type of utility function is easier to elicit than full utilities, and it can lead to a system that is cognitively easier for agents to use.

Formally, a type profile θ has *dichotomous* utility functions if each agent $i \in N$ has a linear utility function with $u_i(x) \in \{0, 1\}$ for all $x \in A$. The preferences of the agents are completely described by their *approval sets* $A_i = \{x \in A : u_i(x) = 1\}$. Profiles of approval sets will be denoted by $A_N = (A_1, \dots, A_n)$. With slight abuse of notation, the type of agent i will now be denoted by $\theta_i = (A_i, C_i)$. The set of agents who approve project x is denoted by $N_x = \{i \in N \colon x \in A_i\}$, its weighted cardinality is called the *score* of x, and denoted by $n_x = \sum_{i \in N_x} C_i$.

We will now introduce a number of mechanisms defined for dichotomous utility functions,[2] and will analyze them in terms of the properties studied in Section 4. As hinted by the results in that section, we find that mechanisms are either efficient or satisfy implementability/individual rationality, but usually not both. However, for the restricted domain of dichotomous utilities, one mechanism turns out to have all these properties. In Figure 1, we illustrate the mechanisms discussed by considering their output on an illuminating type profile.

A simple "mechanism" that can be used as a benchmark is the *uncoordinated rule* which does not make any attempt to aggregate agents' preferences. According to this rule, each agent distributes her contribution equally among the projects she approves:

$$UNC(\theta) = \sum_{i \in N} \sum_{x \in A_i} \frac{C_i}{|A_i|} \cdot x. \qquad \text{(Uncoordinated Rule)}$$

Obviously, this mechanism is implementable (and individually rational), but it violates efficiency since there is no coordination (see Figure 1).

The standard way of attaining efficiency in mechanism design is by maximizing a notion of social welfare. The literature has identified three central versions of this concept: utilitarian welfare, egalitarian welfare, and the Nash product [21]. The five mechanisms that follow are all based on optimizing welfare.

[2] All of the mechanisms we consider have straightforward extensions to general linear utilities.

The simplest rule following this recipe is the *utilitarian rule*. It returns a distribution δ which maximizes the weighted sum of agents' utilities $\sum_{i\in N} C_i\, u_i(\delta)$.[3] These are exactly those distributions in which the endowment is distributed only on the projects with highest score: if any part of the endowment is spent on other projects, utilitarian welfare can be increased by redistributing it to a project with higher score. Note that there might be several projects that have the same score, and so there may be many distributions maximizing utilitarian welfare. If so, for concreteness, we let the endowment be distributed uniformly among them. Let $A^{\max} = \{x \in A\colon n_x \geq n_y \text{ for all } y \in A\}$ be the set of projects with highest score. Then,

$$UTIL(\theta) = \sum_{i\in N} \sum_{x\in A^{\max}} \frac{C_i}{|A^{\max}|} \cdot x. \qquad \text{(Utilitarian Rule)}$$

UTIL satisfies efficiency, because any distribution dominating its output would have strictly higher social welfare. However, *UTIL* is not implementable because all agents have to distribute their contribution to projects in $A^{\max}$, and there may be agents who approve none of these (see Figure 1). For a similar reason, the rule fails to be individually rational.

A natural way to obtain an implementable mechanism while keeping the spirit of utilitarian welfare is to select a distribution that maximizes welfare among implementable distributions only. This is what the conditional utilitarian rule does. Again, it is possible to explicitly describe the solutions of this constrained optimization problem: each agent needs to distribute her contribution among those of her approved projects which have highest score. Again, for concreteness, we let agents split uniformly in case of ties. Formally, we write $A_i^{\max} = \{x \in A_i : n_x \geq n_y \text{ for all } y \in A_i\}$ for the projects with highest score within the approval set of agent i. Then,

$$CUT(\theta) = \sum_{i\in N} \sum_{x\in A_i^{\max}} \frac{C_i}{|A_i^{\max}|} \cdot x. \qquad \text{(Conditional Utilitarian Rule)}$$

By design, this rule is implementable, and we will see in Theorem 3 that it satisfies individual rationality. However, imposing the implementability constraint has destroyed efficiency (see Figure 1). While *CUT* is not efficient, it is easy to see from the definition that *CUT* is *efficient among implementable distributions*: there cannot exist an *implementable* distribution δ' which dominates the output of *CUT*.

The distributions selected by utilitarian mechanisms may lead to large inequality in the utility levels enjoyed by the agents. In some applications, this may be undesirable, and we may want to guarantee every agent a high utility level. A common way to achieve this is to maximize *egalitarian welfare*, the utility level of the worst-off agent. Since this often leads to inefficient outcomes, we can instead use the leximin approach, which maximizes the lowest utility of the agents and then iteratively refines this result by maximizing the second-lowest utility, and so on.

Formally, for $\delta, \delta' \in \Delta(C)$, write $\delta \geq_L \delta'$ if the vector $(u_i(\delta))_{i\in N}$ is lexicographically at least as large as $(u_i(\delta'))_{i\in N}$ after having sorted both vectors into non-decreasing order. Then,

$$EGAL(\theta) = \{\delta \in \Delta(C) : \delta \geq_L \delta' \text{ for all } \delta' \in \Delta(C)\}. \qquad \text{(Egalitarian Rule)}$$

As defined, *EGAL* may return several distributions, but they are all essentially equivalent, because every agent obtains the same utility in any of the returned distributions [cf. 1, Prop. 2]. Thus, it does not in practice matter how ties are broken.

Just like *UTIL*, the rule *EGAL* is efficient but fails to be implementable and individually rational (see Figure 1). In response, we can introduce the *conditional egalitarian rule*, which selects the

[3] This is a variant of the traditional utilitarian rule where agents are weighted by the size of their contribution. This variant is more robust, and is continuous in the contributions C_i. Without weighting, agents with extremely small individual contributions would get the same influence as agents with large individual contributions.

distribution which is leximin-optimal among implementable distributions. Again, this move does not preserve efficiency, but efficiency among implementable distributions.

$$CEG(\theta) = \{\delta \in \Delta(C) : \delta \text{ is implementable and } \delta \geq_L \delta' \text{ for all implementable } \delta' \in \Delta(C)\}. \qquad \text{(Conditional Egalitarian Rule)}$$

The Nash product, which refers to the product of agent utilities, is often seen as a compromise between utilitarian and egalitarian welfare [21]. Maximizing the Nash product has been found to yield "fair" or "proportional" outcomes in many preference aggregation settings, and it also turns out to be attractive in our context. Formally, it is defined as follows.

$$NASH(\theta) = \arg\max_{\delta \in \Delta(C)} \prod_{i \in N} \left(\sum_{x \in A_i} \delta(x) \right)^{C_i} = \arg\max_{\delta \in \Delta(C)} \sum_{i \in N} C_i \log \left(\sum_{x \in A_i} \delta(x) \right). \qquad \text{(Nash Product Rule)}$$

Just like *UTIL* and *EGAL*, this rule is efficient. But, remarkably, it is not necessary to define a "conditional Nash rule": as we will see in Theorem 2, the optimum for the Nash product is always implementable.

All of these mechanisms can be computed efficiently. For *UNC*, *UTIL*, and *CUT* this is clear from their definitions. One can calculate *EGAL* and *CEG* by solving a sequence of linear programs [cf. 1, Prop. 3]. *NASH* can be efficiently approximated using convex programming [6], but it can return distributions with irrational values, so exact computation is not possible.

It is well-known that *UTIL*, *EGAL*, and *NASH* are efficient [see, e.g., 21]. The example in Figure 1 shows that *UNC*, *CUT*, and *CEG* violate efficiency: The *UNC* distribution is dominated by the *NASH* distribution, the *CUT* distribution is dominated by $3.5a + 1.5b$, and the *CEG* distribution is dominated by $2.\bar{6}a + 2.\bar{3}b$. Efficiency is a demanding property and it can be shown that variants of *CUT* which do not distribute uniformly among most approved projects still violate efficiency. The efficiency loss of *CUT* was quantified and bounded by Aziz, Bogomolnaia, and Moulin [2].

5.1 Implementability and Individual Rationality

In this section, we formally verify the claims concerning implementability and individual rationality from above. Our main technical results establish that the Nash product rule satisfies both implementability and individual rationality.

The example in Figure 1 has shown that *UTIL* and *EGAL* are not implementable. It follows from the definitions that *UNC*, *CUT*, and *CEG* are implementable. It is less obvious why *NASH* should be implementable, but this can be seen from its first-order conditions of optimality. The following proof is similar to a result by Guerdjikova and Nehring [18].

Theorem 2. *NASH satisfies implementability.*

Proof. We consider the Karush–Kuhn–Tucker conditions. Write down the Lagrangian

$$\mathcal{L}(\delta, \lambda, \mu_1, \dots, \mu_m) = \sum_{i \in N} C_i \log \left(\sum_{x \in A_i} \delta(x) \right) + \lambda \left(C - \sum_{x \in A} \delta(x) \right) + \sum_{x \in A} \mu_x \delta(x),$$

where $\lambda \in \mathbb{R}$ is the Lagrange multiplier for the constraint $\sum_{x \in A} \delta(x) = C$ and $\mu_x \geq 0$ is the multiplier for the constraint $\delta(x) \geq 0$.

Suppose δ is an optimal solution. By complementary slackness, we must have $\mu_x = 0$ whenever $\delta(x) > 0$. Also, we must have $\partial \mathcal{L} / \partial \delta(x) = 0$, that is, $\sum_{i \in N_x} C_i / u_i(\delta) - \lambda + \mu_x = 0$. By case distinction on whether $\delta(x) > 0$, it follows that $\lambda \delta(x) = \sum_{i \in N_x} C_i \delta(x) / u_i(\delta)$ for all $x \in A$. Hence,

$$\lambda \cdot C = \sum_{x \in A} \lambda \delta(x) = \sum_{x \in A} \sum_{i \in N_x} C_i \delta(x) / u_i(\delta) = \sum_{i \in N} C_i \sum_{x \in A_i} \delta(x) / u_i(\delta) = \sum_{i \in N} C_i = C.$$

Uncoordinated

	a	b	c	d	u_i
δ_1	0.5	·	0.5	·	3
δ_2	0.5	·	·	0.5	3
δ_3	·	0.5	0.5	·	2
δ_4	·	0.5	·	0.5	2
δ_5	1	·	·	·	2
$\sum$	2	1	1	1	12

not efficient

Utilitarian

	a	b	c	d	u_i
δ_1	1	·	·	·	5
δ_2	1	·	·	·	5
δ_3	1	·	·	·	0
δ_4	1	·	·	·	0
δ_5	1	·	·	·	5
$\sum$	5	·	·	·	15

not implementable

Egalitarian

	a	b	c	d	u_i
δ_1	1	·	·	·	2.5
δ_2	1	·	·	·	2.5
δ_3	·	1	·	·	2.5
δ_4	·	1	·	·	2.5
δ_5	0.5	0.5	·	·	2.5
$\sum$	2.5	2.5	·	·	12.5

not implementable

Conditional Utilitarian

	a	b	c	d	u_i
δ_1	1	·	·	·	3.5
δ_2	1	·	·	·	3.5
δ_3	·	0.5	0.5	·	1.5
δ_4	·	0.5	·	0.5	1.5
δ_5	1	·	·	·	3
$\sum$	3	1	0.5	0.5	13

not efficient

Nash Product

	a	b	c	d	u_i
δ_1	1	·	·	·	3
δ_2	1	·	·	·	3
δ_3	·	1	·	·	2
δ_4	·	1	·	·	2
δ_5	1	·	·	·	3
$\sum$	3	2	·	·	13

Conditional Egalitarian

	a	b	c	d	u_i
δ_1	$0.\bar{6}$	·	$0.\bar{3}$	·	$2.\bar{6}$
δ_2	$0.\bar{6}$	·	·	$0.\bar{3}$	$2.\bar{6}$
δ_3	·	1	·	·	$2.\bar{3}$
δ_4	·	1	·	·	$2.\bar{3}$
δ_5	1	·	·	·	$2.\bar{3}$
$\sum$	$2.\bar{3}$	2	$0.\bar{3}$	$0.\bar{3}$	$12.\bar{3}$

not efficient

Fig. 1. Example for dichotomous utilities and uniform contributions with approval profile $A_N = (\{a,c\},\{a,d\},\{b,c\},\{b,d\},\{a\})$. The last row of each table shows the distribution returned by the mechanism. The other rows show a division of this distribution into individual distributions (zeros omitted) and the agents' utilities. Approved projects are highlighted in grey. This is the smallest approval profile in which efficiency requires more than just assigning 0 to Pareto dominated projects.

So $\lambda = 1$, and hence $\sum_{i \in N_x} C_i/u_i(\delta) = 1$ for all $x \in A$ such that $\delta(x) > 0$.

Now, for each $i \in N$, define an individual distribution $\delta_i \in \Delta(C_i)$ with $\delta_i(x) = C_i\delta(x)/u_i(\delta)$ for all $x \in A_i$, and $\delta_i(x) = 0$ otherwise. Clearly, $u_i(\delta_i) = C_i$. To see that $\delta = \sum_{i \in N} \delta_i$, note that for $x \in A$ with $\delta(x) = 0$ we have $\delta_i(x) = 0$ for all $i \in N$, and for $x \in A$ with $\delta(x) > 0$, we have

$$\sum_{i \in N} \delta_i(x) = \sum_{i \in N_x} \delta_i(x) = \sum_{i \in N_x} C_i\delta(x)/u_i(\delta) = \delta(x) \sum_{i \in N_x} C_i/u_i(\delta) = \delta(x). \qquad \square$$

For *NASH*, the individual distribution δ_i of agent i can be interpreted as the distribution δ restricted and rescaled to her approval set A_i [18], i.e.,

$$\delta_i(x) = C_i \frac{\delta(x)}{u_i(\delta)} = C_i \frac{\delta(x)}{\sum_{y \in A_i} \delta(y)} \text{ for all } x \in A.$$

Notably, the agents can thus easily compute their individual distributions from the distribution δ without the need of a central instance telling the agents their individual contributions and even without the agents knowing the other agents' approval sets or contributions. All other implementable mechanisms considered in this paper fail this property, which can easily be seen in Figure 1.

Theorem 3. *CUT* satisfies individual rationality. *UTIL*, *EGAL*, and *CEG* violate individual rationality.

Proof. *CUT*: Let $\delta = CUT(\theta)$ and $\tilde{\delta} = CUT(\tilde{\theta})$ with $\tilde{\theta} = \theta_{-i}$. Due to implementability of *CUT*, the individual rationality condition $u_i(\delta) \geq u_i(\tilde{\delta}) + C_i$ can be simplified to $\sum_{j \in N\setminus\{i\}} u_i(\delta_j) \geq \sum_{j\in N\setminus\{i\}} u_i(\tilde{\delta}_j)$. We now even show the stronger statement that every addend weakly increases, i.e., $u_i(\delta_j) \geq u_i(\tilde{\delta}_j)$ for all $j \in N \setminus \{i\}$ by claiming that $A_j^{\max} \subseteq A_i$ or $\tilde{A}_j^{\max} \cap A_i = \emptyset$. In both cases the inequality is satisfied because $u_i(\delta_j) = C_j \geq u_i(\tilde{\delta}_j)$ or $u_i(\delta_j) \geq 0 = u_i(\tilde{\delta}_j)$ respectively. The theoretical third case $A_j^{\max} \nsubseteq A_i$ and $\tilde{A}_j^{\max} \cap A_i \neq \emptyset$ is not possible: Assume for contradiction that there exists $x \in A_j^{\max} \setminus A_i$ and $y \in \tilde{A}_j^{\max} \cap A_i$. Due to the definition of $\tilde{A}_j^{\max}$ and $x, y \in A_j$ we have $\tilde{n}_y \geq \tilde{n}_x$. Then, as only the weighted approval scores for projects approved by agent i increase in θ compared to $\tilde{\theta}$, we have

$$n_y = \tilde{n}_y + C_i \geq \tilde{n}_x + C_i > \tilde{n}_x = n_x,$$

which is a contradiction to $x \in A_j^{\max}$.

UTIL: Agents 3 and 4 in Figure 1 each get utility 1 by spending their contribution alone instead of utility 0 when coordinating according to the utilitarian rule.

EGAL: For $A_N = (\{a\}, \{b\}, \{a\})$, *EGAL* would split the endowment $1.5a + 1.5b$. If Agent 3 spends her contribution alone and completely on a, *EGAL* would split the endowment of the first two agents $1a + 1b$ instead. Thus Agent 3 could increase her utility from 1.5 to 2.

CEG: Assume $A_N = (\{a\}, \{b\}, \{c\}, \{ab\}, \{ac\})$ with uniform individual contributions. If Agent 4 with $A_4 = \{ab\}$ spends alone, then $CEG(\theta_{-4}) = 1.5a + b + 1.5c$. (From implementability, it follows that $CEG(\theta_{-4})(b) = 1$ and by symmetry $CEG(\theta_{-4})(a) = CEG(\theta_{-4})(c)$.) Thus, Agent 4 has utility $u_4(\theta_{-4}) + 1 = 2.5 + 1 = 3.5$. If Agent 4 joins the mechanism, then $CEG(\theta) = 1.\bar{6}a + 1.\bar{6}b + 1.\bar{6}c$ (optimality is easy to see from the singleton voters), giving Agent 4 utility $u_4(\theta) = 3.\bar{3} < 3.5$. □

Conditional Egalitarian

	a	b	c	u_i
δ_1	1	·	·	1.5
δ_2	·	1	·	1
δ_3	·	·	1	1.5
~~δ_4~~	~~1~~		~~·~~	~~3.5~~
δ_5	0.5	·	0.5	3
$\sum$	1.5	1	1.5	10.5

w/o Agent 4: $CEG(\theta_{-4})$

Conditional Egalitarian

	a	b	c	u_i
δ_1	1	·	·	$1.\bar{6}$
δ_2	·	1	·	$1.\bar{6}$
δ_3	·	·	1	$1.\bar{6}$
δ_4	$0.\bar{3}$	$0.\bar{6}$	·	$3.\bar{3}$
δ_5	$0.\bar{3}$	·	$0.\bar{6}$	$3.\bar{3}$
$\sum$	$1.\bar{6}$	$1.\bar{6}$	$1.\bar{6}$	$11.\bar{6}$

w/ Agent 4: $CEG(\theta)$

Theorem 4. *NASH* satisfies individual rationality.

The proof of this result is technically involved and requires a number of lemmas, whose proofs we defer to the appendix due to space constraints. At a high level, we estimate the rate of change of an agent's utility as her contribution increases, and integrate this quantity as she goes from not participating to participating in the mechanism to obtain the desired result. The estimation entails expressing the logarithm of the utilities as a Taylor expansion and analyzing the relationship between the change in an agent's contribution and the change in these utilities at the distribution returned by *NASH*.

Formally, in order to prove Theorem 4, we have to show that for all $\theta \in \Theta^n$ and $i \in N$, $u_i(NASH(\theta)) \geq u_i(NASH(\theta_{-i})) + C_i$. Consider the function $g\colon \Theta^n \to \Delta(1)$ with $g(\theta) = NASH(\theta)/C_\theta$ for all $\theta \in \Theta^n$, where C_θ denotes the sum of the contributions in θ. We will show that

$$u_i(g(\theta)) \geq \frac{1}{C_\theta}((C_\theta - C_i)u_i(g(\theta_{-i})) + C_i), \tag{1}$$

which is equivalent to the inequality above for *NASH*.

Denote by $\mathcal{P}_\theta \subseteq \mathbb{R}^n$ the polytope of feasible utility profiles scaled by $1/C_\theta$, i.e., $\mathcal{P}_\theta = \{u(\delta) \colon \delta \in \Delta(1)\}$. Note that $\mathcal{P}_\theta$ is convex. For $U \in \mathcal{P}_\theta$, let $F_\theta(U) = \sum_{i \in N} C_i \log U_i$.

Lemma 1. For all $\theta \in \Theta^n$, F_θ has a unique maximizer $U \in \mathcal{P}_\theta$. Moreover, $-C_\theta \log m \leq F_\theta(U) \leq 0$ and $-{}^{C_\theta}/_{C_i} \log m \leq \log U_i \leq 0$ for all $i \in N$.

Since by Lemma 1, F_θ has a unique maximizer for all $\theta \in \Theta^n$, we can define the function $\mathcal{U} \colon \Theta^n \to \mathbb{R}^n_{\geq 0}$ that returns this unique maximizer. Observe that $\mathcal{U}(\theta) \in \mathcal{P}_\theta$ for all $\theta \in \Theta^n$.

Lemma 2. $\mathcal{U}$ is continuous in C_N on $\mathbb{R}^n_{>0}$ and $\mathcal{U}_i$ is weakly increasing in C_i for all $i \in N$.

Lemma 3. For every $\theta \in \Theta^n$ and $U \in \mathcal{P}_\theta$, there is $\varepsilon > 0$ such that for all $dU \in \mathbb{R}^n$ with $|dU| \leq \varepsilon$ and $U + dU \in \mathcal{P}_\theta$, we have $U + t dU \in \mathcal{P}_\theta$ for all $t \in [0, 2]$.

The next three lemmas will be useful for analyzing error terms obtained in the main analysis.

Lemma 4. Let $\theta \in \Theta^n$, $U = \mathcal{U}(\theta)$, and $dU \in \mathbb{R}^n$ such that $U + dU \in \mathcal{P}_\theta$. Then,

$$\sum_{i \in N} C_i \frac{dU_i}{U_i} \leq 0.$$

If also $U - dU \in \mathcal{P}_\theta$, then equality holds.

Lemma 5. Let $\theta \in \Theta^n$, $x \in \mathbb{R}^n$, and $\alpha, \beta > 0$ such that $\sum_{i \in N} C_i x_i = 0$ and $-\alpha \leq x_i \leq \beta$ for all $i \in N$. Then,

$$\sum_{i \in N} C_i x_i^2 \leq \alpha\beta \sum_{i \in N} C_i.$$

Lemma 6. For all $\mu \in (0, 2)$ there is $\varepsilon^* \in (0, 1)$ with the following property: For any $\Phi \colon [0, 2] \to \mathbb{R}$ such that $\Phi(1) = \max_{t \in [0,2]} \Phi(t)$ and

$$\alpha t - (1 + \varepsilon)\beta t^2 \leq \Phi(t) \leq \alpha t - (1 - \varepsilon)\beta t^2,$$

for some $\alpha, \beta \geq 0$ and $\varepsilon \in (0, \varepsilon^*)$ and all $t \in [0, 2]$, it holds that $\alpha \geq \mu \Phi(1)$.

Proof of Theorem 4. We will prove (1). Let $\mu \in (0, 2)$ and let ε^* be such that the conclusion of Lemma 6 holds. *Let*$\varepsilon \in (0, \varepsilon^*)$. Moreover, let $\theta \in \Theta^n$ and $U = \mathcal{U}(\theta)$. Considering the Taylor expansion of log, there is $\varepsilon' > 0$ such that for all $i \in N$ and $|r| < \varepsilon'$,

$$\left| \log(U_i + r) - \log U_i - \frac{r}{U_i} + \frac{1}{2} \left(\frac{r}{U_i} \right)^2 \right| \leq \frac{\varepsilon}{4} \left(\frac{r}{U_i} \right)^2. \tag{2}$$

Now let $C'_N \in \mathbb{R}^n_{>0}$ such that $C'_1 = C_1 + dC_1$ with $0 < dC_1 < \min\{\varepsilon', {}^{\varepsilon}/_{(2+\varepsilon)} C_1\}$ and $C'_i = C_i$ for all $i \in N \setminus \{1\}$, and let $\theta' = (u, C')$. Consider the function $\phi \colon \mathbb{R}^n \to \mathbb{R}$ defined on dU with $|dU| < \varepsilon^*$, such that

$$\phi(dU) := F_{\theta'}(U + dU) - F_\theta(U) - dC_1 \log U_1 = \sum_{i \in N} C_i \frac{dU_i}{U_i} + dC_1 \frac{dU_1}{U_1} - \psi(dU),$$

for some $\psi \colon \mathbb{R}^n \to \mathbb{R}$ with

$$(1 - \varepsilon) \frac{1}{2} \sum_{i \in N} C_i \left(\frac{dU_i}{U_i} \right)^2 \leq \psi(dU) \leq (1 + \varepsilon) \frac{1}{2} \sum_{i \in N} C_i \left(\frac{dU_i}{U_i} \right)^2.$$

The existence of ψ is guaranteed by (2) and the bound on dC_1.

Now let $U' = \mathcal{U}(\theta')$ and $dU' = U' - U$. Note that, since the only term in $\phi(dU)$ that depends on dU is $F_{\theta'}(U + dU)$, dU' maximizes ϕ among all $dU \in \mathbb{R}^n$ with $U + dU \in \mathcal{P}_\theta$. By Lemma 3, there is

$\varepsilon'' > 0$ such that, for all $dU \in \mathbb{R}^n$ with $|dU| \leq \varepsilon''$ and $U + dU \in \mathcal{P}_\theta$, we have $U + r dU \in \mathcal{P}_\theta$ for all $r \in [0,2]$. Since $\mathcal{U}$ is continuous in C_N by Lemma 2, $|dU'|$ will be small if dC_1 is small and we can choose dC_1 even smaller if necessary so that $2|dU'| \leq \min(\varepsilon', \varepsilon'')$. Then, the function $\Phi\colon [0,2] \to \mathbb{R}$ with $\Phi(r) = \phi(r dU')$ is well-defined and satisfies the prerequisites of Lemma 6 with

$$\alpha = \sum_{i \in N} C_i \frac{dU_i'}{U_i} + dC_1 \frac{dU_1'}{U_1} \quad \text{and} \quad \beta = \frac{1}{2} \sum_{i \in N} C_i \left(\frac{dU_i'}{U_i}\right)^2.$$

Hence, it follows from Lemma 6 that

$$\sum_{i \in N} C_i \frac{dU_i'}{U_i} + dC_1 \frac{dU_1'}{U_1} \geq \mu \Phi(1).$$

Since U maximizes F_θ, by Lemma 4, $\sum_{i \in N} C_i \frac{dU_i'}{U_i} \leq 0$. It follows that

$$dC_1 \frac{dU_1'}{U_1} \geq \mu \Phi(1). \tag{3}$$

Next, let $\delta = g(\theta)$. If $\delta(a) = 1$ for some $a \in A$, then $a \in A_i$ for all $i \in N$, since otherwise the lower bound on $F_\theta(U)$ from Lemma 1 would be violated. Then $u_1(\delta) = 1$ and (1) is trivially satisfied. So assume that $\delta(a) < 1$ for all $a \in A$. Thus, for $|t| > 0$ small enough, the distribution δ^t with $\delta^t(a) = (1+t)\delta(a)$ for all $a \in A_1$ and $(1 - {}^{U_1}/_{(1-U_1)}\, t)\delta(a)$ for all $a \in A \setminus A_1$ is in $\Delta(1)$. Let $dU^t = u(\delta^t) - U$. For $|t|$ small enough, we have that $U + dU^t \in \mathcal{P}_\theta$ and $U - dU^t_\theta \in \mathcal{P}_\theta$. Indeed, $U + dU^t = u(\delta^t)$, and for the second statement we can perturb δ infinitesimally in the opposite direction. This is a valid perturbation because $\delta(a) < 1$ for all $a \in A$, and for $a \in A$ such that $\delta(a) = 0$ we have $\delta^t(a) = \delta(a)$. Thus, by Lemma 4, we have

$$\sum_{i \in N} C_i \frac{dU_i^t}{U_i} = 0.$$

So for sufficiently small $|t|$, we have

$$\phi(dU^t) = dC_1 \frac{dU_1^t}{U_1} - \psi(dU^t) \geq dC_1 \frac{dU_1^t}{U_1} - (1+\varepsilon)\frac{1}{2} \sum_{i \in N} C_i \left(\frac{dU_i^t}{U_i}\right)^2.$$

Since $dU_1^t = u_1(\delta^t) - U_1 = (1+t)U_1 - U_1$, we have that ${}^{dU_1^t}/_{U_1} = t$. Similarly, it follows that $-{}^{U_1}/_{(1-U_1)}\, t \leq {}^{dU_i^t}/_{U_i} \leq t$ for all $i \in N$. Thus, applying Lemma 5 with $\alpha = {}^{U_1}/_{(1-U_1)}\, t$, $\beta = t$, and $x_i = {}^{dU_i^t}/_{U_i}$, it follows that

$$\phi(dU^t) \geq dC_1 t - (1+\varepsilon)\frac{1}{2} \frac{U_1 C_\theta}{1 - U_1} t^2.$$

Now let $t := \frac{1-U_1}{U_1 C_\theta} dC_1$. If dC_1 is small enough, then also t is small enough and, recalling that dU' maximizes ϕ among all $dU \in \mathbb{R}^n$ with $U + dU \in \mathcal{P}_\theta$, we get

$$\Phi(1) = \phi(dU') \geq \phi(dU^t) \geq \frac{1}{2}(1-\varepsilon)\frac{1-U_1}{U_1 C_\theta}(dC_1)^2.$$

Thus, by (3), we get

$$dC_1 \frac{dU_1'}{U_1} \geq \frac{\mu}{2}(1-\varepsilon)\frac{1-U_1}{U_1 C_\theta}(dC_1)^2,$$

from which it follows from $dC_1 > 0$ that

$$dU_1' \geq \frac{\mu}{2}(1-\varepsilon)\frac{1-U_1}{C_\theta} dC_1.$$

Since $\mu \in (0,2)$ was arbitrary and $\varepsilon > 0$ can be chosen arbitrarily small, it follows that

$$dU_1' \geq \frac{1-U_1}{C_\theta} dC_1.$$

Now, let $C^s_N \in \mathbb{R}^n_{>0}$ such that $C^s_1 = C_1 + s$ and $C^s_i = C_i$ for all $i \in N \setminus \{1\}$, $\theta^s = (u, C^s)$, and $\tilde{\mathcal{U}}(s) = \mathcal{U}(\theta^s)$. By the above inequality, the lower right derivative of $\tilde{\mathcal{U}}_1$ at s is at least ${(1-\tilde{\mathcal{U}}_1(s))}/{C_{\theta^s}}$. Integrating this estimate from 0 to dC_1 yields

$$-\int_0^{dC_1} \frac{\frac{\partial \tilde{\mathcal{U}}_1(s)}{\partial s}}{1-\tilde{\mathcal{U}}_1(s)} ds \leq -\int_0^{dC_1} \frac{1}{C_\theta + s} ds$$

from which we get

$$\log(1-\tilde{\mathcal{U}}_1(dC_1)) - \log(1-\tilde{\mathcal{U}}_1(0)) \leq -(\log(C_\theta + dC_1) - \log(C_\theta)).$$

Exponentiation yields ${(1-\tilde{\mathcal{U}}_1(dC_1))}/{(1-\tilde{\mathcal{U}}_1(0))} \leq {C_\theta}/{(C_\theta + dC_1)}$. Rewriting the variable θ as θ', we have

$$\tilde{\mathcal{U}}_1(dC_1) \geq \frac{1}{C_{\theta'} + dC_1}(C_{\theta'}\tilde{\mathcal{U}}_1(0) + dC_1) = \frac{1}{C_{\theta'} + dC_1}(C_{\theta'}\mathcal{U}_1(\theta') + dC_1). \tag{4}$$

Finally, to prove (1), let $\theta \in \Theta^n$, $C_1 > \varepsilon > 0$, and $\theta^\varepsilon = (u, C^\varepsilon)$ with $C^\varepsilon_1 = \varepsilon$ and $C^\varepsilon_i = C_i$ for all $i \in N \setminus \{1\}$. Let $x \in g(\theta)$ and $U = u(x)$. By the monotonicity part of Lemma 2 and taking $\theta' = \theta^\varepsilon$ and $dC_1 = C_1 - \varepsilon$ in (4), we have that

$$U_1 \geq \frac{1}{C_\theta}\left(C_{\theta^\varepsilon} U_1(\theta^\varepsilon) + C_1 - \varepsilon\right) \geq \frac{1}{C_\theta}\left((C_\theta - C_1 + \varepsilon)U_1(\theta_{-1}) + C_1 - \varepsilon\right).$$

Since this inequality holds for all $C_1 > \varepsilon > 0$, it follows that

$$U_1 \geq \frac{1}{C_\theta}\left((C_\theta - C_1)U_1(\theta_{-1}) + C_1\right),$$

which proves (1). □

Curiously, we are not aware of any other rules that satisfy implementability and efficiency, or individual rationality and efficiency.

5.2 Strategyproofness

When agents are strategic, they may try to misrepresent their preferences in a way that induces the mechanism to choose a more-preferred distribution. Mechanisms that are immune to strategic misrepresentation are called *strategyproof.*

Definition 4 (Strategyproofness)**.** A mechanism is *strategyproof* if for all θ, θ' with $\theta = \theta'$ except $u_i \neq u_i'$, $u_i(f(\theta)) \geq u_i(f(\theta'))$.

Note that, under this definition, we are only interested in preventing misrepresentation of the utility function u_i. The other part of an agents' type is their contribution C_i which they might also misreport (more precisely, underreport), but this worry is captured by the notion of individual rationality.

In most mechanism design settings, strategyproofness is only obtained by degenerate mechanisms that ignore most of the information, such as dictatorships. However, in the domain of dichotomous preferences, more positive results are known. For example, in social choice, approval voting is known to be strategyproof [7]. In our setting, there also exist attractive strategyproof mechanisms. In particular, *UNC*, *CUT*, and *UTIL* are all strategyproof [2]. Remarkably, in the conditional utilitarian rule *CUT*, we have an example of a rule that is individually rational, implementable, and strategyproof. It is also efficient among implementable distributions, but it fails to be efficient outright. Indeed, of the strategyproof rules we have listed, only *UTIL* is efficient. However, *UTIL* fails to be implementable and it fails to be individually rational. In fact, *UTIL* can be unfair to an extreme extent: if one agent contributes the majority of the endowment, then *UTIL* will exclusively fund projects approved by the majority contributor. Thus, *UTIL* can leave some contributors with zero utility, and thus fails a property that Bogomolnaia, Moulin, and Stong [6] call *positive share*.

Definition 5 (Positive share)**.** A mechanism satisfies *positive share* if for all θ and $i \in N$, $u_i(f(\theta)) > 0$.

In 2005, Bogomolnaia, Moulin, and Stong [6] conjectured that, like *UTIL*, all efficient and strategyproof mechanisms will fail positive share—and hence many other desirable properties such as implementability or individual rationality, which imply positive share. They "submit as a challenging conjecture the following statement: there is no strategyproof and ex ante efficient mechanism guaranteeing positive shares." Bogomolnaia et al. were able to prove impossibility theorems of this type only when substituting much stronger versions of strategyproofness or of positive share, and additionally requiring anonymity and neutrality. Even so, their proofs were rather involved, and one of them required that $m \geq 17$. As to whether a mechanism satisfying the original conditions exists, they left it as "a challenging open question to which we suspect the answer is negative when $[m]$ and $[n]$ are large enough."

Here, we confirm Bogomolnaia et al.'s conjecture.

Theorem 5. No mechanism satisfies efficiency, strategyproofness, and positive share when $m \geq 4$ and $n \geq 6$.

To our surprise, Bogomolnaia et al.'s suspicion that an impossibility would require a large number of voters and projects turned out to be false. In addition, our proof goes through with a significantly weaker form of strategyproofness than the one stated in Definition 4.

We proved Theorem 5 using computer-aided theorem proving techniques and specifically using SAT solving. The basic idea is to reduce the statement in question to a finite—yet very large—problem, which is encoded as a formula in propositional logic and then shown to be unsatisfiable by a SAT solver. The formula's variables describe the mechanism in explicit form, with variables for each possible type profile, and we add constraints to enforce the axioms. We then extract a minimal unsatisfiable set of constraints from the formula and translate this back into a human-readable proof of the result. This approach has been employed successfully to prove a number of impossibility theorems in social choice theory [see, e.g., 8, 17, 25].

On first sight, our problem has a continuous flavor, since the mechanisms that we consider return real-valued distributions. This suggests encodings into integer linear programming, or into SMT, which has previously been used to prove a strong impossibility theorem in probabilistic social choice [8]. A drawback of these continuous methods is that they can (presently) only handle comparatively small instances. Solving times tend to become prohibitive once we search for an impossibility on a domain of more than a few thousand profiles. Discrete encodings of social choice problems into SAT can often be solved for hundreds of thousands of profiles.

Our problem can be discretized by only considering the *support* of the distribution returned by our mechanism. Thus, for 4 alternatives, there are only $2^4 - 1 = 15$ possible outcomes per profile (rather than infinitely many). Note that the positive share axiom only refers to the support. Less obviously, Aziz, Brandl, and Brandt [3] have proved that whether a distribution is efficient or not depends only on its support. The only remaining axiom is strategyproofness, which depends on the precise distributions returned by the mechanism. However, it turns out that impossibility still holds when only considering particularly clear-cut manipulations. In the encoding, we *only* consider manipulations in which the manipulator enforces a distributions in which the *entire* endowment is distributed on his approved projects, i.e., by manipulating he obtains the maximum utility of C.

Even after discretizing, the formulas involved are very big, and further reduction techniques are needed. Without imposing anonymity, there are $15^6 \approx 11$ million different profiles with $n = 6$ and $m = 4$, and we need to use 15 variables for each profile (one for each support), giving 170 million variables in total. It is much easier to obtain a result when we impose anonymity and neutrality, which was also done by Bogomolnaia, Moulin, and Stong [6]. A mechanism is *anonymous* if it is invariant under renaming agents, and it is *neutral* if a permutation of the projects induces the same permutation in the mechanism's output.

When we consider anonymous and neutral mechanisms, the number of essentially different profiles reduces to 2197. In fact, with these extra axioms, the impossibility holds even for $n = 5$, for which there are only 736 essentially different profiles. Solving the resulting formula is almost instantaneous with a modern SAT solver. After extracting a minimal unsatisfiable set, we were astonished to find that it only referred to two different profiles, giving a short and elegant proof.

Theorem 6. No anonymous and neutral mechanism satisfies efficiency, strategyproofness, and positive share when $m \geq 4$ and $n \geq 5$.

Proof. We prove the incompatibility for $m = 4$ and $n = 5$. The proof can be adapted to larger values by adding agents approving all projects or by adding projects which no-one approves.

Assume there is a strategyproof mechanism f satisfying efficiency and positive share. Now consider a profile θ with uniform contributions $C_i = 1$ for all agents $i \in N$ and the approval profile

$$A_N = (\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{a\}).$$

Let $\delta = f(\theta)$ be the distribution returned by the mechanism. Because f is anonymous and neutral, since b and c are symmetric, we must have $\delta(b) = \delta(c)$, and this value must be positive by positive share for Agent 4. It follows that $u_1(\delta) < C$ because a positive amount is spent on project c, which Agent 1 does not approve.

Suppose Agent 1 states her approval set as $\{b, d\}$. The resulting profile θ' has approval profile

$$A'_N = (\{b, d\}, \{a, c\}, \{a, d\}, \{b, c\}, \{a\}).$$

Let $\delta' = f(\theta')$ be the distribution now returned by the mechanism. Suppose first that both $\delta'(c)$ and $\delta'(d)$ are positive, say $\delta'(c) \geq \epsilon$ and $\delta'(d) \geq \epsilon$ for some $\epsilon > 0$. Then δ' is Pareto dominated by the distribution obtained from δ' by moving ϵ from c to a and ϵ from d to b. This contradicts efficiency of f, so either $\delta'(c) = 0$ or $\delta'(d) = 0$. Now c and d are symmetric projects in θ', and thus we must have $\delta'(c) = \delta'(d)$ by anonymity and neutrality of f. Thus $\delta'(c) = \delta'(d) = 0$. Hence, the entire budget is distributed between projects a and b, and so $u_1(\delta') = C$, where we take Agent 1's utility as reported in profile θ, and in particular $u_1(\delta') > u_1(\delta)$.

Hence, Agent 1 has successfully manipulated, which contradicts strategyproofness. $\square$

This short proof heavily uses symmetry arguments. Without anonymity and neutrality, the proofs become much more complicated. In the appendix, we give a proof that still uses anonymity, but that does not need neutrality. Without either of the axioms, a tractable formula can be obtained by only including profiles similar to the ones used in the proofs of the weaker statements. The resulting computer-generated proof, which reasons about manipulations between hundreds of type profiles, is available on request.

Mechanisms satisfying notions such as positive share or implementability try to be "fair" to each agent, and aim for an outcome that makes every agent reasonably happy. There is an obvious strategy to try to exploit this tendency: agents may pretend to be less happy than they are. In our setting, this would correspond to approving fewer projects.[4] We can show, by a proof similar to the one above, that every efficient mechanism that satisfies positive share can be manipulated using this technique. The proof uses neutrality and, in contrast to Theorem 5, we do not know whether this can be dropped. The proof can be found in the appendix.

Theorem 7. Every anonymous and neutral mechanism which satisfies efficiency and positive share can be manipulated by an agent reporting a subset of their truthful approval set, when $m \geq 5$ and $n \geq 5$.

Interestingly, Theorem 5 implies there is no efficient and strategyproof mechanism which approximates egalitarian welfare (since the best attainable egalitarian welfare in any profile is always at least C_i, and we show that every efficient and strategyproof mechanism will sometimes return a distribution with egalitarian welfare 0).

Another way to potentially manipulate the resulting distribution is to split one's contribution into k parts and pretend to be k agents (with the same utility functions) rather than only one. It follows from the definitions that all considered mechanisms are immune to this kind of manipulation.

5.3 Monotonicity

Our model gives rise to a number of natural monotonicity axioms. While for all the considered mechanisms increasing the contribution of an agent cannot decrease her utility, this may result in a utility decrease for another agent (avoiding this was proposed as an axiom in the context of cake cutting by Moulin and Thomson [22]).

In this section, we will briefly discuss monotonicity with respect to *projects*, i.e., if a project becomes more popular, it should not receive less resources.

Definition 6 (Project monotonicity)**.** A mechanism f is *project monotonic* if for all type profiles θ with approval sets $(A_1, \ldots, A_n)$, agents $i \in N$, projects $x \notin A_i$, and $\theta' = \theta$ except $A'_i = A_i \cup \{x\}$: $f(\theta')(x) \geq f(\theta)(x)$.

It follows from the definitions that *UTIL* and *CUT* satisfy project monotonicity. Perhaps surprisingly, the other considered mechanisms fail project monotonicity.

Theorem 8. *EGAL*, *CEG*, and *NASH* violate project monotonicity.

Proof. For θ with $A_N = (\{a\}, \{abc\}, \{bd\}, \{cd\})$ and uniform contributions, $EGAL(\theta) = CEG(\theta) = 2a + 2d$. If Agent 1 additionally approves project d, i.e., $A'_1 = \{a, d\}$, we get $EGAL(\theta') = CEG(\theta') = 0.8a + 0.8b + 0.8c + 1.6d$. Hence, the contribution for d decreased from 2 to 1.6.

CEG and *EGAL*

	a	b	c	d	u_i
δ_1	1	·	·	·	2
δ_2	1	·	·	·	2
δ_3	·	·	·	1	2
δ_4	·	·	·	1	2
$\sum$	2	·	·	2	8

θ with $A_1 = \{a\}$

CEG and *EGAL*

	a	b	c	d	u_i
δ_1	·	·	·	1	2.4
δ_2	0.8	·	0.2	·	2.4
δ_3	·	0.8	·	0.2	2.4
δ_4	·	·	0.6	0.4	2.4
$\sum$	0.8	0.8	0.8	1.6	9.6

θ' with $A'_1 = \{a, d\}$

[4] This notion of "subset-strategyproofness" has also been studied in the context of proportional multiwinner elections [24]. The corresponding notion of *superset*-strategyproofness has been studied by Aziz, Bogomolnaia, and Moulin [2], who found that *CUT* and *EGAL* satisfy it, while *NASH* fails it.

For *NASH*, consider θ with $A_N = (\{a\}, \{a,b\}, \{a,c\}, \{b,c,d\}, \{b,d\}, \{c,d\})$ and contributions $C_i = 1$ for $1 \leq i \leq 3$ and $C_i = 2$ for $4 \leq i \leq 6$, we have $NASH(\theta) = 3a + 6d$. If Agent 1 additionally approves project d, i.e., $A_1' = \{a,d\}$, we get $NASH(\theta') = 2\kappa\, a + \kappa\, b + \kappa\, c + (9 - 4\kappa)\, d$ with $\kappa = {}^{(7-\sqrt{22})}/_{3}$. Hence, the contribution for project d decreased from 6 to $9 - 4\kappa \approx 5.92055\ldots$.

NASH

	a	b	c	d	u_i
δ_1	1	·	·	·	3
δ_2	1	·	·	·	3
δ_3	1	·	·	·	3
δ_4	·	·	·	2	6
δ_5	·	·	·	2	6
δ_6	·	·	·	2	6
$\sum$	3	·	·	6	27

θ with $A_1 = \{a\}$

NASH (rounded)

	a	b	c	d	u_i
δ_1	0.21	·	·	0.79	7.46
δ_2	$0.\bar{6}$	$0.\bar{3}$	·	·	2.31
δ_3	$0.\bar{6}$	·	$0.\bar{3}$	·	2.31
δ_4	·	0.21	0.21	1.59	7.46
δ_5	·	0.23	·	1.77	6.69
δ_6	·	·	0.23	1.77	6.69
$\sum$	1.54	0.77	0.77	5.92	32.92

θ' with $A_1' = \{a, d\}$

□

6 CONCLUSION

We have proposed a novel mechanism design setting that is concerned with the collective distribution of individual contributions. While results for general utility functions are rather negative, the results for dichotomous utility functions are very encouraging (see Table 3). We identified two attractive distribution mechanisms (*NASH* and *CUT*) that satisfy implementability and individual rationality. On top of implementability, *NASH* has the remarkable property that each individual distribution is identical to the total distribution restricted and rescaled to the corresponding approval set. While *NASH* satisfies the standard notion of efficiency, *CUT* only guarantees efficiency among all implementable distributions (which may be sufficient if non-implementable distribution are ruled out *per se*). On the other hand, *CUT* satisfies strategyproofness while *NASH* can be manipulated, even when restricting attention to "free-rider" manipulations. Both *NASH* and *CUT*—as well as all the other mechanism we consider—can be efficiently computed (computing the *NASH* distribution *exactly* is not possible, but it can be approximated efficiently using standard optimization software).

We believe that efficient distribution mechanisms like the ones discussed in our paper are not only of theoretical interest but can be easily deployed in the real world to allow donors to coordinate their philanthropic efforts.

	UNC	UTIL	EGAL	NASH	CUT	CEG
Efficiency	–	✓	✓	✓	(imp.)	(imp.)
Implementability	✓	–	–	✓	✓	✓
Individual Rationality	✓	–	–	✓	✓	–
Strategyproofness	✓	–	–	–	✓	–
Project Monotonicity	✓	✓	–	–	✓	–

Table 3. Axiomatic properties of distribution mechanisms for dichotomous utility functions. *CUT* and *CEG* are not efficient, but they return distributions that are efficient among all implementable distributions.

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A OMITTED PROOFS

A.1 Proof of Lemma 1

Assume for contradiction that there are two distinct $U', U'' \in \mathcal{P}_\theta$ which maximize F_θ. As a positive linear combination of strictly concave functions, F_θ is a strictly concave function. Hence, for $U = 1/2\,(U' + U'') \in \mathcal{P}_\theta$, by strict concavity of F_θ, we have

$$F_\theta(U) > \frac{1}{2}\left(F_\theta(U') + F_\theta(U'')\right) = F_\theta(U'),$$

which contradicts the assumption that U' maximizes F_θ over $\mathcal{P}_\theta$.

Let $\delta \in \Delta(1)$ be the uniform distribution over A and $U \in \mathcal{P}_\theta$ a maximizer of F_θ. Observe that for $U^\delta = u(\delta) \in \mathcal{P}_\theta$, we have $U_i^\delta \geq 1/m$ for all $i \in N$. Hence, $F_\theta(U) \geq F_\theta(U^\delta) \geq -C_\theta \log m$. Moreover, for all $i \in N$, $U_i \leq 1$ and hence, $F_\theta(U) \leq 0$.

Lastly, let $i \in N$. Clearly, $\log U_i$ is upper bounded by $\log 1 = 0$. For the lower bound, observe that

$$C_i \log U_i \geq -C_\theta \log m - \sum_{j \in N \setminus \{i\}} C_j \log U_j \geq -C_\theta \log m,$$

where the first inequality follows from the lower bound on $F_\theta(U)$ and the second inequality follows from the upper bound on $\log U_j$. The desired bound is obtained after division by C_i.

A.2 Proof of Lemma 2

First we show that $\mathcal{U}$ is continuous in C_N on $\mathbb{R}^n_{>0}$. Let $\theta \in \Theta^n$, $\theta = (u, C_N)$ with $C_N \in \mathbb{R}^n_{>0}$, $(C_N^k)_{k \in \mathbb{N}} \subseteq \mathbb{R}^n_{>0}$ converging to C_N, and $\theta^k = (u, C_N^k)$. Further, let $U^k = \mathcal{U}(\theta^k)$ and $U = \mathcal{U}(\theta)$. Observe that since C_N^k converges to C_N, by Lemma 1, $1 \geq U_i^k \geq \lambda > 0$ for all i and some $\lambda > 0$ and large enough k. Hence, by passing to a subsequence if necessary, we may assume that U^k converges to U^* for some $U^* \in \mathcal{P}_\theta$. Since the family of functions F_θ, F_{θ^k}, $k \in \mathbb{N}$, is uniformly equicontinuous on $[\lambda, 1]^n$, it follows that $F_{\theta^k}(U^k)$ converges to $F_\theta(U^*)$. Moreover, as U^k maximizes F_{θ^k}, we have $F_{\theta^k}(U^k) \geq F_{\theta^k}(U)$, which converges to $F_\theta(U)$. Hence, U^* maximizes F_θ, which, by Lemma 1, implies that $U^* = U$ and hence, U^k converges to U.

Let $\theta, \theta' \in \Theta^n$ and $t > 0$ such that $\theta' = (u, C')$ with $C_1' = C_1 + t$ and $C_i = C_i'$ for all $i \in N \setminus \{1\}$. We show that $\mathcal{U}_1(\theta') \geq \mathcal{U}_1(\theta)$. Let $U = \mathcal{U}_1(\theta)$ and $U' = \mathcal{U}_1(\theta')$ and assume for contradiction that $U_1' < U_1$. Then,

$$F_{\theta'}(U') = \sum_{i \in N} C_i \log U_i' + t \log U_1' < \sum_{i \in N} C_i \log U_i + t \log U_1 = F_{\theta'}(U),$$

where the inequality follows from the assumption that $U_1' < U_1$ and the fact that U is a maximizer of F_θ. This contradicts the assumption that U' maximizes $F_{\theta'}$.

A.3 Proof of Lemma 3

Since $\mathcal{P}_\theta$ is a polytope, it is an intersection of a finite number of closed half-spaces H_i. Observe that the desired property holds for each H_i. Indeed, if the point U is in the interior of H_i, we can take ε to be half of the distance from U to the boundary of H_i, while if U is on the boundary of H_i, the entire ray $\{U + tdU \mid t \geq 0\}$ is contained in H_i and we can take ε to be any positive real number. It follows that the desired property also holds for the intersection of the half-spaces H_i, which is $\mathcal{P}_\theta$.

A.4 Proof of Lemma 4

Consider the function $\tau\colon [0, 1] \to \mathbb{R}$ with $\tau(t) = F_\theta(U + tdU)$ and observe that τ attains its maximum at 0. Since $U_i > 0$ for all $i \in N$ by Lemma 1, τ is differentiable at 0. Hence, the right derivative of τ

at 0 is non-positive, i.e.,

$$\frac{\partial \tau}{\partial t}\Big|_{t=0} = \frac{\partial}{\partial t}\left(\sum_{i\in N} C_i \log(U_i + t dU_i)\right)\Big|_{t=0} = \sum_{i\in N} C_i \frac{dU_i}{U_i} \leq 0.$$

If additionally $U - dU \in \mathcal{P}_\theta$, the first part implies $-\sum_{i\in N} C_i \frac{dU_i}{U_i} \leq 0$, from which equality follows.

A.5 Proof of Lemma 5

Since $-\alpha \leq x_i \leq \beta$, we have $\left|x_i - \frac{\beta-\alpha}{2}\right| \leq \frac{\beta+\alpha}{2}$. It follows that

$$\begin{aligned}\sum_{i\in N} C_i x_i^2 &= \sum_{i\in N} C_i \left(x_i - \frac{\beta-\alpha}{2}\right)^2 - \left(\frac{\beta-\alpha}{2}\right)^2 \sum_{i\in N} C_i \\ &\leq \left(\frac{\beta+\alpha}{2}\right)^2 \sum_{i\in N} C_i - \left(\frac{\beta-\alpha}{2}\right)^2 \sum_{i\in N} C_i \\ &= \alpha\beta \sum_{i\in N} C_i,\end{aligned}$$

as claimed.

A.6 Proof of Lemma 6

We first prove an auxiliary lemma.

Lemma 7. Let $\lambda^* \in (0, 1/2)$. Then, there are $\varepsilon^* \in (0,1)$ and $t \in [1,2]$ such that

$$t - \lambda\frac{1+\varepsilon}{1-\varepsilon}t^2 > 1 - \lambda \quad \text{for all } \lambda \in [0,\lambda^*] \text{ and } \varepsilon \in (0,\varepsilon^*).$$

Proof. The inequality in the statement can be rewritten as $\lambda < \frac{t-1}{\frac{1+\varepsilon}{1-\varepsilon}t^2-1}$. Choose an arbitrary $t \in (1, \frac{1}{\lambda^*} - 1)$. We have $t \in [1,2]$ and $\lambda^* < \frac{1}{1+t}$. Since $\lim_{\varepsilon\to 0} \frac{t-1}{\frac{1+\varepsilon}{1-\varepsilon}t^2-1} = \frac{1}{1+t}$, we can choose $\varepsilon^* \in (0,1)$ such that $\lambda^* < \frac{t-1}{\frac{1+\varepsilon}{1-\varepsilon}t^2-1}$ for all $\varepsilon \in (0,\varepsilon^*)$. It follows that $\lambda < \frac{t-1}{\frac{1+\varepsilon}{1-\varepsilon}t^2-1}$ for all $\lambda \in [0,\lambda^*]$ and $\varepsilon \in (0,\varepsilon^*)$, as desired. □

We now proceed to prove Lemma 6. If $\mu \leq 1$, then by choosing any $\varepsilon^* \in (0,1)$, we have $\mu\Phi(1) \leq \Phi(1) \leq \alpha$ by assumption. Assume henceforth that $\mu > 1$. For any $\varepsilon^* \in (0,1)$ that we choose later, note that if $\alpha = 0$, then taking the given $\varepsilon \in (0,\varepsilon^*)$ yields $\Phi(1) \leq 0$, so $\alpha \geq \mu\Phi(1)$ always holds. Hence it suffices to consider $\alpha > 0$. Let $\lambda^* := 1 - \frac{1}{\mu} > 0$ and choose $\varepsilon^* > 0$ and $t^* \in [1,2]$ such that

$$t^* - \lambda\frac{1+\varepsilon}{1-\varepsilon}(t^*)^2 > 1 - \lambda$$

for all $\lambda \in [0,\lambda^*]$ and $\varepsilon \in (0,\varepsilon^*)$, which is possible by Lemma 7.

Let $\lambda := \frac{\alpha-\Phi(1)}{\alpha} \geq 0$. Assume for the sake of contradiction that the desired conclusion is not true, i.e., $\alpha < \mu\Phi(1)$. This is equivalent to $\lambda < \lambda^*$. Since the function $\Psi(t) := \alpha t - \Phi(t)$ satisfies $\beta(1-\varepsilon)t^2 \leq \Psi(t) \leq \beta(1+\varepsilon)t^2$, by substituting $t = t^*$ and $t = 1$, we have $\Psi(t^*) \leq \Psi(1)\frac{1+\varepsilon}{1-\varepsilon}(t^*)^2$. It follows that

$$\Phi(t^*) = \alpha t^* - \Psi(t^*) \geq \alpha\left(t^* - \frac{\Psi(1)}{\alpha}\frac{1+\varepsilon}{1-\varepsilon}(t^*)^2\right) = \alpha\left(t^* - \lambda\frac{1+\varepsilon}{1-\varepsilon}(t^*)^2\right) > \alpha(1-\lambda) = \Phi(1).$$

This contradicts the assumption that $\Phi(1) = \max_{t\in[0,2]} \Phi(t)$.

A.7 Proof of Theorem 7

We prove the incompatibility for $m = 5$ and $n = 5$, and the proof can be adapted to larger values as before.

Assume that f is a mechanism satisfying efficiency and positive share. Now consider a profile θ with uniform contributions ($C_i = 1$ for all agents $i \in N$) and the approval profile

$$A_N = (\{a\}, \{abc\}, \{abd\}, \{ace\}, \{de\}).$$

Let $\delta = f(\theta)$ be the distribution returned by the mechanism. Since f is efficient, we must have $\delta(b) = \delta(c) = 0$, because otherwise a Pareto improvement can be obtained by redistributing resources from either of these alternatives to a. Since the profile is symmetric under the permutation $\sigma = (b\ c)(d\ e)$, we must have $\delta(b) = \delta(c)$ and $\delta(d) = \delta(e)$ because f is anonymous and neutral. By positive share for Agent 5, we must have $\delta(d) = \delta(e) > 0$. It follows that $u_4(\delta) < C$ because a positive amount is spent on project d, which Agent 4 does not approve.

Now, suppose that the fourth agent pretends not to approve a, so we get the profile θ' with the following approvals:

$$A'_N = (\{a\}, \{abc\}, \{abd\}, \{ce\}, \{de\}).$$

Let $\delta' = f(\theta')$ be the distribution now returned by the mechanism. Again, by efficiency, we must have $\delta'(b) = 0$ since we can otherwise redistribute resources from b to a to get a Pareto improvement. Next, suppose that both $\delta'(c)$ and $\delta'(d)$ are positive, say $\delta'(c) \geq \epsilon$ and $\delta'(d) \geq \epsilon$ for some $\epsilon > 0$. Then δ' is Pareto dominated by the distribution obtained from δ' by moving ϵ from c to a and ϵ from d to e. This contradicts efficiency of f, so either $\delta'(c) = 0$ or $\delta'(d) = 0$. Since projects c and d are symmetric in θ', we must have $\delta'(c) = \delta'(d) = 0$. Hence, δ' distributes the entire endowment between projects a and e, and so $u_4(\delta') = C$.

Thus, Agent 4 has successfully manipulated f by reporting a subset of her true approval set.

A.8 Proof of Theorem 5 assuming anonymity

Throughout this proof, we will only consider profiles with uniform contributions. Recall that the *support* of a distribution δ is the set of projects x for which $\delta(x) > 0$. In this proof, we will sometimes use shorthands like xyz to refer to the set $\{x, y, z\}$.

Let $A = \{a, b, c, d\}$ be the set of projects. For each pair $\{x, y\}$ of projects from $\{b, c, d\}$, consider the following profile, where z is the project in the singleton $\{b, c, d\} \setminus \{x, y\}$.

$$P_{x,y} = (\{x\}, \{y\}, \{a, x\}, \{a, y\}, \{x, z\}, \{y, z\}).$$

If, in this profile, the mechanism gives a positive amount of resources to both a and z, then we can redistribute these resources to x and y in a way that leads to a Pareto improvement, contradicting efficiency. By positive share, both x and y need to get a positive amount. Thus, there are three options for the support used at profile $P_{x,y}$:

(α) $\{x, y\}$,
(β) $\{a, x, y\}$,
(γ) $\{x, y, z\}$.

We will prove, using strategyproofness, that option α is impossible, that option β can hold at only one of the profiles $P_{b,c}, P_{c,d}, P_{d,b}$, and that option γ can also hold at only one of these profiles. This is a contradiction.

Each part of the proof refers to a table of profiles (Tables 4, 5, and 6). Each row specifies a profile. The last column lists all supports that are compatible with efficiency and positive share. We say that a support is *chosen* at a profile if it is the support of the distribution returned by f at that profile. Each of the three proof parts begins by making an assumption on which support is chosen at the

	A_1	A_2	A_3	A_4	A_5	A_6	possible supports
θ^1 :	ay	az	xy	zy	xy	y	$ay, \underline{zy}, \underline{azy}$
θ^2 :	axy	az	xy	zy	xy	y	$ay, \underline{zy}, \underline{azy}$
θ^3 :	ax	az	xy	zy	xy	y	$ay, axy, \underline{azy}$
θ^4 :	ax	azy	xy	zy	xy	y	$\underline{ay}, xy, axy$
θ^5 :	ax	ay	xy	zy	xy	y	$\underline{ay}, xy, axy$
θ^6 :	ax	ay	xy	zy	x	y	$xy, \underline{axy}$
θ^7 :	ax	ay	zx	zy	x	y	$xy, \underline{axy}, \underline{xzy}$

Table 4. Profiles for the proof, part α.

profile of the first row: namely, the proof assumes it is one of the underlined supports. The proof then goes through the profiles in the table, using strategyproofness to deduce which support(s) are possibly chosen in the other profiles; these possible supports are underlined. We say that an agent is *happy* with the support chosen at a profile if the agent approves all projects in that support.

Option α is impossible. We prove that $f(P_{x,y})$ does not have support $\{x, y\}$.

Consider profile θ^1. In this profile, a and z are symmetric, and so without loss of generality we may assume that zy or azy is chosen at θ^1. (That is, if ay is chosen, then the proof that follows can be relabeled to yield the same conclusion.) Thus, Agent 1 is not happy and manipulates to obtain θ^2. By strategyproofness, Agent 1 cannot be happy at θ^2 so either zy or azy must be chosen. Agent 1 manipulates again to obtain θ^3, where Agent 1 cannot be happy so azy is chosen. Note that Agent 2 in θ^4 is happy with azy. Thus, to avoid a manipulation of Agent 2, that agent needs to already be happy in θ^4, so there ay is chosen. To avoid manipulation of Agent 2 in θ^5, Agent 2 needs to also be happy in θ^5, so ay is chosen there. Agent 5 is not happy and manipulates from θ^5 to θ^6, and cannot be happy there, so axy is chosen. Agent 3 is not happy and manipulates to θ^7 and cannot be happy there, so xy cannot be chosen. This is the desired conclusion, because $\theta^7 = P_{x,y}$.

Option β applies at most once. We prove that if $f(P_{x,y})$ has support $\{a, x, y\}$, then $f(P_{y,z})$ does not have support $\{a, y, z\}$. Be relabeling the proof, one can establish that if $f(P_{y,z})$ has support $\{a, y, z\}$ then $f(P_{z,x})$ does not have support $\{a, z, x\}$; and that if $f(P_{z,x})$ has support $\{a, z, x\}$ then $f(P_{x,y})$ does not have support $\{a, x, y\}$. These three statements give the desired conclusion.

Consider profile $\theta^8 = P_{x,y}$, and assume that axy is chosen. To avoid Agent 1 manipulating, Agent 1 needs to be happy in θ^9, so xy is chosen. To avoid Agent 1 manipulating, Agent 1 needs to be happy in θ^{10}, so xy is chosen. To avoid Agent 5 manipulating, Agent 5 needs to be happy in θ^{11}, so xy is chosen. Agent 4 is not happy and manipulates from θ^{11} to θ^{12}, and cannot be happy there so xy of zxy is chosen. Agent 4 is not happy and manipulates to θ^{13}, and cannot be happy there so zxy is chosen. To avoid Agent 3 manipulating, Agent 3 needs to be happy in θ^{14}, so zy is chosen. To avoid Agent 3 manipulating, Agent 3 needs to be happy in θ^{15}, so zy is chosen. To avoid Agent 1 manipulating, Agent 1 needs to be happy in θ^{16}, so zy is chosen. Agent 2 is not happy and manipulates to θ^{17}, and cannot be happy there so zy or azy is chosen. Agent 2 is not happy and manipulates to θ^{18}, and cannot be happy there so azy is chosen. To avoid Agent 4 manipulating, Agent 4 needs to be happy in θ^{19}, so ay is chosen. To avoid Agent 4 manipulating, Agent 4 needs to be happy in θ^{20}, so ay is chosen. Agent 5 is not happy and manipulates to θ^{21}, and cannot be happy there so ay or axy is chosen. Agent 5 is not happy and manipulates to θ^{22}, and cannot be happy there so axy is chosen. To avoid Agent 2 manipulating, Agent 2 needs to be happy in θ^{23}, so xy is

	A_1	A_2	A_3	A_4	A_5	A_6	possible supports
θ^8 :	ax	ay	zx	zy	x	y	xy, $\underline{axy}$, xzy
θ^9 :	axy	ay	zx	zy	x	y	$\underline{xy}$, zxy
θ^{10} :	xy	ay	zx	zy	x	y	$\underline{xy}$, zxy
θ^{11} :	xy	ay	zx	zy	xy	y	zy, $\underline{xy}$, zxy
θ^{12} :	xy	ay	zx	azy	xy	y	zy, $\underline{xy}$, $\underline{zxy}$
θ^{13} :	xy	ay	zx	az	xy	y	zy, azy, $\underline{zxy}$
θ^{14} :	xy	ay	zxy	az	xy	y	ay, $\underline{zy}$, azy
θ^{15} :	xy	ay	zy	az	xy	y	ay, $\underline{zy}$, azy
θ^{16} :	zy	ay	zy	az	xy	y	ay, $\underline{zy}$, azy
θ^{17} :	zy	axy	zy	az	xy	y	ay, $\underline{zy}$, $\underline{azy}$
θ^{18} :	zy	ax	zy	az	xy	y	ay, $\underline{azy}$, axy
θ^{19} :	zy	ax	zy	azy	xy	y	$\underline{ay}$, xy, axy
θ^{20} :	zy	ax	zy	ay	xy	y	$\underline{ay}$, xy, axy
θ^{21} :	zy	ax	zy	ay	zxy	y	$\underline{ay}$, xy, $\underline{axy}$
θ^{22} :	zy	ax	zy	ay	zx	y	xy, $\underline{axy}$, zxy
θ^{23} :	zy	axy	zy	ay	zx	y	zy, $\underline{xy}$, zxy
θ^{24} :	zy	xy	zy	ay	zx	y	zy, $\underline{xy}$, zxy
θ^{25} :	azy	xy	zy	ay	zx	y	zy, $\underline{xy}$, $\underline{zxy}$
θ^{26} :	azy	xy	z	ay	zx	y	zy, $\underline{xy}$, $\underline{zxy}$
θ^{27} :	az	xy	z	ay	zx	y	zy, azy, $\underline{zxy}$

Table 5. Profiles for the proof, part β.

chosen. To avoid Agent 2 manipulating, Agent 2 needs to be happy in θ^{24}, so xy is chosen. Agent 1 is not happy and manipulates to θ^{25}, and cannot be happy there so xy or zxy is chosen. Agent 3 is not happy and manipulates to θ^{26}, and cannot be happy there so xy or zxy is chosen. Agent 1 is not happy and manipulates to θ^{27}, and cannot be happy there so azy is not chosen. This is the desired conclusion, since $\theta^{27} = P_{y,z}$ up to reordering voters, and we assumed that f is anonymous.

Option γ applies at most once. Consider profile θ^{28}. Note that projects x and z are symmetric in it. We will prove that if at θ^{28} we choose xy or zxy, then at $\theta^{40} = P_{x,y}$ we cannot choose xzy. On the other hand if at θ^{28} we choose zy, then the proof can be relabelled to show that at $P_{z,y}$ we cannot choose zxy. Thus, we prove that $\{x, y, z\}$ can be chosen at at most one of the profiles $P_{x,y}$ and $P_{z,y}$. By further relabellings of the proof, like in part β, we then find that $\{x, y, z\}$ can be chosen at at most one of $P_{x,y}$, $P_{y,z}$, and $P_{z,x}$, as desired.

Consider profile θ^{28}, and assume that xy or zxy is chosen. Agent 1 is not happy and manipulates to θ^{29}, and cannot be happy there so zxy is chosen. Agent 2 is not happy and manipulates to θ^{30}, and cannot be happy there so zxy is chosen. Agent 1 is not happy and manipulates to θ^{31}, and cannot be happy there so zxy is chosen. To avoid Agent 3 manipulating, Agent 3 needs to be happy in θ^{32}, so zy is chosen. To avoid Agent 3 manipulating, Agent 3 needs to be happy in θ^{33}, so zy is chosen. Agent 2 is not happy and manipulates to θ^{34}, and cannot be happy there so zy or azy is chosen. Agent 2 is not happy and manipulates to θ^{35}, and cannot be happy there so azy is chosen. To avoid Agent 1 manipulating, Agent 1 needs to be happy in θ^{36}, so ay is chosen. To avoid Agent 1 manipulating, Agent 1 needs to be happy in θ^{37}, so ay is chosen. Agent 4 is not happy and manipulates to θ^{38}, and cannot be happy there so ay or axy is chosen. Agent 5 is not

	A_1	A_2	A_3	A_4	A_5	A_6	possible supports
θ^{28} :	zy	zy	zx	xy	xy	y	$zy, \underline{xy}, \underline{zxy}$
θ^{29} :	az	zy	zx	xy	xy	y	$zy, \underline{zxy}$
θ^{30} :	az	azy	zx	xy	xy	y	$zy, \underline{zxy}$
θ^{31} :	az	ay	zx	xy	xy	y	$zy, azy, \underline{zxy}$
θ^{32} :	az	ay	zxy	xy	xy	y	$ay, \underline{zy}, azy$
θ^{33} :	az	ay	zy	xy	xy	y	$ay, \underline{zy}, azy$
θ^{34} :	az	axy	zy	xy	xy	y	$ay, \underline{zy}, \underline{azy}$
θ^{35} :	az	ax	zy	xy	xy	y	$ay, \underline{azy}, axy$
θ^{36} :	azy	ax	zy	xy	xy	y	$\underline{ay}, xy, axy$
θ^{37} :	ay	ax	zy	xy	xy	y	$\underline{ay}, xy, axy$
θ^{38} :	ay	ax	zy	zxy	xy	y	$\underline{ay}, xy, \underline{axy}$
θ^{39} :	ay	ax	zy	zxy	x	y	$xy, \underline{axy}$
θ^{40} :	ay	ax	zy	zx	x	y	$xy, \underline{axy}, zxy$

Table 6. Profiles for the proof, part γ.

happy and manipulates to θ^{39}, and cannot be happy there so axy is chosen. Agent 4 is not happy and manipulates to θ^{40}, and cannot be happy there so zxy is not chosen. This is the promised conclusion.