Computing Dominance-Based Solution Concepts

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Two common criticisms of Nash equilibrium are its dependence on very demanding epistemic assumptions and its computational intractability. We study the computational properties of less demanding set-valued solution concepts that are based on varying notions of dominance. These concepts are intuitively appealing, they always exist, and admit unique minimal solutions in important subclasses of games. Examples include Shapley's saddles, Harsanyi and Selten's primitive formations, Basu and Weibull's CURB sets, and Dutta and Laslier's minimal covering sets. We propose two generic algorithms for computing these concepts and investigate for which classes of games and which properties of the underlying dominance notion the algorithms are sound and efficient.

1 Introduction

Saddle points, i.e., combinations of actions such that no player can gain by deviating, are among the earliest solution concepts considered in game theory (see, e.g., von Neumann and Morgenstern, 1944). In two-player zero-sum games (henceforth *matrix games*), every saddle point happens to coincide with the optimal outcome both players can guarantee in the worst case and thus enjoys a very strong normative foundation. Unfortunately, however, not every matrix game possesses a saddle point. In order to remedy this situation, Borel (1921) introduced *mixed*—i.e., randomized—strategies and von Neumann (1928) proved that every matrix game contains a mixed saddle point (or equilibrium) that moreover maintains the appealing normative properties of saddle points. The existence result was later generalized to arbitrary normal-form games by Nash (1951), at the expense of its normative foundation. Since then, Nash equilibrium has commonly been criticized for resting on very demanding epistemic assumptions such as the common knowledge of von Neumann-Morgenstern utilities.[1]

Shapley (1953a,b) showed that the existence of saddle points (and even uniqueness in the case of matrix games) can also be guaranteed by moving to *minimal sets* of actions rather than randomizations over them.[2] Shapley defines a *generalized saddle point (GSP)* to be a tuple of subsets of actions for each player that satisfies a simple external stability condition: Every action not contained in a player's subset is dominated by some action in the set, given that the remaining players choose actions from their respective sets. A GSP is minimal if it does not contain another GSP. Minimal GSPs, which Shapley calls *saddles*, also satisfy internal stability in the sense that no two actions within a set dominate each other, given that the remaining players choose actions from their respective sets. While Shapley was the first to conceive GSPs, he was not the only one. Apparently unaware of Shapley's work, Samuelson (1992) uses the very related concept of a *consistent pair* to point out epistemic inconsistencies in the concept of iterated weak dominance. Also, *weakly admissible sets* as defined by McKelvey and Ordeshook (1976) in the context of spatial voting games and the *minimal covering set* as defined by Dutta (1988) in the context of majority tournaments are GSPs (Duggan and Le Breton, 1996a). In a regrettably unpublished paper, Duggan and Le Breton (1996b) extend Shapley's approach to normal-form games and define a D-set as a minimal tuple of sets that is internally and externally stable with respect to a so-called *dominance structure* D. Depending on D, a number of different solution concepts can be defined. For the case of *strict* dominance (S), Shapley (1964) showed that every matrix game admits a *unique* S-set. Duggan and Le Breton (1996a) proved the same statement for weak dominance (W) and very weak dominance (V) in a subclass of symmetric matrix games that we refer to as confrontation games.

In recent years, the computational complexity of game-theoretic solution concepts has come under increasing scrutiny. One of the most prominent results in this stream of research is that the problem of finding Nash equilibria in bimatrix games is PPAD-complete (Chen et al., 2009; Daskalakis et al., 2009) and thus unlikely to admit an efficient algorithm. This result has sparked the search for alternative, computationally tractable, solution concepts. Despite the fact that Shapley's saddles were devised as early as 1953 (Shapley, 1953a,b) and are thus almost as old as Nash equilibrium (Nash, 1951), surprisingly little is known about their computational properties. Common notions of dominance have widely been studied from a computational perspective in the form of *iterated dominance* (Gilboa et al., 1993; Knuth et al., 1988; Conitzer and Sandholm, 2005; Brandt et al., 2011a). D-sets are "refinements" of iterated D-dominance and *cannot* be found by iterated elimination of dominated actions.

In this paper, we propose two generic algorithms (a greedy and a sophisticated one) for computing D-sets and study their soundness and efficiency for various dominance struc-

[1] See, e.g., Luce and Raiffa (1957, pp. 74–76), Fishburn (1978), Bernheim (1984), Pearce (1984), Myerson (1991, pp. 88–91), Börgers (1993), Aumann and Brandenburger (1995), Perea (2007), Jungbauer and Ritzberger (2011).

[2] The main results of the 1953 reports later reappeared in revised form (Shapley, 1964).

	S	B	S^*	C_D	C_M	V	V^*
normal-form games	poly	poly	poly				
matrix games	unique	unique	unique				
symmetric matrix games				unique	unique	exp	
confrontation games				unique			exp
tournament games		unique		unique			

Table 1: Summary of results. For a given dominance structure D and a class of games (ordered by set inclusion), the table shows bounds on the *asymptotic number* of D-sets (unique, polynomial, or exponential). If a cell is highlighted in dark gray, the greedy algorithm finds all D-sets in the given class in polynomial time. If it is highlighted in light gray, the analogous statement holds for the sophisticated algorithm. If a cell spans several columns, the corresponding D-sets coincide within the respective class of games. C_D and C_M are only defined for symmetric matrix games.

tures D and subclasses of games. In addition to the dominance structures mentioned above, we study their mixed counterparts (denoted by D^* for a given dominance structure D), Börgers dominance (B) (Börgers, 1993), covering (C_M) (McKelvey, 1986; Dutta and Laslier, 1999), and deep covering (C_D) (Duggan, 2011). We then define abstract properties that, when satisfied by a dominance structure within a particular class of games, allow for our algorithms to be sound and efficient. More specifically, our results yield

- greedy algorithms for computing all S-sets (aka saddles), S^*-sets (aka primitive formations and equivalent to CURB sets in two-player games), and B-sets of a given normal-form game, and
- sophisticated algorithms for computing the unique C_M-set and the unique C_D-set of a given symmetric matrix game. Within the subclass of confrontation games, these algorithms coincide and also yield the W-set and the V-set (aka weak saddles).

Our algorithms subsume existing algorithms for computing saddles in matrix games (Shapley, 1964), minimal covering sets in binary symmetric matrix games (Brandt and Fischer, 2008), and CURB sets in two-player games (Benisch et al., 2010). Interestingly, the sophisticated algorithms rely on the repeated computation of Nash equilibria via linear programming, even though the corresponding solution concepts are purely ordinal. For the remaining combinations of dominance structures and classes of games, we show that these classes admit an exponential number of D-sets. This renders the computation of *all* such sets infeasible.[3] Our results are summarized in Table 1.

2 Preliminaries

A (finite) *game in normal form* is a tuple $\Gamma = (N, (A_i)_{i\in N}, (u_i)_{i\in N})$, where $N = \{1, \dots, n\}$ is a nonempty finite set of *players* and for each player $i \in N$, A_i is a nonempty finite set of *actions* available to player i, and $u_i : (\prod_{i\in N} A_i) \to \mathbb{R}$ is a function mapping each action profile (i.e., combination of actions) to a real-valued *utility* for player i.

A two-player game $\Gamma = (\{1,2\}, (A_1, A_2), (u_1, u_2))$ is a *matrix game* (or *zero-sum game*) if $u_1(a_1, a_2) + u_2(a_1, a_2) = 0$ for all $(a_1, a_2) \in A_1 \times A_2$. It is *symmetric* if $A_1 = A_2$ and $u_1(a_1, a_2) = u_2(a_2, a_1)$ for all $(a_1, a_2) \in A_1 \times A_1$. Whenever we are concerned with symmetric matrix games, we slightly deviate from the notation used in the rest of the paper for notational convenience: The set $A_1 = A_2$ of actions is denoted by A and the utility function of player 1 is denoted by u. A symmetric matrix game can be conveniently represented by a skew-symmetric matrix containing the utilities of player 1. For a subset $B \subseteq A$ of actions, $\Gamma|_B$ denotes the *subgame* of $\Gamma = (A, u)$ restricted to B, i.e., $\Gamma|_B$ is the symmetric matrix game $(B, u|_{B\times B})$.

Confrontation games are symmetric matrix games characterized by the fact that the two players get the same utility if and only if they play the same action (Duggan and Le Breton, 1996a). Formally, a symmetric matrix game $\Gamma = (A, u)$ is called confrontation game if for all $a, b \in A$, $u(a, b) = 0$ if and only if $a = b$.[4] If moreover $u(a, b) \in \{-1, 0, 1\}$ for all $a, b \in A$, we have a *tournament game*.[5]

Let $\Delta(M)$ denote the set of all probability distributions over a finite set M. A (mixed) *strategy* of a player $i \in N$ is an element of $\Delta(A_i)$. A *Nash equilibrium* is a combination of strategies such that no player can benefit by unilaterally changing his strategy. The *essential set* $ES(\Gamma)$ is the set of all actions that are played with positive probability in some Nash equilibrium of Γ (Dutta and Laslier, 1999). As every normal-form game has a Nash equilibrium (Nash, 1951), the essential set is never empty.

3 Dominance-Based Solution Concepts

In this section, we formally define the dominance structures and solution concepts considered in this paper. Furthermore, we introduce a number of properties that will be critical for our algorithmic results.

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[3] In the case of very weak dominance, it has also been shown that finding *some* V-set is computationally intractable in two-player games (Brandt et al., 2011b). Whether V-sets and V^*-sets can be computed efficiently in matrix games remains an open problem.

[4] Duggan and Le Breton (1996a) refer to this property as the *off-diagonal property*.

[5] The term *tournament game* refers to the fact that such a game $\Gamma = (A, u)$ can be represented by a tournament graph with vertex set A and edge set $\{(a, b) : u(a, b) = 1\}$. In a similar fashion, a confrontation game can be represented by a *weighted* tournament graph.

3.1 Dominance Structures

We need the following notation, which will be used throughout the paper. Let A_N denote the n-tuple $(A_1, \ldots, A_n)$ containing all action sets. An n-tuple $X = (X_1, \ldots, X_n)$ is said to be *nonempty* if $X_i \neq \emptyset$ for all $i \in N$. For a nonempty n-tuple $X = (X_1, \ldots, X_n)$, we write $X \subseteq A_N$ if $X_i \subseteq A_i$ for all $i \in N$. To simplify the exposition, we will frequently abuse terminology and refer to an n-tuple $X \subseteq A_N$ as a "set." For every player i, we furthermore let X_{-i} denote the set $\prod_{j \in N \setminus \{i\}} X_j$ of all opponent action profiles where each opponent $j \in N \setminus \{i\}$ is restricted to play only actions from $X_j \subseteq A_j$.

Consider a player $i \in N$. Whether an action (or a combination of actions) in A_i dominates another action in A_i naturally depends on which actions the other players have at their disposal. This is reflected in the following definition, in which a dominance structure is defined as a mapping from opponent action profiles to dominance relations on A_i. In order to accommodate for Börgers dominance (Börgers, 1993) and mixed dominance structures, we furthermore have dominance relations on A_i relate *sets of actions* to individual actions.

Definition 1. *Let $\Gamma = (N, (A_i)_{i \in N}, (u_i)_{i \in N})$ be a game in normal form and $X \subseteq A_N$. For each player $i \in N$, a* dominance structure *D maps X_{-i} to a subset of $2^{A_i} \times A_i$ such that $(X_i, a_i) \in D(X_{-i})$ implies $(Y_i, a_i) \in D(X_{-i})$ for all Y_i with $X_i \subseteq Y_i \subseteq A_i$.*

For $X_i \subseteq A_i$ and $a_i \in A_i$, we write $X_i \; D(X_{-i}) \; a_i$ if $(X_i, a_i) \in D(X_{-i})$. If X_i consists of a single action x_i, we write $x_i \; D(X_{-i}) \; a_i$ instead of $\{x_i\} \; D(X_{-i}) \; a_i$ to avoid cluttered notation.

We go on to define the main dominance structures considered in this paper, together with their mixed counterparts that allow for randomized strategies.

Definition 2. *Let $\Gamma = (N, (A_i)_{i \in N}, (u_i)_{i \in N})$ be a game in normal form and $i \in N$. Furthermore, let $X \subseteq A_N$ and $a_i \in A_i$.*

- strict dominance (S): *$X_i \; S(X_{-i}) \; a_i$ if there exists $x_i \in X_i$ with $u_i(x_i, x_{-i}) > u_i(a_i, x_{-i})$ for all $x_{-i} \in X_{-i}$.*
- weak dominance (W)*: $X_i \; W(X_{-i}) \; a_i$ if there exists $x_i \in X_i$ with $u_i(x_i, x_{-i}) \geq u_i(a_i, x_{-i})$ for all $x_{-i} \in X_{-i}$ and the inequality is strict for at least one $x_{-i} \in X_{-i}$.*
- very weak dominance (V)*: $X_i \; V(X_{-i}) \; a_i$ if there exists $x_i \in X_i$ with $u_i(x_i, x_{-i}) \geq u_i(a_i, x_{-i})$ for all $x_{-i} \in X_{-i}$.*
- Börgers dominance (B)*: $X_i \; B(X_{-i}) \; a_i$ if $X_i \; W(Y_{-i}) \; a_i$ for all nonempty $Y_{-i} \subseteq X_{-i}$.*
- mixed strict dominance (S^*): *$X_i \; S^*(X_{-i}) \; a_i$ if there exists $s_i \in \Delta(X_i)$ with $u_i(s_i, x_{-i}) > u_i(a_i, x_{-i})$ for all $x_{-i} \in X_{-i}$.*
- mixed weak dominance (W^*)*: $X_i \; W^*(X_{-i}) \; a_i$ if there exists $s_i \in \Delta(X_i)$ with $u_i(s_i, x_{-i}) \geq u_i(a_i, x_{-i})$ for all $x_{-i} \in X_{-i}$ and the inequality is strict for at least one $x_{-i} \in X_{-i}$.*

- mixed very weak dominance (V^*): *X_i $V^*(X_{-i})$ a_i if there exists $s_i \in \Delta(X_i)$ with $u_i(s_i, x_{-i}) \geq u_i(a_i, x_{-i})$ for all $x_{-i} \in X_{-i}$.*

Börgers dominance has a mixed counterpart as well, requiring that X_i $W^*(Y_{-i})$ a_i for all $Y_{-i} \subseteq X_{-i}$. However, mixed Börgers dominance coincides with mixed strict dominance (Duggan and Le Breton, 1996a).

The following dominance structures are only well-defined in symmetric matrix games, and we will refer to them as *symmetric* dominance structures.

Definition 3. *Let $\Gamma = (A, u)$ be a symmetric matrix game, $X, Y \subseteq A$, and $a \in A$.*

- covering (C_M): *X $C_M(Y)$ a if there exists $x \in X \cap Y$ with*
  - *$u(x, a) > 0$ and*
  - *$u(x, y) \geq u(a, y)$ for all $y \in Y$.*
- deep covering (C_D): *X $C_D(Y)$ a if there exists $x \in X \cap Y$ with*
  - *$u(x, a) > 0$,*
  - *$u(x, y) \geq u(a, y)$ for all $y \in Y$, and*
  - *$u(x, y) > u(a, y)$ for all $y \in Y$ with $u(a, y) = 0$.*

Covering was introduced by McKelvey (1986) and later generalized by Dutta and Laslier (1999), and deep covering is a generalization of a notion by Duggan (2011).

For two dominance structures D and D', we write $D \subseteq D'$ and say that D is *coarser* than D' and that D' is *finer* than D, if $D(X_{-i}) \subseteq D'(X_{-i})$ for all $X \subseteq A_N$. The following relations follow immediately from the respective definitions: $S \subseteq B \subseteq W \subseteq V$, $C_D \subseteq C_M$, and $D \subseteq D^*$ for all $D \in \{S, W, V\}$.

3.2 Solutions and D-sets

Generalizing a classic cooperative solution concept by von Neumann and Morgenstern (1944), a set of actions X can be said to be *stable* if it consists precisely of those alternatives not dominated by X (see also Wilson, 1970). This fixed-point characterization can be split into two conditions of internal and external stability: First, there should be no reason to restrict the selection by excluding some action from it; second, there should be an argument against each proposal to include an outside action into the selection.

Definition 4. *Let D be a dominance structure and $\Gamma = (N, (A_i)_{i \in N}, (u_i)_{i \in N})$ a game in normal form. A tuple $X \subseteq A_N$ is a D-*solution *in Γ if for every $i \in N$,*

$$X_i \setminus \{x_i\} \; D(X_{-i}) \; x_i \text{ for no } x_i \in X_i, \text{ and} \tag{1}$$

$$X_i \; D(X_{-i}) \; a_i \text{ for all } a_i \in A_i \setminus X_i. \tag{2}$$

	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
a_1	0	2	−2	1	2			
a_2	−2	0	2	1	2			
a_3	2	−2	0	1	1		3	
a_4	−1	−1	−1	0	1			
a_5	−2	−2	−1	−1	0			
a_6						0	3	−3
a_7			−3			−3	0	3
a_8						3	−3	0

	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
b_1	0	0	0	0	1	1	1	1
b_2	0	0	1	−1	0	1	1	−1
b_3	0	−1	0	1	1	1	−1	1
b_4	0	1	−1	0	0	−1	1	1
b_5	−1	0	−1	0	0	1	1	1
b_6	−1	−1	−1	1	−1	0	−1	1
b_7	−1	−1	1	−1	−1	1	0	−1
b_8	−1	1	−1	−1	−1	−1	1	0

D	D-set
S	$(\{a_1,a_2,a_3,a_4,a_5\},\{a_1,a_2,a_3,a_4,a_5\})$
W,V,B	$(\{a_1,a_2,a_3,a_4\},\{a_1,a_2,a_3,a_4\})$
S^*,W^*,V^*	$(\{a_1,a_2,a_3\},\{a_1,a_2,a_3\})$

D	D-set
C_D	$(\{b_1,b_2,b_3,b_4,b_5\},\{b_1,b_2,b_3,b_4,b_5\})$
C_M	$(\{b_1,b_2,b_3,b_4\},\{b_1,b_2,b_3,b_4\})$
V	$(\{b_1\},\{b_1\})$

Figure 1: Example games with unique D-sets for several dominance structures D. The game on the left is a confrontation game and the game on the right is a symmetric matrix game.

We refer to (1) and (2) as *internal* and *external D-stability*, respectively. We are mainly interested in *inclusion-minimal* D-solutions. Following Duggan and Le Breton (1996b), we call them *D-sets*.

Definition 5. *A* D-set *is a D-solution X such that there does not exist a D-solution Y with $Y \subseteq X$ and $Y \neq X$.*

Figure 1 contains examples of D-sets for all dominance structures considered in this paper.

Various set-valued solution concepts that have been proposed in the literature can be characterized as D-sets for some dominance structure D. Shapley's (1964) *saddles* and *weak saddles* for matrix games correspond to S- and V-sets, respectively, Dutta and Laslier's (1999) *minimal covering sets* for symmetric matrix games correspond to C_M-sets, and Duggan's (2011) deep covering sets for binary symmetric matrix games correspond to C_D-sets. Furthermore, mixed refinements of Shapley's saddles, as proposed by Duggan and Le Breton (2001) for binary symmetric matrix games, correspond to S^*- and W^*-sets.

Two further solution concepts that fit into our framework are Harsanyi and Selten's (1988) *formations* and Basu and Weibull's (1991) *CURB sets*. The respective dominance structures are defined in terms of best response sets. An action a_i is *rationally dominated* with respect to a set X_{-i} of opponent action profiles if it is *not* a best response to any mixed opponent strategy with support in X_{-i}. A subtle difference occurs if there are more than two players (and therefore more than one opponent). While in *correlated rational dominance* (R_c), opponents are allowed to play joint, i.e., correlated, mixtures (and thus to act like a single opponent), *uncorrelated rational dominance* (R_u) restricts opponents to independent mixtures.

A tuple of sets is a CURB set if and only if it is externally stable with respect to R_u, and *minimal CURB sets* coincide with R_u-sets. Similarly, a tuple of sets is a formation if and only if it is externally stable with respect to R_c, and *primitive formations* are R_c-sets. Since it is well known that an action is not a best response to some correlated opponent strategy if and only if it is dominated by a mixed strategy (see, e.g., Pearce, 1984, Lemma 3), the dominance structures R_c and S^* coincide. As a consequence, all our results concerning S^*-sets directly apply to primitive formations as well. The same is true for minimal CURB sets in two-player games, due to the equivalence of R_c and R_u for $n = 2$.[6]

3.3 Properties of Dominance Structures

We now define a number of properties in order to formalize for which dominance structures, D-sets can be computed efficiently. An action $a_i \in A_i$ is said to be D-maximal with respect to X_{-i} if it is not D-dominated by A_i.

Definition 6. *Let D be a dominance structure and $X \subseteq A_N$. The D-maximal elements of A_i with respect to X_{-i} are defined as*

$$\max(D(X_{-i})) = A_i \setminus \{a_i \in A_i : A_i\ D(X_{-i})\ a_i\}.$$

Definition 7. *Let $X \subseteq A_N$ and $a_i \in A_i$. A dominance structure D satisfies*

- monotonicity *(MON) if $X_i\ D(X_{-i})\ a_i$ implies $X_i\ D(Y_{-i})\ a_i$ for all nonempty $Y_{-i} \subseteq X_{-i}$,*
- computational tractability *(COM) if $X_i\ D(X_{-i})\ a_i$ can be checked in polynomial time, and*
- maximal domination *(MAX) if $A_i\ D(X_{-i})\ a_i$ implies* $\max(D(X_{-i}))\ D(X_{-i})\ a_i$.
- singularity *(SING) if $X_i\ D(X_{-i})\ a_i$ implies the existence of an action $x_i \in X_i$ with $x_i\ D(X_{-i})\ a_i$.*

It is easily seen that S, B, and V are monotonic, and that W is not. S and W satisfy maximal domination because the relations $S(X_{-i})$ and $W(X_{-i})$—restricted to pairs of singletons—are transitive and irreflexive. On the other hand, V violates MAX because $\max(V(X_{-i}))$ may be empty. It directly follows from the definitions that S, W, V, C_M, and C_D are singular. As shown in Sections 5 and 6, all the dominance structures introduced in Section 3.1 are computationally tractable.

The following properties are defined for symmetric dominance structures.

[6] It was recently shown that CURB sets are computationally intractable in games with more than two players (Hansen et al., 2010). In fact, even checking uncorrelated rational dominance is coNP-hard.

	a_1	a_2	a_3
a_1	0	1	0
a_2	−1	0	1
a_3	0	−1	0

Figure 2: Symmetric matrix game without W- and W^*-solutions.

Definition 8. *Let $X, X' \subseteq A$ and $a, b, c \in A$. A symmetric dominance structure D satisfies*

- weak monotonicity *(weak MON) if $a\ D(X)\ b$ implies $a\ D(Y)\ b$ for all $Y \subseteq X$ with $a \in Y$,*
- transitivity *(TRA) if $a\ D(X)\ b$, $b\ D(X')\ c$, and $a \in X \cap X'$ imply $a\ D(X \cap X')\ c$,*
- computational tractability of finding subsets *(SUB-COM) if a nonempty subset of a D-set can be computed in polynomial time, and*
- uniqueness *(UNI) if every symmetric matrix game has a unique D-set.*

Monotonicity turns out to be sufficient for the *existence of solutions*: If a dominance structure D satisfies MON, a D-solution can be constructed by iteratively eliminating actions that are dominated. When the elimination process comes to an end, MON ensures that the resulting set is externally D-stable. Note, however, that these solutions need not be minimal (see, e.g., Figure 1).[7] The same is true for symmetric dominance structures satisfying weak monotonicity. As weak dominance and mixed weak dominance are not monotonic, the above argument does not apply to those dominance structures. In fact, there are games without any W- or W^*-solution (see Figure 2 for an example). For this reason, we do not consider the W- and W^*-solutions in this paper.[8]

Another beneficial property of (weakly) monotonic dominance structures is that *minimal* externally stable sets also happen to be internally stable. This is again due to the fact that the iterative elimination of dominated actions preserves external stability.

[7] Under fairly general conditions, D-solutions obtained by iterated elimination of dominated actions are *maximal* (Duggan and Le Breton, 1996b). The maximal S^*-solution of a two-player game, for instance, consists of all *rationalizable* actions (Pearce, 1984; Bernheim, 1984).

[8] The fact that W-solutions may fail to exist was first observed by Samuelson (1992). There are at least three approaches to restore the existence of W-solutions. First, one can ignore internal stability and define W-solutions as externally W-stable sets (Duggan and Le Breton, 2001). The properties of W-sets defined in this way are similar to those of V-sets: The number of W-sets may be exponential, even in symmetric matrix games, and a number of natural problems concerning W-sets are computationally intractable (Brandt et al., 2011b). Second, one can look for restricted classes of games in which W-solutions are guaranteed to exist. One such class is the class of confrontation games, where the W-set is unique and coincides with the V-set. Third, one can consider the so-called *monotonic kernel* of W, which turns out to be identical to B (Duggan and Le Breton, 1996b).

Proposition 1. *Let D be a dominance structure satisfying MON or weak MON. Then, a set $X \subseteq A_N$ is a D-set if and only if it is a minimal externally D-stable set.*

Our proofs will frequently exploit this equivalence of D-sets and minimal externally D-stable sets. In particular, we will make use of the following easy corollary.

Corollary 1. *Let D be a dominance structure satisfying MON or weak MON and let $X \subseteq A_N$ be externally D-stable. Then, there exists a D-set Y with $Y \subseteq X$.*

4 General Results

We will now study how to compute minimal solutions for the dominance structures introduced in the previous section. To this end, we introduce two generic algorithms: a *greedy* and a *sophisticated* one. In principle, these algorithms can be applied to any game and any of the dominance structures introduced in Section 3.1. The goal of this section is to identify, for each algorithm, which properties of a dominance structure guarantee that the algorithm is sound and efficient. In addition, we will construct a family of games that admit an exponential number of minimal solutions.

4.1 Generic Greedy Algorithm

Shapley (1964) has shown that every matrix game possesses a unique S-set and described an algorithm, attributed to Harlan Mills, to compute this set. The idea behind this algorithm is that given a subset of the S-set, the S-set itself can be computed by iteratively adding actions that are maximal, i.e., not dominated with respect to the current subset of actions of the other player. We generalize Mills' algorithm in two directions. First, we identify general conditions on a dominance structure D that ensure that this greedy approach works. Second, we consider arbitrary n-player normal-form games, thereby losing uniqueness of D-sets, and devise an algorithm that computes *all* D-sets of such games in polynomial time.

We start by looking at some structural properties of externally stable sets. For monotonic dominance structures satisfying maximal domination, externally stable sets are closed under intersection.[9]

Proposition 2. *Let D be a dominance structure satisfying MON and MAX. If X and Y are externally D-stable and $X \cap Y \neq \emptyset$, then $X \cap Y$ is externally D-stable.*

Proof. Suppose that X and Y are externally D-stable and $X \cap Y \neq \emptyset$. In order to show that $X \cap Y$ is externally D-stable, fix $i \in N$ and consider $a_i \in A_i \setminus (X_i \cap Y_i)$. Without loss of generality, assume that $a_i \notin X_i$. As X is an externally D-stable, $X_i \; D(X_{-i}) \; a_i$, and thus $A_i \; D(X_{-i}) \; a_i$. Now MON implies $A_i \; D(X_{-i} \cap Y_{-i}) \; a_i$. Since $a_i \in A_i \setminus (X_i \cap Y_i)$ was chosen arbitrarily, $\max(D(X_{-i} \cap Y_{-i})) \subseteq X_i \cap Y_i$. Moreover, maximal domination implies $\max(D(X_{-i} \cap Y_{-i})) \; D(X_{-i} \cap Y_{-i}) \; a_i$, which finally yields $(X_i \cap Y_i) \; D(X_{-i} \cap Y_{-i}) \; a_i$. □

[9] $X \cap Y$ is to be read componentwise. Hence, $X \cap Y \neq \emptyset$ if and only if $X_i \cap Y_i \neq \emptyset$ for all $i \in N$.

Algorithm 1 Minimal externally D-stable set containing X^0

procedure `min_ext`$(\Gamma, (X_1^0, \dots, X_n^0))$
  for all $i \in N$ **do**
    $X_i \leftarrow X_i^0$
  end for
  repeat
    for all $i \in N$ **do**
      $Y_i \leftarrow \max(D(X_{-i})) \setminus X_i$
      $X_i \leftarrow X_i \cup Y_i$
    end for
  until $\bigcup_{i=0}^n Y_i = \emptyset$
  return $(X_1, \dots, X_n)$

One particularly useful consequence of Proposition 2 is the uniqueness of minimal externally D-stable sets containing given sets of actions.

Corollary 2. *Let $X^0 \subseteq A_N$. Under the assumptions of Proposition 2, the minimal externally D-stable set containing X^0 is unique: If Y and Z are externally D-stable with $X^0 \subseteq Y$ and $X^0 \subseteq Z$, then $Y \subseteq Z$ or $Z \subseteq Y$.*

Proof. Suppose that both Y and Z are minimal among all externally D-stable sets containing X^0, and that neither $Y \subseteq Z$ nor $Z \subseteq Y$. As both Y and Z contain X^0, $Y \cap Z$ is nonempty and Proposition 2 implies that $Y \cap Z$ is externally D-stable. This contradicts minimality of both Y and Z. □

If D moreover satisfies computational tractability, the minimal externally D-stable set containing X^0 can be computed efficiently by greedily adding undominated actions.

Proposition 3. *Let $X^0 \subseteq A_N$. If D satisfies MON, MAX, and COM, the minimal externally D-stable set containing X^0 can be computed in polynomial time.*

Proof. We show that Algorithm 1 computes the minimal externally D-stable set containing X^0 and runs in polynomial time. Algorithm 1 starts with X^0 and iteratively adds all actions that are not yet dominated. As D satisfies COM, these actions can be computed efficiently. Moreover, the number of loops is bounded by $\sum_{i=1}^n |A_i|$.

Let X^{min} be the minimal externally D-stable set containing X^0. We show that during the execution of Algorithm 1, the set X is always a subset of X^{min}. At the end of the algorithm, $\bigcup_{i=0}^n Y_i = \emptyset$ implies that $\max(D(X_{-i})) \subseteq X_i$ for all $i \in N$. As D satisfies MAX, this shows that X is externally D-stable.

We prove $X \subseteq X^{min}$ by induction on $|X| = \sum_{i=1}^n |X_i|$. At the beginning of the algorithm, $X = X^0 \subseteq X^{min}$ by definition of X^{min}. Now assume that $X \subseteq X^{min}$ at the beginning of a particular iteration. We have to show that for all $i \in N$, $Y_i \subseteq X_i^{min}$. Let $a_i \in$

Algorithm 2 All D-sets

```
procedure D_set(Γ)
  C ← ∅
  for all (a_1, ..., a_n) ∈ ∏_{i∈N} A_i do
    C ← C ∪ min_ext(Γ, ({a_1}, ..., {a_n}))
  end for
  return { X ∈ C : X is inclusion-minimal in C }
```

$Y_i = \max(D(X_{-i})) \setminus X_i$, and assume for contradiction that $a_i \notin X_i^{min}$. Since X^{min} is externally D-stable, $X_i^{min}\ D(X_{-i}^{min})\ a_i$. By the induction hypothesis, $X_{-i} \subseteq X_{-i}^{min}$, which together with MON implies $X_i^{min}\ D(X_{-i})\ a_i$. It follows that $A_i\ D(X_{-i})\ a_i$, contradicting the assumption that $a \in \max(D(X_{-i}))$. □

Whenever X^0 is contained in a D-set, Algorithm 1 returns this D-set. This property can be used to construct an algorithm to compute all D-sets of a game: Call Algorithm 1 for every possible combination of singleton sets of actions of the different players. The result is a set of externally D-stable sets, and the D-sets of the game are the inclusion-minimal elements of this set. This idea is made precise in Algorithm 2.

Theorem 1. *If D satisfies MON, MAX, and COM, all D-sets can be computed in polynomial time.*

Proof. We show that Algorithm 2 computes all D-sets of Γ and runs in polynomial time. Polynomial running time follows immediately because Algorithm 1 is invoked $|A|$ times, and inclusion-maximality can be checked easily.

As for correctness, we first show that every D-set X is an element of the set C. To see this, note that Proposition 2 implies that X is the minimal externally D-stable set containing $(\{x_1\}, \ldots, \{x_n\})$ for every $(x_1, \ldots, x_n) \in \prod_{i=1}^n X_i$. By definition, each D-set is inclusion-minimal.

To show that all inclusion-minimal elements of C are D-sets, it is sufficient to observe that all elements of C are externally D-stable. □

4.2 Generic Sophisticated Algorithm

Algorithm 2 is not sound for all dominance structures considered in this paper. For instance, very weak dominance violates maximal domination and therefore does not satisfy the conditions of Theorem 1. The example given in Figure 3 shows that, even in tournament games, Algorithm 2 fails to find the unique V-set. The failure of the greedy algorithm can be traced back to the following problem: For some dominance structures D, actions that are D-undominated with respect to a given set X_{-i} might be D-dominated with respect to a larger set $Y_{-i} \supset X_{-i}$. This subtlety is clearly problematic for the iterative approach, as it is no longer obvious which undominated actions should be added to a subset of a

	a_1	a_2	a_3	a_4	a_5	a_6
a_1	0	1	−1	1	1	−1
a_2	−1	0	1	1	−1	1
a_3	1	−1	0	−1	1	1
a_4	−1	−1	1	0	1	−1
a_5	−1	1	−1	−1	0	1
a_6	1	−1	−1	1	−1	0

Figure 3: Tournament game with unique V-set $(\{a_1, a_2, a_3\}, \{a_1, a_2, a_3\})$. Initiating Algorithm 1 with any pair $(\{a_i\}, \{a_j\})$ results in a proper superset of the V-set.

D-set. We will identify conditions on dominance structures D that allow for the following sophisticated method: Instead of adding all D-undominated actions, merely add actions contained in a D-set of the *subgame* induced by the D-undominated actions. This immediately gives rise to a recursive algorithm, whose running time may however be exponential. If a nonempty subset of a D-set can be found efficiently, an efficient algorithm can be constructed.

In this section, we will only be concerned with symmetric dominance structures D satisfying uniqueness. If a symmetric matrix game Γ has a unique D-set (X_1, X_2), it is easily verified that $X_1 = X_2$.[10] In this case, the set $X_1 = X_2$ will be denoted by $S_D(\Gamma)$. The following lemma is the key ingredient for the sophisticated algorithm.

Lemma 1. *Let Γ be a symmetric matrix game and D a symmetric dominance structure satisfying weak MON, TRA, SING, and UNI. Let furthermore $X \subseteq S_D(\Gamma)$ and define $\Gamma_X = \Gamma|_{\max(D(X))\setminus X}$. Then, $S_D(\Gamma_X) \subseteq S_D(\Gamma)$.*

Proof. Let $A' = \max(D(X)) \setminus X$. We can assume that A' is nonempty, as otherwise $S_D(\Gamma_X) = S_D(\Gamma|_{A'})$ is empty and there is nothing to prove.

Now partition the set A' of undominated actions into two sets $C = A' \cap S_D(\Gamma)$ and $C' = A' \setminus S_D(\Gamma)$ of actions contained in $S_D(\Gamma)$ and actions not contained in $S_D(\Gamma)$. We will show that (C, C) is an externally D-stable in Γ_X. Then, Corollary 1 and UNI imply that $S_D(\Gamma_X) \subseteq C$ and, therefore, $S_D(\Gamma_X) \subseteq S_D(\Gamma)$.

In order to show that (C, C) is externally D-stable in Γ_X, consider some $z \in C'$. Since $z \notin S_D(\Gamma)$, singularity of D implies that there exists $y \in S_D(\Gamma)$ with $y\ D(S_D(\Gamma))\ z$. It is easy to see that $y \notin X$, since otherwise weak MON would imply that $y\ D(X)\ z$, contradicting the assumption that $z \in A'$. On the other hand, assume that $y \in S_D(\Gamma) \setminus (X \cup C)$. Then there is some $x \in X$ such that $x\ D(X)\ y$. However, according to TRA, $x\ D(X)\ y$ and $y\ D(S_D(\Gamma))\ z$ imply $x\ D(X)\ z$, again contradicting the assumption that $z \in A'$. Thus

[10]Suppose $X_1 \neq X_2$. Then, (X_2, X_1) is another D-set in Γ.

Algorithm 3 D-set of a symmetric matrix game

```
procedure D_set_symm(Γ)
  X ← subset of S_D(Γ)
  repeat
    Γ_X ← Γ|_{max(D(X))\X}
    X' ← subset of S_D(Γ_X)
    X ← X ∪ X'
  until max(D(X)) \ X = ∅
  return (X, X)
```

$y \in C$, and using weak MON again, $y\ D(S_D(\Gamma))\ z$ and $z \in A'$ imply $y\ D(A')\ z$. Hence C is externally D-stable in Γ_X. □

Two further properties are required to turn the insight of Lemma 1 into an efficient algorithm: First, we need a polynomial-time subroutine to compute a nonempty subset of the unique D-set; second, the dominance structure D itself must be computationally tractable.

Theorem 2. *If D satisfies weak MON, TRA, SING, UNI, SUB-COM, and COM, the D-set of a symmetric matrix game can be computed in polynomial time.*

Proof. We show that Algorithm 3 computes $S_D(\Gamma)$ and runs in polynomial time. In each iteration, at least one action is added to the set X, so the algorithm is guaranteed to terminate after at most $|A|$ iterations. Each iteration consists of (i) computing the set $\max(D(X)) \setminus X$ of undominated actions and (ii) finding a subset of $S_D(\Gamma_X)$. Since D satisfies COM and SUB-COM, both tasks can be performed in polynomial time.

As for correctness, we show by induction on the number of iterations that $X \subseteq S_D(\Gamma)$ holds at any time. When the algorithm terminates, X is externally D-stable, which together with the induction hypothesis implies that $X = S_D(\Gamma)$. The base case is trivial. Now assume that $X \subseteq S_D(\Gamma)$ at the beginning of a particular iteration. Then $X \cup X' \subseteq X \cup S_D(\Gamma_X) \subseteq S_D(\Gamma)$, where the first inclusion is due to $X' \subseteq S_D(\Gamma_X)$ and the second inclusion follows from Lemma 1 and the induction hypothesis. □

4.3 Games with an Exponential Number of D-Sets

Our algorithms do not apply to all dominance structures considered in this paper. In fact, some dominance structures give rise to an *exponential* number of D-sets, even in symmetric matrix games. We need the following lemma, which is easily established.

Lemma 2. *Let $\Gamma = (A, u)$ be a symmetric matrix game. Define Γ' as the matrix game that is identical to Γ except that player 1 has an additional action $\hat{a}$ that always yields a utility of 1. That is, $\Gamma' = (\{1, 2\}, (A \cup \{\hat{a}\}, A), (u_1, u_2))$ with $u_1(a, b) = u(a, b)$ for all $a, b \in A$,*

$u_1(\hat{a}, a) = 1$ *for all* $a \in A$*, and* $u_2 = -u_1$*. Then, there exists no subset* $X \subseteq A$ *such that* $X\ V^*(X)\ \hat{a}$*.*

A D-set (X_1, X_2) is *symmetric* if $X_1 = X_2$. It is straightforward to verify that every symmetric matrix game has a symmetric D-set for all dominance structures D. For $D \in \{V, V^*\}$, any symmetric matrix game with multiple symmetric D-sets can be used to show that the number of D-sets may be exponential in general.

Theorem 3. *Let* $D \in \{V, V^*\}$*. If there exists a symmetric matrix game with at least two symmetric* D*-sets, then there exists a family of symmetric matrix games such that the number of* D*-sets is exponential in the size of the game.*

Proof. Let $D \in \{V, V^*\}$ and consider a symmetric matrix game $\Gamma = (A, u)$ with $k \geq 2$ symmetric D-sets. We construct a family $(\Gamma_i)_{i\in\mathbb{N}}$ such that $\Gamma_i = (A^i, u^i)$ is a symmetric matrix game and the number of symmetric D-sets in Γ_i is $k^{|A^i|/|A|}$.

Let $\Gamma_1 = \Gamma$. For $i \geq 1$, $\Gamma_{i+1} = (A^{i+1}, u^{i+1})$ is defined inductively as follows.

$$A^{i+1} = A^{i,0} \cup A^{i,1} \cup A^{i,2},$$

where for each $\ell \in \{0, 1, 2\}$, $A^{i,\ell}$ is a copy of A^i. For $a \in A^{i,\ell}$ and $b \in A^{i,\ell'}$, the utility function u^{i+1} is defined by

$$u^{i+1}(a, b) = \begin{cases} \ \ u^i(a, b) & \text{if } \ell = \ell', \\ -1 & \text{if } \ell' = \ell + 1, \\ \ \ 1 & \text{if } \ell' = \ell + 2, \end{cases}$$

where $\ell + c$ should be understood to mean $\ell + c \mod 3$. If M_i is the matrix representing Γ_i, $\mathbb{1}$ is the $|A^i| \times |A^i|$ matrix containing only ones, and $-\mathbb{1}$ is $(-1) \cdot \mathbb{1}$, then the game Γ_{i+1} is represented by the block matrix

$$M_{i+1} = \begin{pmatrix} M_i & -\mathbb{1} & \mathbb{1} \\ \mathbb{1} & M_i & -\mathbb{1} \\ -\mathbb{1} & \mathbb{1} & M_i \end{pmatrix}.$$

We will show that, for each $i \geq 1$, the symmetric D-sets of Γ_{i+1} can be characterized in terms of the symmetric D-sets of Γ_i. The following notation will be useful. For $X \subseteq A^{i+1}$ and $\ell \in \{0, 1, 2\}$, let $X_\ell = X \cap A^{i,\ell}$ denote the part of X that lies in block ℓ. We claim that for each $i \geq 1$,

$$\begin{aligned} &(X, X) \text{ is a symmetric } D\text{-set in } \Gamma_{i+1} \qquad \text{if and only if} \\ &(X_\ell, X_\ell) \text{ is a symmetric } D\text{-set in } \Gamma_i \text{ for all } \ell \in \{0, 1, 2\}. \end{aligned} \tag{3}$$

Before proving this equivalence, we make the following observation.

$$\text{If } (X, X) \text{ is a } D\text{-set in } \Gamma_{i+1}\text{, then } X_\ell \neq \emptyset \text{ for all } \ell \in \{0, 1, 2\}. \tag{4}$$

To see this, let $x \in X$ be an arbitrary action and choose ℓ such that $a \in X_\ell$. Consider the game where the actions of player 2 are restricted to X_ℓ. As $u^{i+1}(a,b) = 1$ for all $a \in X_{\ell+1}$ and $b \in X_\ell$, Lemma 2 implies that no action in $X_{\ell+1}$ is D-dominated by X_ℓ. Therefore, at least one of the actions in $X_{\ell+1}$ has to be contained in X, i.e., $X_{\ell+1} \neq \emptyset$. Repeating the argument, $X_{\ell+1} \neq \emptyset$ implies $X_{\ell+2} \neq \emptyset$, which proves (4).

We are now ready to prove the equivalence (3). For the direction from left to right, assume that (X, X) is a D-set in Γ_{i+1} and let $\ell \in \{0, 1, 2\}$. We need to show that (X_ℓ, X_ℓ) is a D-set in Γ_i. By (4), we know that $X_\ell \neq \emptyset$. To show that (X_ℓ, X_ℓ) is externally D-stable, consider some $a \in A^{i,\ell} \setminus X_\ell$. As (X, X) is externally D-stable in Γ_{i+1}, $X \; D(X) \; a$. However, the definition of u^{i+1} ensures that none of the actions in $X_{\ell+1} \cup X_{\ell+2}$ is essential for this dominance relation to hold. It thus follows that $X_\ell \; D(X) \; a$. Monotonicity of D finally yields $X_\ell \; D(X_\ell) \; a$. For minimality of (X_ℓ, X_ℓ), note that the existence of an externally D-stable set $(X', Y') \neq (X_\ell, X_\ell)$ in Γ_i with $X', Y' \subseteq X_\ell$ would contradict the minimality of (X, X) in Γ_{i+1}.

For the direction from right to left, (X, X) is externally D-stable in Γ_{i+1} because each (X_ℓ, X_ℓ) is externally D-stable in Γ_i. Furthermore (X, X) is minimal, as a proper subset of (X, X) that is externally D-stable in Γ_{i+1} would yield an externally D-stable subset of (X_ℓ, X_ℓ) for some $\ell \in \{0, 1, 2\}$, contradicting the minimality of (X_ℓ, X_ℓ) in Γ_i.

Let k_i denote the number of symmetric D-sets in Γ_i. It follows from (3) that $k_{i+1} = k_i^3$ for all $i \geq 1$. As $k_1 = k$, this yields $k_i = k^{3^{i-1}}$. As $|A^i| = 3^{i-1}|A|$, the number of D-sets in $\Gamma_i = (A^i, u^i)$ equals $k_i = k^{|A^i|/|A|}$. In particular, k_i is exponential in $|A^i|$. □

The construction used in the proof of Theorem 3 also works for weak dominance and mixed weak dominance. Moreover, it is easily seen that the games $(\Gamma_i)_{i \in \mathbb{N}}$ are confrontation games whenever Γ is a confrontation game.

5 Greedy Algorithms

In this section, we investigate the consequences of Theorem 1 on S-, B-, and S^*-sets.

Corollary 3. *All S-sets of a normal-form game can be computed in polynomial time.*

Proof. According to Theorem 1, it suffices to show that S satisfies MON, MAX, and COM. It can easily be verified that S satisfies MON and MAX. Furthermore, S satisfies COM because $x_i \; S(X_{-i}) \; a_i$ can be checked efficiently by simply comparing $u_i(x_i, x_{-i})$ and $u_i(a_i, x_{-i})$ for each $x_{-i} \in X_{-i}$. □

The same is true for Börgers dominance.

Corollary 4. *All B-sets of a normal-form game can be computed in polynomial time.*

Proof. According to Theorem 1, it suffices to show that B satisfies MON, MAX, and COM. As was the case for S, it can easily be checked that B satisfies MON and MAX.

Algorithm 4 Checking Börgers dominance

```
procedure Boergers_dom(Γ, (X_i)_{i∈N}, a_i)
  Y_{-i} ← X_{-i}
  repeat
    if X_i W(Y_{-i}) a_i then
      Choose x_i ∈ X_i such that x_i W(Y_{-i}) a_i
      C(x_i) ← {y_{-i} ∈ Y_{-i} : u_i(x_i, y_{-i}) > u_i(a_i, y_{-i})}
      Y_{-i} ← Y_{-i} \ C(x_i)
    else
      return "no"
    end if
  until Y_{-i} = ∅
  return "yes"
```

It remains to be shown that B satisfies COM. Consider the following procedure, formalized in Algorithm 4, which checks whether $X_i\ B(X_{-i})\ a_i$ holds. First check whether X_i weakly dominates a_i. If no, then X_i does not Börgers-dominate a_i either. If yes, we can find an action $x_i \in X_i$ with $u_i(x_i, x_{-i}) \geq u_i(a_i, x_{-i})$ for all $x_{-i} \in X_{-i}$. Define $C(x_i)$ as the set of all tuples $x_{-i} \in X_{-i}$ for which the latter inequality is strict. $C(x_i)$ is nonempty by definition of W. It follows that $x_i\ W(Y_{-i})\ a_i$ for all Y_{-i} with $Y_{-i} \cap C(x_i) \neq \emptyset$. We can therefore restrict attention to subsets of $Y_{-i} \setminus C(x_i)$ and "recursively" check whether $X_i\ B(Y_{-i} \setminus C(x_i))\ a_i$. It is easily verified that this procedure correctly checks Börgers-dominance and runs in polynomial time. □

The requirements for the greedy algorithm are also met by mixed strict dominance.

Corollary 5. *All S^*-sets of a normal-form game can be computed in polynomial time.*

Proof. According to Theorem 1, it suffices to show that S^* satisfies MON, MAX, and COM. It is easily verified that S^* satisfies MON. Furthermore, S^* satisfies COM because $X_i\ S^*(X_{-i})\ a_i$ can be checked efficiently with the help of a linear program (see Proposition 1 by Conitzer and Sandholm, 2005).

We now show that S^* satisfies MAX. Without loss of generality, assume that $u_i(a) \geq 0$ for all $i \in N$ and $a \in \prod_{i=1}^n A_i$. The following geometric interpretation will be useful. For an action a_i of player $i \in N$, define $u_i(a_i, X_{-i}) = (u_i(a_i, x_{-i}))_{x_{-i} \in X_{-i}}$ as the vector of possible utilities for player i if he plays a_i and the other players play some $x_{-i} \in X_{-i}$. For a set $Y_i \subseteq A_i$ of actions of player i, denote by $u_i(Y_i, X_{-i}) = \cup_{y_i \in Y_i} u_i(y_i, X_{-i})$ the union of all such vectors, and write $m = |X_{-i}|$ for their dimension. For a set of vectors $V \subseteq \mathbb{R}^m_{\geq 0}$, define $L[V]$ to be the *lower contour set* of $conv(V)$, i.e.,

$$L[V] = \bigcup \{\, x \in \mathbb{R}^m_{\geq 0} :\ \exists v \in conv(V) \text{ with } v \geq x \,\},$$

where $v \geq x$ is to be read componentwise.

The underlying intuition is that each action whose vector of utilities lies in the *interior* of $L[V]$ is strictly dominated by some strategy in $\Delta(V)$. More formally, $X_i\ S^*(X_{-i})\ a_i$ if and only if $u_i(a_i, X_{-i}) \in int(L[u_i(X_i, X_{-i})])$.

Now suppose $A_i\ S^*(X_{-i})\ a_i$. We have to show that $\max(S^*(X_{-i}))\ S^*(X_{-i})\ a_i$, i.e., there exists $s_i \in \Delta(\max(S^*(X_{-i})))$ with $u_i(s_i, x_{-i}) > u_i(a_i, x_{-i})$ for all $x_{-i} \in X_{-i}$. Since $A_i\ S^*(X_{-i})\ a_i$, we know that there must be some $s_i \in \Delta(A_i)$ with $u_i(s_i, x_{-i}) > u_i(a_i, x_{-i})$ for all $x_{-i} \in X_{-i}$. It thus suffices to show that

$$L[u_i(\max(S^*(X_{-i})), X_{-i})] = L[u_i(A_i, X_{-i})],$$

such that an action is strictly dominated by some strategy in $\Delta(\max(S^*(X_{-i})))$ if and only if it is strictly dominated by some strategy in $\Delta(A_i)$.

The inclusion from left to right is trivial since $\max(S^*(X_{-i})) \subseteq A_i$. For the inclusion from right to left, recall that a convex and compact set in $\mathbb{R}^m$ is equal to the convex hull of the set of its extreme points. As both $L[u_i(A_i, X_{-i})]$ and $L[u_i(\max(S^*(X_{-i})), X_{-i})]$ are compact and convex, it remains to be shown that no point in $u_i(A_i \setminus \max(S^*(X_{-i})), X_{-i})$ is an extreme point of $L[u_i(A_i, X_{-i})]$. This follows from the fact that any such point is strictly dominated by some $a_i^* \in \Delta(A_i)$. Indeed, the definition of $\max(S^*(X_{-i}))$ ensures that for each $a_i \in A_i \setminus \max(S^*(X_{-i}))$, there exists $a_i^* \in \Delta(A_i)$ with $u_i(a_i^*, x_{-i}) > u_i(a_i, x_{-i})$ for all $x_{-i} \in X_{-i}$. □

6 Sophisticated Algorithms

In this section, we investigate the consequences of Theorem 2 and Theorem 3 on C_D-, C_M-, V-, and V^*-sets.

Let us first consider C_M and C_D, which are only defined in symmetric matrix games. For this class of games, it turns out that both structures yield unique minimal solutions. In fact, we can show the following more general result.

Proposition 4. *Let D be a symmetric dominance structure satisfying weak monotonicity. If $D \subseteq C_M$, then every symmetric matrix game has a unique D-set.*

It is unknown whether $D \subseteq C_M$ is *necessary* for the uniqueness of D-sets in symmetric matrix games. Various symmetric dominance structures that are finer than C_M have been shown to admit disjoint minimal solutions, sometimes involving rather elaborate combinatorial arguments. Examples include *unidirectional covering* (Brandt and Fischer, 2008) and *extending* (Brandt, 2011; Brandt et al., 2012).

In order to prove Proposition 4, we need the following two lemmata.[11]

Lemma 3. *Let $D \subseteq C_M$ be a symmetric dominance structure satisfying weak MON. Let furthermore $\Gamma = (A, u)$ be a symmetric matrix game and $X, Y \subseteq A$. If (X, X) and (Y, Y) are externally D-stable sets, then $(X \cap Y, X \cap Y)$ is externally D-stable.*

[11] These lemmata are adapted from Duggan and Le Breton (1996a), who proved the analogous statements for W-sets in confrontation games.

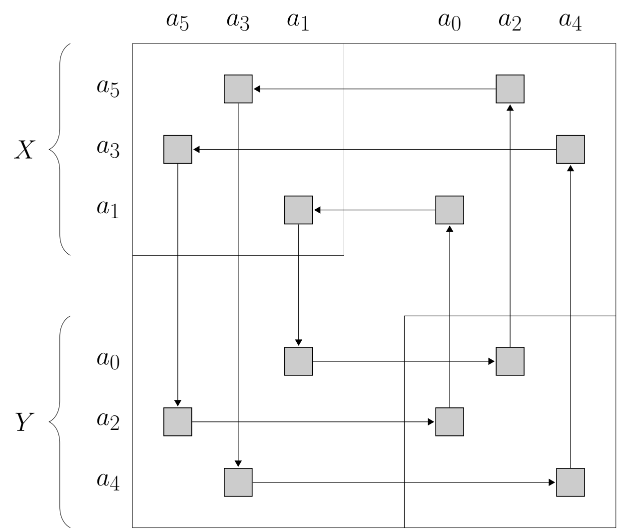

Figure 4: Construction in the proof of Lemma 3. An arrow from a profile (a_i, a_j) to another profile $(a_{i'}, a_{j'})$ indicates that $u(a_i, a_j) \geq u(a_{i'}, a_{j'})$.

Proof. We first show that $X \cap Y \neq \emptyset$. Assume for contradiction that $X \cap Y = \emptyset$ and let $a_0 \in Y$. As (X, X) is externally D-stable and $D \subseteq C_M$, there exists $a_1 \in X$ with $a_1\ C_M(X)\ a_0$. As (Y, Y) is externally D-stable and $D \subseteq C_M$, there exists $a_2 \in Y$ with $a_2\ C_M(X)\ a_1$. Repeatedly applying these arguments yields an infinite sequence $(a_0, a_1, a_2, ...)$ such that

- for all *even* $i \geq 0$, $a_i \in Y$ and $a_{i+1}\ C_M(X)\ a_i$, and
- for all *odd* $i \geq 1$, $a_i \in X$ and $a_{i+1}\ C_M(Y)\ a_i$.

Since A is finite, this sequence must contain repetitions. Without loss of generality, assume that $a_k = a_0$ for some even $k > 0$. We can therefore construct the following chain of inequalities (consult Figure 4 for an example with $k = 6$):

$$\begin{aligned} u(a_0, a_1) &\leq u(a_1, a_1) \leq u(a_1, a_0) \leq u(a_2, a_0) \leq u(a_2, a_{k-1}) \\ &\leq u(a_3, a_{k-1}) \leq u(a_3, a_{k-2}) \leq u(a_4, a_{k-2}) \leq \dots \\ &\leq u(a_0, a_2) \leq u(a_0, a_1) \end{aligned}$$

It follows that all utilities in this chain of inequalities are equal. Moreover, since Γ is a symmetric matrix game, we have that $u(a_1, a_1) = 0$ and hence that all utilities in this chain are zero. In particular, $u(a_1, a_0) = 0$, which contradicts the assumption that $a_1\ C_M(Y)\ a_0$. This proves that $X \cap Y \neq \emptyset$.

In order to show that $(X \cap Y, X \cap Y)$ is externally D-stable, take an arbitrary $a_0 \notin X \cap Y$. Without loss of generality, assume that $a_0 \notin X$. As (X, X) is externally D-stable and $D \subseteq C_M$, there exists $a_1 \in X$ with $a_1\ C_M(X)\ a_0$. If $a_1 \notin Y$, there exists $a_2 \in Y$ with $a_2\ C_M(X)\ a_1$. This construction finally yields some $a_k \in X \cap Y$, for otherwise we have a contradiction as in the first part of the proof. Repeated application of weak MON now yields $a_k \; D(X \cap Y) \; a_i$ for all $i < k$. In particular, $a_k \; D(X \cap Y) \; a_0$, as desired. □

The proof of the following lemma is similar to that of Lemma 3 and we omit it.

Lemma 4. *Under the assumptions of Lemma 3, if (X, Y) is externally D-stable, then $(X \cap Y, X \cap Y)$ is externally D-stable.*

We are now ready to prove Proposition 4.

Proof of Proposition 4. Lemma 4 implies that every D-set (X, Y) satisfies $X = Y$, as otherwise $(X \cap Y, X \cap Y)$ would be a smaller externally D-stable set. Similarly, Lemma 3 implies that there cannot exist two D-sets (X, X) and (Y, Y) with $X \neq Y$. □

Since C_D is coarser than C_M and both C_D and C_M are weakly monotonic, it immediately follows that C_D and C_M satisfy uniqueness, which is one of the main ingredients of the sophisticated algorithm.

Corollary 6. *Every symmetric matrix game has a unique C_D-set and a unique C_M-set.*

We now show that the other requirements for the sophisticated algorithm are satisfied as well.

Corollary 7. *The C_D-set and the C_M-set of a symmetric matrix game can be computed in polynomial time.*

Proof. According to Theorem 2, it is sufficient to show that both C_M and C_D satisfy weak MON, TRA, SING, UNI, SUB-COM, and COM.

It is easily verified that both C_M and C_D satisfy weak MON, TRA, SING, and COM. UNI was shown in Corollary 6. Finally, Dutta and Laslier (1999) have shown that the essential set $ES(\Gamma)$ is a (nonempty) subset of $S_{C_M}(\Gamma)$ and ES can be computed in polynomial time using linear programming (see Brandt and Fischer, 2008). This proves that C_M satisfies SUB-COM. The same is true for C_D because $S_{C_M}(\Gamma) \subseteq S_{C_D}(\Gamma)$. □

Let us now turn to very weak dominance. In contrast to S-sets, V-sets are not unique in matrix games. It is in fact easily seen that even a symmetric matrix game can have multiple very weak saddles: If all action profiles yield the same utility, then every single profile constitutes a V-set. Therefore, Theorem 3 implies that there are symmetric matrix games with an exponential number of V-sets. The following corollary is an immediate consequence.

Corollary 8. *Computing all V-sets of a game requires exponential time in the worst case, even for symmetric matrix games.*

	a_1	a_2	a_3	a_4	a_5	a_6
a_1	0	3	−3	−1	−1	2
a_2	−3	0	3	−1	2	−1
a_3	3	−3	0	2	−1	−1
a_4	1	1	−2	0	3	−3
a_5	1	−2	1	−3	0	3
a_6	−2	1	1	3	−3	0

Figure 5: Confrontation game with two symmetric V^*-sets: $(\{a_1, a_2, a_3\}, \{a_1, a_2, a_3\})$ and $(\{a_4, a_5, a_6\}, \{a_4, a_5, a_6\})$. In the former V^*-set, a_4 is dominated by $\frac{2}{3}a_1 + \frac{1}{3}a_3$, a_5 is dominated by $\frac{1}{3}a_2 + \frac{2}{3}a_3$, and a_6 is dominated by $\frac{1}{3}a_1 + \frac{2}{3}a_2$. In the latter V^*-set, a_1 is dominated by $\frac{2}{3}a_5 + \frac{1}{3}a_6$, a_2 is dominated by $\frac{2}{3}a_4 + \frac{1}{3}a_5$, and a_3 is dominated by $\frac{1}{3}a_4 + \frac{2}{3}a_6$.

Moreover, Brandt et al. (2011b) have shown that a number of natural problems like *finding* V-sets, checking whether a given action is contained in *some* V-set, or deciding whether there is a unique V-set are computationally intractable.

In confrontation games, the picture is different: Duggan and Le Breton (1996a) have shown that these games have a unique V-set, which moreover coincides with the (unique) C_M-set.[12] The following positive result now follows from Corollary 7.

Corollary 9. *The unique V-set of a confrontation game can be computed in polynomial time.*

V^*-sets, on the other hand, are not even unique in confrontation games, as witnessed in Figure 5. Applying Theorem 3 again, we get the following.

Corollary 10. *Computing all V^*-sets of a game requires exponential time in the worst case, even for confrontation games.*

In order to guarantee the uniqueness of V^*-sets, we thus have to restrict the class of games even further. Duggan and Le Breton (2001) have shown that tournament games have a unique V^*-set, which moreover coincides with the (unique) C_M-set.

Corollary 11. *The unique V^*-set of a tournament game can be computed in polynomial time.*

[12] To be precise, Duggan and Le Breton (1996a) have shown that the W-set of a confrontation is unique and coincides with the C_M-set. However, it can be shown that the proofs carry over for very weak dominance. As a consequence, V-sets, W-sets, and C_M-sets all coincide in confrontation games.

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