Some Remarks on Dodgson's Voting Rule

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Key words Social Choice Theory, Voting Rules

Sparked by a remarkable result due to Hemaspaandra et al. [1], the voting rule attributed to Charles Dodgson (aka Lewis Carroll) has become one of the most studied voting rules in computational social choice. However, the computer science literature often neglects that Dodgson's rule has some serious shortcomings as a choice procedure. This short note contains four examples revealing Dodgson's deficiencies.

Given a finite set of voters who each possesses a strict individual ranking over a finite set of alternatives, a *Condorcet winner* is an alternative that is preferred to every other alternative by some majority of the voters. A Condorcet winner is unique whenever it exists, which unfortunately is not always the case. For this reason, a variety of voting rules that select the Condorcet winner whenever one exists but differ in their treatment of the remaining cases has been proposed. One of these rules is commonly called *Dodgson's rule*. An alternative is called a *Dodgson winner* if it can be made a Condorcet winner by interchanging as few adjacent alternatives in the individual rankings as possible. For an example, consider the preference profile on the left hand side of Table 1. Each column represents a group of voters with the same preferences, given in descending order. The size of each group is specified in the first row. In this particular example, there are twelve voters and four alternatives (A, B, C, and D), none of which is a Condorcet winner. When three of the eight voters in the first four columns switch alternatives A and C in their rankings, A is preferred to every other alternative by some strict majority (i.e., at least seven voters) and thus becomes a Condorcet winner. Since no other alternative can be made a Condorcet winner by switching at most three pairs, alternative A is the unique Dodgson winner.

Fishburn attributed this rule to Charles Dodgson, even though he notes that "since Dodgson's function has serious defects, it may be a bit unfair to label [Dodgson's rule] with his name in view of the fact that the idea of counting inversions was cautiously proposed as a part of a more complex procedure" [2, p. 474]. Tideman adds that "Dodgson did not actually propose the rule that has been given his name. Rather, he used it implicitly to criticize other rules" [3, p. 194]. In fact, in the very same pamphlet in which Dodgson initiates the counting of inversions, he states "the conclusion I come to is that, in the case of persistent cyclical majorities, there ought to be 'no election"' [4]. Apparently, Dodgson was unaware of Condorcet's work [5], and his papers are therefore remarkable precursors of the theory of voting [6].

2	2	2	2	2	1	1
D	B	C	D	A	A	D
C	C	A	B	B	D	A
A	A	B	C	C	B	B
B	D	D	A	D	C	C

Winner: A (3 switches)

6	6	6	6	6	3	3
D	B	C	D	A	A	D
C	C	A	B	B	D	A
A	A	B	C	C	B	B
B	D	D	A	D	C	C

Winner: D (6 switches)

Table 1 Dodgson's rule fails homogeneity. Alternative A is the Dodgson winner in the left preference profile. Tripling all voters (as shown in the profile on the right) makes D the Dodgson winner.

In the remainder of this note, I will summarize four severe shortcomings of Dodgson's rule scattered in the literature on political science and social choice theory. A Dodgson winner may become a loser when cloning

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voters *(i)*, cloning alternatives *(iv)*, or letting the winner rise in individual rankings *(ii)*. On the other hand, a unique Dodgson winner may remain the unique winner when all individual rankings are reversed *(iii)*.

***(i)* Homogeneity** A voting rule is *homogeneous* if uniformly replicating all voters does not affect the election outcome. According to Smith "homogeneity seems an extremely natural requirement; if each voter suddenly splits into m voters, each of whom has the same preferences as the original, it would be hard to imagine how the 'collective preference' would change" [7, p. 1029]. Nurmi points out that "inhomogeneous systems pose a major challenge to representative arrangements since the outcomes ensuing from the representative body depend not only on the correspondence between the voters' and their representatives' views but also on the size of the representative body" [8, p. 12]

The example given in Table 1 shows that Dodgson's rule does not satisfy homogeneity. This was first observed by Fishburn [2] who consequently suggested a homogeneous variant of Dodgson's rule.[1] Unfortunately, even Fishburn's homogeneous version of Dodgson's rule fails to satisfy the three properties discussed in the following.

***(ii)* Monotonicity** A voting rule is *monotonic* if a winning alternative remains a winning alternative if it rises in one or more individual rankings (and everything else stays fixed).[2] According to Fishburn, "monotonicity is clearly a desirable condition for social selection procedures, and I am not aware that it has been seriously challenged in the literature. [...] Almost all reasonable-sounding election procedures that have been proposed in the literature and which do not involve determination of [the election outcome] by successive elimination are monotonic" [12, p. 119/131]. In earlier work, Fishburn states: "Since increased support for a candidate should not penalize that candidate, violations of monotonicity seem rather serious" [2, p. 477]. Fishburn [12] has shown that Dodgson's rule satisfies monotonicity only when there are no more than three alternatives. The preference profiles given in Table 2 exemplify Dodgson's failure of monotonicity for the case of four alternatives.

15	9	9	5	5
C	B	A	A	B
A	D	B	C	A
D	C	D	B	C
B	A	C	D	D

Winner: A (3 switches)

15	9	9	5	5
C	B	A	A	A
A	D	B	C	B
D	C	D	B	C
B	A	C	D	D

Winner: C (2 switches)

Table 2 Dodgson's rule fails monotonicity (example due to Fishburn [12]). Alternative A is the Dodgson winner in the left preference profile. If at least two of the voters in the last column switch A and B in their ranking, so that A becomes their most preferred alternative (as shown in the profile on the right), alternative C becomes the Dodgson winner.

***(iii)* Smith Set Principle** A *dominating set* is a non-empty set of alternatives such that every alternative in the set is preferred to every alternative outside the set by some majority of the voters. A voting rule satisfies the *Smith set principle* if it only chooses alternatives that are contained in all dominating sets. Good [13] has shown that there always exists a unique inclusion-minimal dominating set, which is now commonly known as the *Smith set* [7]. If there exists a Condorcet winner, it is the unique element in the Smith set. Fishburn asserts: "I personally regard Smith's Condorcet Principle as the most compelling extension of the basic [Condorcet] principle. [...] Moreover, I find it hard to imagine an argument against Smith's Condorcet Principle that would not also be an argument against Condorcet's Principle" [2, p. 479]

Dodgson's rule not only fails the Smith set principle but may even select the *Condorcet loser*, i.e., an alternative that loses every pairwise comparison (see Table 3).[3] Fishburn considers rules that fail to satisfy the Smith set principle to be "'dubious' extensions of the basic Condorcet criterion" [2, p. 480].

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[1] Another example (which as well as Fishburn's original example requires more than four alternatives) appears in Ratliff [9]. The four-alternative example given by Tideman [10] is unfortunately flawed.

[2] This condition should not be confused with the much stronger condition of *Maskin monotonicity* [11].

[3] Occasionally Dodgson's rule is interpreted so as to only apply to alternatives in the Smith set [14]. This can be traced back to a comment in Dodgson's original description [4].

10	8	7	4
D	*B*	*C*	*D*
A	*C*	*A*	*C*
B	*A*	*B*	*A*
C	*D*	*D*	*B*

Winner: *D* (3 switches)

10	8	7	4
C	*D*	*D*	*B*
B	*A*	*B*	*A*
A	*C*	*A*	*C*
D	*B*	*C*	*D*

Winner: *D* (no switches)

Table 3 Dodgson's rule selects the Condorcet loser in the left preference profile and therefore does not satisfy the Smith set principle (example due to Nurmi [8]). Since in this particular example, there is even a fixed majority of voters (those in the second and third column) that rank alternative *D* last, Dodgson's rule does not even satisfy a weaker version of the Smith set principle called the *mutual majority criterion*. As another consequence of this example, Dodgson's rule fails the *reversal symmetry* criterion, which states that a unique election winner has to become a loser when all individual rankings are reversed [15]. When reversing all individual rankings in the example given in Table 3, alternative *D* becomes the Condorcet winner and therefore remains the unique Dodgson winner.

(iv) **Independence of Clones** Two alternatives are clones of each other if they are ranked next to each other in every individual ranking, i.e., both alternatives perform identically in pairwise comparisons with any other alternative. A voting rule is *independent of clones* if a losing alternative cannot be made a winning alternative by introducing clones.[4] Table 4 shows that Dodgson's rule is not independent of clones.

5	4	3
A	*B*	*C*
B	*C*	*A*
C	*A*	*B*

Winner: *A* (2 switches)

5	4	3
A	*B*	*C*
B	*C*	C'
C	C'	*A*
C'	*A*	*B*

Winner: *B* (3 switches)

Table 4 Dodgson's rule fails independence of clones. Alternative *A* is the Dodgson winner in the left preference profile. After introducing alternative C', a clone of *C*, alternative *B* becomes the Dodgson winner.

In a comparative study of Condorcet-consistent voting rules, Fishburn concludes that "the least attractive function in my opinion is [Dodgson's rule] since it is very difficult to compute in some situations and fails to satisfy either monotonicity or Smith's Condorcet Principle" [2, p. 488]. In fact, as was shown by Bartholdi, III et al. [17] and Hemaspaandra et al. [1], computing a Dodgson winner is not only NP-hard, but not even in NP, unless the polynomial hierarchy collapses.

Since social choice theory is rife with impossibility results, it is worth mentioning that there exist numerous voting rules that easily satisfy *all* mentioned criteria and can furthermore be computed in polynomial time, e.g., ranked pairs [3], Schulze's rule [18], the essential set [19], or the minimal covering set [20].

Acknowledgements I thank Felix Fischer and Paul Harrenstein for helpful comments. This material is based upon work supported by the Deutsche Forschungsgemeinschaft under grant BR 2312/3-2.

References

1. E. Hemaspaandra, L. Hemaspaandra, and J. Rothe, Exact analysis of Dodgson elections: Lewis Carroll's 1876 voting system is complete for parallel access to NP, Journal of the ACM **44**(6), 806–825 (1997).
2. P. C. Fishburn, Condorcet social choice functions, SIAM Journal on Applied Mathematics **33**(3), 469–489 (1977).

[4] Please consult Tideman [3, 10] for a more stringent definition. Laffond et al. [16] proposed a more sophisticated version of this property called *composition-consistency*.

3. T. N. Tideman, Independence of clones as a criterion for voting rules, Social Choice and Welfare **4**(3), 185–206 (1987).

4. C. L. Dodgson, A Method for Taking Votes on More than Two Issues (Clarendon Press, 1876).

5. M. de Condorcet, Essai sur l'application de l'analyse à la probabilité des décisions rendues à la pluralité des voix (Imprimerie Royale, 1785), facsimile published in 1972 by Chelsea Publishing Company, New York.

6. D. Black, The Theory of Committees and Elections (Cambridge University Press, 1958).

7. J. H. Smith, Aggregation of preferences with variable electorate, Econometrica **41**(6), 1027–1041 (1973).

8. H. Nurmi, A comparison of some distance-based choice rules in ranking environments, Theory and Decision **57**, 5–24 (2004).

9. T. C. Ratliff, A comparison of Dodgson's method and Kemeny's rule, Social Choice and Welfare **18**, 79–89 (2001).

10. T. N. Tideman, Collective Decisions And Voting: The Potential for Public Choice (Ashgate, 2006).

11. H. Nurmi, Monotonicity and its cognates in the theory of choice, Public Choice **121**(1), 25–49 (2004).

12. P. C. Fishburn, Monotonicity paradoxes in the theory of elections, Discrete Applied Mathematics **4**(2), 119–134 (1982).

13. I. J. Good, A note on Condorcet sets, Public Choice **10**, 97–101 (1971).

14. C. Klamler, The Dodgson ranking and its relation to Kemeny's method and Slater's rule, Social Choice and Welfare **23**, 91–102 (2004).

15. D. G. Saari and S. Barney, Consequences of reversing preferences, Mathematical Intelligencer **25**, 17–31 (2003).

16. G. Laffond, J. Lainé, and J.-F. Laslier, Composition-consistent tournament solutions and social choice functions, Social Choice and Welfare **13**, 75–93 (1996).

17. J. Bartholdi, III, C. A. Tovey, and M. A. Trick, Voting schemes for which it can be difficult to tell who won the election, Social Choice and Welfare **6**(3), 157–165 (1989).

18. M. Schulze, A new monotonic and clone-independent single-winner election method, Voting Matters **17**, 9–19 (2003).

19. J.-F. Laslier, Aggregation of preferences with a variable set of alternatives, Social Choice and Welfare **17**, 269–282 (2000).

20. B. Dutta, Covering sets and a new Condorcet choice correspondence, Journal of Economic Theory **44**, 63–80 (1988).