Consistent Probabilistic Social Choice

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Two fundamental axioms in social choice theory are consistency with respect to a variable electorate and consistency with respect to components of similar alternatives. In the context of traditional non-probabilistic social choice, these axioms are incompatible with each other. We show that in the context of probabilistic social choice, these axioms uniquely characterize a function proposed by Fishburn (Rev. Econ. Stud., 51(4), 683–692, 1984). Fishburn’s function returns so-called maximal lotteries, i.e., lotteries that correspond to optimal mixed strategies of the underlying plurality game. Maximal lotteries are guaranteed to exist due to von Neumann’s Minimax Theorem, are almost always unique, and can be efficiently computed using linear programming.

1. Introduction

Many important properties in the theory of social choice concern the consistency of aggregation functions under varying parameters. What happens if two electorates are merged? How should an aggregation function deal with components of similar alternatives? How should choices from overlapping agendas be related to each other? These considerations have led to a number of consistency axioms that these functions should ideally satisfy.\(^1\) Unfortunately, social choice theory is rife with impossibility results which have revealed the incompatibility of many of these properties. Young and Levenglick (1978), for example, have pointed out that every social choice function that satisfies Condorcet-consistency violates consistency with respect to variable electorates. On the other hand, it follows from results by Young (1975) and Laslier (1996) that all Pareto-optimal social choice functions

\(^1\)Consistency conditions have found widespread acceptance well beyond social choice theory and have been applied successfully to characterize various concepts in mathematical economics such as proportional representation rules (Balinski and Young, 1978), Nash’s bargaining solution (Lensberg, 1988), the Shapley value (Hart and Mas-Colell, 1989), and Nash equilibrium (Peleg and Tijs, 1996). Young (1994) and Thomson (2014) provide excellent overviews and give further examples.
that are consistent with respect to variable electorates are inconsistent with respect to components of similar alternatives.

The main result of this paper is that, in the context of probabilistic social choice, two natural and well-known consistency conditions are not only compatible with each other, but uniquely characterize an appealing probabilistic social choice function. Probabilistic social choice functions yield lotteries over alternatives (rather than sets of alternatives) and were first formally studied by Zeckhauser (1969), Fishburn (1972), and Intriligator (1973). Perhaps one of the best known results in this context is Gibbard’s characterization of strategyproof probabilistic social choice functions (Gibbard, 1977). An important corollary of Gibbard’s characterization, attributed to Sonnenschein, is that random dictatorships are the only strategyproof and ex post efficient probabilistic social choice functions. In random dictatorships, one of the voters is picked at random and his most preferred alternative is implemented as the social choice. While Gibbard’s theorem might seem as an extension of classic negative results on strategyproof non-probabilistic social choice functions (Gibbard, 1973; Satterthwaite, 1975), it is in fact much more positive (see, also Barberà, 1979).

In contrast to deterministic dictatorships, the uniform random dictatorship (henceforth, random dictatorship), in which every voter is picked with the same probability, enjoys a high degree of fairness and is in fact used in many subdomains of social choice that are concerned with the fair assignment of objects to agents (see, e.g., Abdulkadiroğlu and Sönmez, 1998; Bogomolnaia and Moulin, 2004; Che and Kojima, 2010; Budish et al., 2013). Apart from issues of allocative fairness, probabilistic social choice has recently gained increasing interest in social choice (see, e.g., Ehlers et al., 2002; Bogomolnaia et al., 2005; Chatterji et al., 2014) and political science (see, e.g., Goodwin, 2005; Dowlen, 2009; Stone, 2011).  

Two important factors concerning the acceptability of social choice lotteries are the degree of randomness and risk aversion on behalf of the voters. It lies in the nature of preference aggregation that some situations call for randomization or other means of tie-breaking. We postpone a discussion of how much randomization is required to the end of this section, when concrete probabilistic social choice functions have been introduced, and first focus on risk aversion and its relationship to the frequency of preference aggregation. If the aggregation procedure is not frequently repeated, the law of large numbers does not apply and risk-averse voters might prefer a sure outcome to a lottery whose expectation they actually prefer to the sure outcome. Hence, probabilistic social choice seems particularly suitable for novel models of preference aggregation that have been made possible by technological advance and the emergence of the Internet, which easily allow for frequent aggregation intervals. In recurring elections with a fixed set of alternatives, voters need not resubmit their preferences in every aggregation interval; rather preferences can be centrally stored and only changed if desired. For example, probabilistic social choice functions could

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Interestingly, the use of lotteries for the selection of political decision makers goes back to the world’s first democracy in Athens where it was widely regarded as a principal characteristic of democracy (Headlam, 1933).
help a group of coworkers with the daily decision of where to have lunch. Other examples include Internet radio stations that decide, which song should be played next based on the preferences of the listeners, or movie streaming websites that recommend a ‘movie of the day’ based on the preferences of (likewise) customers. There is a steadily growing community of computer scientists, economists, and mathematicians who explore a new research area called computational social choice, in which existing and new preference aggregation rules are considered for collective optimization, planning, and recommendation (see, e.g., Brandt et al., 2013, 2015).

Another reason to study probabilistic social choice functions is that the resulting lotteries do not necessarily have to be interpreted as probability distributions. They can, for instance, also be seen as fractional allocations of divisible objects such as time shares or monetary budgets. The axioms considered in this paper are equally natural for these interpretations than they are for the probabilistic interpretation.

In this paper, we consider two consistency axioms, non-probabilistic versions of which have been widely studied in the literature. The first one, population-consistency, requires that, whenever two electorates agree on a lottery, this lottery should also be returned by the union of both electorates. The second axiom, composition-consistency, requires that the probability of an alternative should be unaffected by replacing another alternative with a component of alternatives that bear the same relationship to all alternatives outside of the component. Moreover, the relative probability of an alternative within the component should be directly proportional to the probability that the alternative receives when the component is considered in isolation. Despite their intuitive appeal, these axioms can be motivated by the desire to restrict the influence of a central planner on the outcome by partitioning the electorate into subelectorates or by introducing similar variants of alternatives, respectively.

We show that the only probabilistic social choice function satisfying these properties is the function that returns all maximal lotteries for a given preference profile. Maximal lotteries, which were proposed by Fishburn (1984), are equivalent to mixed maximin strategies of the symmetric zero-sum game given by the pairwise majority margins. Whenever there is an alternative that is preferred to any other alternative by some majority of voters (a so-called Condorcet winner), the lottery that assigns probability one to this alternative is the unique maximal lottery. This is in contrast to random dictatorship, which only returns a degenerate lottery if all voters unanimously favor the same alternative. At the same time, maximal lotteries satisfy consistency with respect to variable electorates which has been identified by Young (1974a), Nitzan and Rubinstein (1981), Saari (1990b), and others as the defining property of Borda’s scoring rule. As such, the characterization can be seen as one possible resolution of the historic dispute between the founding fathers of social choice theory, the Chevalier de Borda and the Marquis de Condorcet, which dates back to the 18th century.4

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4See Section 5 for a list of papers in which maximal lotteries were considered independently of Fishburn’s work.

4In this sense, our main theorem is akin to the characterization of Kemeny’s rule by Young and Levenglick
In comparison to random dictatorship, maximal lotteries involve a much lower degree of randomness. There is strong empirical and experimental evidence that most real-world preference profiles for political elections admit a Condorcet winner (see, e.g., Regenwetter et al., 2006; Laslier, 2010; Gehrlein and Lepelley, 2011). Under these circumstances, maximal lotteries are degenerate and assign all probability to a single alternative. Randomization is only required in the less likely case of cyclical majorities.

2. Preliminaries

Let \( U \) be an infinite universal set of alternatives. The set of feasible sets (or agendas) from which alternatives are to be chosen is the set of finite and non-empty subsets of \( U \), denoted by \( \mathcal{F}(U) \). The set of all linear (i.e., complete, transitive, and antisymmetric) preference relations over some set \( A \in \mathcal{F}(U) \) will be denoted by \( \mathcal{L}(A) \).

For some finite set \( X \), we denote by \( \Delta(X) \) the set of all probability distributions with rational values over \( X \). A (fractional) preference profile \( R \) for a given agenda \( A \) is an element of \( \Delta(\mathcal{L}(A)) \), which can be associated with the \((|A|!−1)\)-dimensional (rational) unit simplex. We interpret \( R(\succeq) \) as the fraction of voters with preference relation \( \succeq \in \mathcal{L}(A) \) and depict preference profiles by tables in which each column represents a preference relation \( \succeq \) with \( R(\succeq) > 0 \). The table below shows an example profile on three alternatives.\(^5\)

\[
\begin{array}{ccc}
1/2 & 1/3 & 1/6 \\
a & a & b \\
b & c & c \\
c & b & a \\
\end{array}
\]

(Example 1)

The set of all preference profiles for a fixed agenda \( A \) is denoted by \( \mathcal{R}|_A \) and \( \mathcal{R} \) is defined as the set of all preference profiles, i.e., \( \mathcal{R} = \bigcup_{A \in \mathcal{F}(U)} \mathcal{R}|_A \). For \( B \subseteq A \) and \( R \in \mathcal{R}|_A \), \( R|_B \) is the restriction of \( R \) to alternatives in \( B \), i.e., for all \( \succeq \in \mathcal{L}(B) \),

\[
(R|_B)(\succeq) = \sum_{\succeq' \in \mathcal{L}(A): \succeq \subseteq \succeq'} R(\succeq').
\]

For all \( x, y \in A \), \( R(x, y) = (R|_{\{x, y\}})((\{x, y\})) \) is called the fractional collective preference of \( x \) over \( y \). Informally, \( R(x, y) \) is the fraction of voters who prefer \( x \) to \( y \). In Example 1, \( R(a, b) = 5/6 \).

\(^5\)Note that our representation of preference profiles implicitly assumes that a PSCF is anonymous (i.e., all voters are treated identically) and homogeneous (i.e., duplication of the electorate does not affect the outcome). Similar models (sometimes even assuming a continuum of voters) have for example been considered by Young (1974b, 1975), Young and Levenglick (1978), Saari (1995), Dasgupta and Maskin (2008), Che and Kojima (2010), and Budish and Cantillion (2012).
Elements of $\Delta(A)$ are called \textit{lotteries} and will be written as convex combinations of alternatives. If $p$ is a lottery, $p_x$ is the probability that $p$ assigns to alternative $x$.

A \textit{probabilistic social choice function (PSCF)} $f$ is an (upper hemi-) continuous function that, for any agenda $A \in \mathcal{F}(U)$, maps a preference profile $R \in \mathcal{R}|_A$ to a convex subset of $\Delta(A)$.\footnote{Fishburn (1973, pp. 248–249) argues that the set of lotteries returned by a probabilistic social choice function should be convex because it would be unnatural if two lotteries are socially acceptable while a randomization between them is not.} A PSCF is thus a collection of mappings from high-dimensional simplexes to low-dimensional simplexes. Two further properties that we demand from any PSCF are \textit{unanimity} and \textit{decisiveness}. Unanimity states that in the case of one voter and two alternatives, the preferred alternative should be chosen with probability one.\footnote{This is the only condition we impose that actually interprets the preference relations. It is equivalent to \textit{ex post} efficiency for agendas of size two and is slightly weaker than faithfulness (Young, 1974a).}

Since we only consider fractional preference profiles, this amounts to for all $x, y \in U$ and $R \in \mathcal{R}|_{\{x,y\}}$,

$$f(R) = \{x\} \text{ whenever } R(x, y) = 1. \quad \text{(unanimity)}$$

Decisiveness requires that the set of preference profiles where $f$ is multi-valued is negligible in the sense that for all $A \in \mathcal{F}(U)$,

$$\{R \in \mathcal{R}|_A : |f(R)| = 1\} \text{ is dense in } \mathcal{R}|_A. \quad \text{(decisiveness)}$$

In other words, for every preference profile that yields multiple lotteries, there is an arbitrarily close preference profile that only yields a single lottery.

Probabilistic social choice functions considered in the literature usually satisfy these conditions and are therefore well-defined PSCFs. For example, consider \textit{random dictatorship (RD)}, in which one voter is picked uniformly at random and his most-preferred alternative is returned. Formally, $RD$ returns the unique lottery, which is determined by multiplying fractions of voters with their respective top choices, i.e., for all $A \in \mathcal{F}(U)$ and $R \in \mathcal{R}|_A$,

$$RD(R) = \left\{ \sum_{\succ \in \mathcal{L}(A)} R(\succ) \cdot \max(\succ) \right\} \quad \text{(random dictatorship)}$$

where $\max(\succ)(A)$ denotes the unique maximal element of $A$ with respect to $\succ$, i.e., $\max(\succ)(A) \succ x$ for all $x \in A$. For the preference profile $R$ given in Example 1,

$$RD(R) = \{5/6 a + 1/6 b\}.$$
3. Population-consistency and Composition-consistency

The consistency conditions we consider are generalizations of the corresponding conditions for SCFs, i.e., the axioms coincide with their non-probabilistic counterparts.

The first axiom relates choices from varying electorates to each other. More precisely, it requires that whenever a lottery is chosen simultaneously by two electorates, this lottery should also be chosen by the union of both electorates. For example, consider the two preference profiles

\[
\begin{align*}
R' & \quad \begin{array}{c}
1/2 \\
\end{array} a \\
1/2 \\
b \\
c \\
a \\
\end{array} \\
\begin{array}{c}
b \\
c \end{array} c \\
c \\
\begin{array}{c}
1/2 \\
\end{array} \\
\begin{array}{c}
a \\
b \end{array} \\
c \\
\begin{array}{c}
c \\
a \\
b \end{array} \\
\end{align*}
\]

\[
R'' \quad \begin{array}{c}
1/4 \\
\end{array} a \\
1/4 \\
b \\
\begin{array}{c}
c \end{array} c \\
\begin{array}{c}
1/2 \\
\end{array} \\
\begin{array}{c}
\begin{array}{c}
a \\
b \end{array} \end{array} \\
\begin{array}{c}
c \\
\end{array} \begin{array}{c}
a \end{array} \\
\begin{array}{c}
b \end{array} \begin{array}{c}
a \end{array} \\
\end{align*}
\]

(Example 2)

Population-consistency then demands that any lottery that is chosen in both \(R'\) and \(R''\) (say, \(1/2a + 1/2b\)) also has to be chosen when both preference profiles are merged. Formally, a PSCF satisfies population-consistency if for all \(A \in \mathcal{F}(U)\), \(R', R'' \in \mathcal{R}_A\), and any convex combination \(R\) of \(R'\) and \(R''\), i.e., \(R = \lambda R' + (1 - \lambda)R''\) for some \(\lambda \in [0,1]\),

\[
f(R') \cap f(R'') \subseteq f(R). \quad \text{(population-consistency)}
\]

Population-consistency is arguably one of the most natural axioms for variable electorates and is usually considered in a slightly stronger version, known as reinforcement or simply consistency, where the inclusion in the equation above is replaced with equality whenever the left-hand-side is non-empty (see also Remark 4). It was first considered independently by Smith (1973), Young (1974a), and Fine and Fine (1974a,b) and features prominently in the characterization of scoring rules by Smith (1973) and Young (1975). Population-consistency and its variants have found wide-spread acceptance in the social choice literature (see, e.g., Young, 1974b; Fishburn, 1978; Young and Levenglick, 1978; Saari, 1990a, 1995; Myerson, 1995; Congar and Merlin, 2012).

The second axiom prescribes how PSCFs should deal with decomposable preference profiles. For two agendas \(A', B \in \mathcal{F}(U)\) and \(A = A' \cup B\), \(B\) is a component in \(R \in \mathcal{R}_A\) if the alternatives in \(B\) are adjacent in all preference relations that appear in \(R\), i.e., for all \(a \in A \setminus B\) and \(b, b' \in B\), \(a \succeq b\) if and only if \(a \succeq b'\) for all \(\succeq \in \mathcal{L}(A)\) with \(R(\succeq) > 0\).

Intuitively, the alternatives in \(B\) can be seen as variants or clones of the same alternative because they have exactly the same relationship to all alternatives that are not in \(B\). For example, consider the following preference profile \(R\) in which \(B = \{b, b'\}\) forms a component.
The ‘essence’ of $R$ is captured by $R|_{A'}$, where $A' = \{a, b\}$ contains only one of the cloned alternatives. It seems reasonable to demand that a PSCF should assign the same probability to $a$ (say, $\frac{1}{2}$) independently of the number of clones of $b$ and the internal relationship between these clones. This condition will be called cloning-consistency and was first proposed by Tideman (1987) (see also Zavist and Tideman, 1989). Its origins can be traced back to earlier, more general, decision-theoretic work by Arrow and Hurwicz (1972) and Maskin (1979) where it is called deletion of repetitious states as well as early work on majoritarian SCFs by Moulin (1986).

When having a second look at Example 4, it may appear strange that cloning-consistency does not impose any restrictions on the probabilities that $f$ assigns to the clones. While clones behave completely identical with respect to uncloned alternatives, they are not indistinguishable from each other. It seems that the relationships between clones ($R|_B$) should be taken into account as well. For example, one would expect that $f$ assigns more probability to $b$ than to $b'$ because two thirds of the voters prefer $b$ to $b'$. An elegant and mathematically appealing way to formalize this intuition is to require that the probabilities of the clones $b$ and $b'$ are directly proportional to the probabilities that $f$ assigns to these alternatives when restricting attention to the component $\{b, b'\}$. This condition, known as composition-consistency, is due to Laffond et al. (1996) and was studied in detail for majoritarian SCFs (see, e.g., Laslier, 1996, 1997; Brandt, 2011; Brandt et al., 2011; Horan, 2013).\footnote{More generally, modular decompositions of discrete structures have found widespread applications in operations research and combinatorial optimization (see, e.g., Möhring, 1985).}

For a formal definition of composition-consistency, let $A', B \in \mathcal{F}(U)$ and $A = A' \cup B$ such that $A' \cap B = \{b\}$. For $p \in \Delta(A')$ and $q \in \Delta(B)$, we define $p \times_b q \in \Delta(A)$ by letting

$$
(p \times_b q)_x = \begin{cases} 
    p_x & \text{if } x \in A \setminus B, \\
    p_b q_x & \text{if } x \in B.
\end{cases}
$$

The operator $\times_b$ is extended to sets of lotteries $X \subseteq \Delta(A')$ and $Y \subseteq \Delta(B)$ by applying it to all pairs of lotteries in $X \times Y$, i.e., $X \times_b Y = \{p \times_b q \in \Delta(A) : p \in X \text{ and } q \in Y\}$.

Then, a PSCF $f$ satisfies composition-consistency if for all $R \in \mathcal{R}|_A$ such that $B$ is a component in $R$,

\[
  f(R|_{A'}) \times_b f(R|_B) = f(R). \quad \text{(composition-consistency)}
\]
In Example 4 above, \( \frac{1}{2}a + \frac{1}{2}b \in f(R_{A'}) \), \( \frac{2}{3}b + \frac{1}{3}b' \in f(R_{B}) \), and composition-consistency would imply that \( \frac{1}{2}a + \frac{1}{3}b + \frac{1}{6}b' \in f(R) \).

4. Non-Probabilistic Social Choice

In the context of non-probabilistic SCFs (i.e., PSCFs that only return the convex hull of degenerate lotteries), there is some friction between population-consistency and composition-consistency. In fact, the conflict between these notions can be traced back to the well-documented dispute between the pioneers of social choice theory, the Chevalier de Borda and the Marquis de Condorcet (see, e.g., Black, 1958; Young, 1988, 1995; McLean and Hewitt, 1994). Borda proposed a score-based voting rule—Borda’s rule—that can be axiomaticallly characterized using population-consistency (Young, 1974a). It then turned out that the entire class of scoring rules (which apart from Borda’s rule also includes plurality) can be characterized using population-consistency (Smith, 1973; Young, 1974b, 1975). Condorcet, on the other hand, advocated Condorcet-consistency, which requires that an SCF selects a Condorcet winner whenever one exists. As Condorcet already pointed out, Borda’s rule fails to be Condorcet-consistent. Worse, Young and Levenglick (1978) even showed that no Condorcet-consistent SCF satisfies population-consistency (the defining property of scoring rules).\(^9\) Laslier (1996), on the other hand, showed that no Pareto-optimal rank-based rule—a generalization of scoring rules—satisfies composition-consistency while this property is satisfied by various Condorcet-consistent SCFs (Laffond et al., 1996). One of the few SCFs that satisfies both properties is the rather indecisive Pareto rule (which returns all alternatives that are not Pareto-dominated). Since our definition of PSCFs already incorporates a certain degree of decisiveness, we obtain the following impossibility.

(The proofs of all theorems are deferred to the Appendix.)

**Theorem 1.** There is no non-probabilistic SCF that satisfies population-consistency and composition-consistency.

In light of this result it is perhaps surprising that, for probabilistic social choice, both axioms are not only mutually compatible but even uniquely characterize a PSCF.

5. Characterization of Maximal Lotteries

Maximal lotteries were first considered by Kreweras (1965) and independently proposed and studied in more detail by Fishburn (1984). Interestingly, maximal lotteries or variants thereof have been rediscovered again by economists (Laffond et al., 1993),\(^10\) mathematicians

\(^9\)Theorem 2 by Young and Levenglick (1978) actually uses the strong variant of population-consistency, but their proof also holds for population-consistency as defined in this paper.

\(^10\)Laffond et al. (1993) have explored the support of maximal lotteries in some detail. Laslier (2000) has characterized the support using monotonicity, Fishburn’s C2, regularity, inclusion-minimality, the strong superset property, and a weak version of cloning-consistency.
(Fisher and Ryan, 1995), political scientists (Felsenthal and Machover, 1992), and computer scientists (Rivest and Shen, 2010).

In order to define maximal lotteries, we need some notation. For $A \in \mathcal{F}(U)$, $R \in \mathcal{R}|_A$, and $x, y \in A$, the *majority margin* $g_R(x, y)$ is defined as the difference between the fraction of voters who prefer $x$ to $y$ and the fraction of voters who prefer $y$ to $x$, i.e.,

$$g_R(x, y) = R(x, y) - R(y, x).$$

Thus, $g_R(x, y) = -g_R(y, x)$ and $g_R(x, y) \in [-1, 1]$ for all $x, y \in A$. A *(weak)* Condorcet winner is an alternative $x$ that is maximal in $A$ according to $g_R$ in the sense that $g_R(x, y) \geq 0$ for all $y \in A$. If $g_R(x, y) > 0$ for all $y \in A \setminus \{x\}$, $x$ is called a *strict* Condorcet winner. A PSCF is Condorcet-consistent if $x \in f(R)$ whenever $x$ is a Condorcet winner in $R$.

It is well known from the Condorcet paradox that maximal elements may fail to exist. As shown by Kreweras (1965) and Fishburn (1984), this drawback can however be remedied by considering lotteries over alternatives. The function $g_R$ can be extended to a bilinear form on the set of lotteries by computing the *expected* majority margin. For $p, q \in \Delta(A)$, define

$$g_R(p, q) = \sum_{x, y \in A} p_x q_y g_R(x, y).$$

The set of maximal lotteries is then defined as the set of “probabilistic Condorcet winners.” Formally, for all $A \in \mathcal{F}(U)$ and $R \in \mathcal{R}|_A$,

$$ML(R) = \{p \in \Delta(A) : g_R(p, q) \geq 0 \text{ for all } q \in \Delta(A)\}.$$  

As an example, consider the preference profile given in Example 1 of Section 2. Alternative $a$ is a strict Condorcet winner and $ML(R) = \{a\}$. This is in contrast to $RD(R) = \{5/6 a + 1/6 b\}$, which puts positive probability on any first-ranked alternative no matter how small the corresponding fraction of voters.

The Minimax Theorem implies that $ML(R)$ is non-empty for all $R \in \mathcal{R}$ (von Neumann, 1928). In fact, $g_R$ can be interpreted as the payoff matrix of a symmetric zero-sum game—the so-called *plurality game*—and maximal lotteries as the mixed maximin strategies (or Nash equilibria) of this game. Since maximin strategies can be found by solving a linear program, maximal lotteries can be efficiently computed. Interestingly, $ML(R)$ is a singleton in many cases. In particular, this holds if there is an odd number of voters (Laffond et al., 1997; Le Breton, 2005). Moreover, we show that the set of preference profiles that yield a unique maximal lottery is open and dense, which implies that the set of profiles with multiple maximal lotteries is nowhere dense and thus negligible. As a consequence, $ML$ satisfies decisiveness on top of the other properties we demand from a PSCF (such as unanimity, continuity, and convex-valuedness) and therefore constitutes a well-defined PSCF.

In contrast to the non-probabilistic case where majority rule is known as the only reasonable neutral SCF on two alternatives (May, 1952; Dasgupta and Maskin, 2008), there is an infinite number of decisive PSCFs even when restricting attention to only two alternatives.
(see, e.g., Saunders, 2010). Within our framework of fractional preference profiles, a PSCF on two alternatives is a convex-valued continuous correspondence from the unit interval to itself. Unanimity fixes the function values at the endpoints of the unit interval. Decisiveness requires that the points where the function is multi-valued are isolated. Population-consistency implies that the function is monotonic and composition-consistency is trivially satisfied. The two natural extreme cases of functions that meet these requirements are a probabilistic version of simple majority rule and the proportional lottery (Figure 1). Interestingly, these two extreme points are taken by maximal lotteries and random dictatorship as for all $x, y \in U$, and $R \in \mathcal{R} \{x,y\}$,

$$ML(R) = \begin{cases} \{x\} & \text{if } R(x,y) > \frac{1}{2}, \\ \{y\} & \text{if } R(x,y) < \frac{1}{2}, \\ \Delta(\{x,y\}) & \text{otherwise,} \end{cases} \quad \text{and} \quad RD(R) = \{R(x,y) x + R(y,x) y\}.$$

![Figure 1: Maximal lotteries and random dictatorship on two-element agendas.](image)

Once we consider more than two alternatives, composition-consistency is applicable and, in conjunction with population-consistency, not only requires the PSCF in question to be identical to simple majority rule on two-element agendas, but even completely characterizes $ML$ for any number of alternatives.\(^{12}\)

**Theorem 2.** A PSCF $f$ satisfies population-consistency and composition-consistency if and only if $f = ML$.

The proof of Theorem 2 is rather involved but quite instructive as it rests on a number of lemmas that might be of independent interest (see Appendix C). The high-level structure is

\(^{11}\)Fishburn and Gehrlein (1977) compare these two-alternative PSCFs on the basis of expected voter satisfaction and find that the simple majority rule outperforms the proportional rule.

\(^{12}\)This also shows that $RD$ violates composition-consistency, which can be seen by Example 4 in Section 3. However, $RD$ does satisfy cloning-consistency (see Remark 5).
as follows. The fact that \( ML \) satisfies population-consistency and composition-consistency follows relatively easily from basic linear algebra.

For the converse direction, we first show that population-consistency and composition-consistency characterize \( ML \) on two-element agendas. Interestingly, the proof uses composition-consistency on agendas of size three. For agendas of more than two alternatives, we assume that \( f \) is a population-consistent and composition-consistent PSCF and then show that \( f \subseteq ML \) and \( ML \subseteq f \). The first statement takes up the bulk of the proof and is shown by assuming for contradiction that there is a preference profile for which \( f \) yields a lottery that is not maximal. We then identify a set of preference profiles with full dimension for which \( f \) returns the uniform lottery over a subset of alternatives and which has the uniform profile, i.e., the preference profile in which every preference relation is expressed by the same fraction of voters, in its interior. Along the way we show that \( f \) has to be Condorcet-consistent for all preference profiles that are close to the uniform profile. It follows that there has to be an \( \varepsilon \)-ball around some strict Condorcet profile (close to the uniform profile), for which \( f \) returns the uniform lottery over a subset as well as the lottery with probability one on the Condorcet winner. This contradicts decisiveness. For the inclusion of \( ML \) in \( f \), we take an arbitrary preference profile and an arbitrary vertex of the set of maximal lotteries for this profile and then construct a sequence of preference profiles that converges to the original profile and whose maximal lotteries converge to the specified maximal lottery. From \( f \subseteq ML \) and continuity, we obtain that \( f \) has to select this lottery in the original preference profile. Lastly, convexity implies that \( ML \subseteq f \).

6. Remarks

We conclude the paper with a number of remarks.

**Remark 1 (Independence of axioms).** The axioms used in Theorem 2 are independent from each other. \( RD \) satisfies population-consistency, but violates composition-consistency (see also Remark 5). When defining \( ML^3 \) via the third power of the majority margin \( g_R^3(x,y) = (g_R(x,y))^3 \), \( ML^3 \) satisfies composition-consistency, but violates population-consistency.\(^{13} \) Also the properties implicit in the definition of PSCFs are independent. The PSCF that returns all maximal lotteries for the profile in which all preference relations are reversed violates unanimity but satisfies decisiveness, population-consistency, and composition-consistency. When not requiring decisiveness, returning all \( ex \ post \) efficient lotteries is consistent with the remaining axioms.

**Remark 2 (Size of Universe).** The proof of Theorem 2 exploits the infinity of the universe. \( ML^3 \), as defined in the previous remark, satisfies population-consistency for up to three alternatives and composition-consistency. This implies that the statement of Theorem 2 requires a universe that contains at least four alternatives.

\(^{13}\)Such variants of \( ML \) were already considered by Fishburn (1984).
Remark 3 (Uniqueness). The set of profiles in which $ML$ is not single-valued is negligible in the sense specified in the definition of PSCFs. When extending the set of fractional profiles to the reals, it can also be shown that maximal lotteries are almost always unique by using an argument similar to that of Harsanyi (1973a).

Remark 4 (Strong population-consistency). $ML$ does not satisfy the stronger version of population-consistency in which the set inclusion is replaced with equality (see Section 3). This can be seen by observing that every lottery is maximal for the union of any two electorates whose preferences are completely opposed to each other. When there are at least three alternatives, it is possible to find two such preference profiles which yield the same unique maximal lottery and strong population-consistency is violated. However, whenever $ML$ is single-valued (which, as seen in Remark 3, is almost always the case), strong population-consistency is equivalent to population-consistency and therefore satisfied by $ML$.

Remark 5 (Cloning-consistency and Condorcet-consistency). Cloning-consistency, which was informally defined in Section 3, can be formally defined as follows. Let $A', B \in \mathcal{F}(U)$ and $A = A' \cup B$ such that $A' \cap B = \{b\}$, $R \in \mathcal{R}_{|A}$, and $B$ a component in $R$. Then, a PSCF $f$ satisfies cloning-consistency if $\{(p_x)_{x \in A \setminus B} : p \in f(R)\} = \{(p_x)_{x \in A \setminus B} : p \in f(R|_{A'})\}$. Cloning-consistency, which is weaker than composition-consistency, suffices for the proof of Theorem 2 when additionally requiring Condorcet-consistency. It is therefore possible to alternatively characterize $ML$ using population-consistency, cloning-consistency, and Condorcet-consistency. As above, the axioms are independent from each other. $RD$ satisfies all axioms except Condorcet-consistency. $ML^3$, as defined in Remark 1, satisfies all axioms except population-consistency. The PSCF that is identical to $ML^3$ for agendas of size three and otherwise identical to $ML$ satisfies all axioms except cloning-consistency.

Remark 6 (Agenda-consistency). $ML$ also satisfies agenda-consistency, which requires that the set of all lotteries that are chosen from two overlapping agendas should be identical to the set of lotteries that are chosen from the union of both agendas (and whose support is contained in both agendas). The inclusion from left to right is known as Sen’s $\gamma$ or expansion whereas the inclusion from right to left is Sen’s $\alpha$ or contraction (Sen, 1971). Numerous impossibility results, including Arrow’s well-known theorem, have revealed that agenda-consistency is prohibitive in non-probabilistic social choice when paired with minimal further assumptions such as non-dictatorship and Pareto-optimality (e.g., Sen, 1977, 1986; Campbell and Kelly, 2002).

Remark 7 (Domain extensions). In contrast to $RD$, which at least requires that every voter has a unique top choice, $ML$ does not require the asymmetry, completeness,

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14Strong populations-consistency is quite demanding. It is for example violated by rather basic functions such as the Pareto rule.

15Sen’s $\alpha$ actually goes back to Chernoff (1954) and Nash (1950), where it is called independence of irrelevant alternatives (not to be confused with Arrow’s IIA) (see Monjardet, 2008, for more details).

16Pattanaik and Peleg (1986) obtain a similar impossibility for probabilistic social choice using an interpretation of Sen’s $\alpha$ that is stronger than ours.
or even transitivity of individual preferences (and still satisfies population-consistency and composition-consistency in these more general domains).

**Remark 8 (Efficiency).** It has already been observed by Fishburn (1984) that ML is *ex post* efficient, i.e., Pareto-dominated alternatives always receive probability zero in all maximal lotteries. Aziz et al. (2013a) strengthened this statement by showing that ML even satisfies SD-efficiency (also known as ordinal efficiency) as well as the even stronger notion of PC-efficiency (see Aziz et al., 2015). Random serial dictatorship (the canonical generalization of RD to weak preferences) violates SD-efficiency (Bogomolnaia and Moulin, 2001; Bogomolnaia et al., 2005).

**Remark 9 (Strategyproofness).** In contrast to RD, ML is not strategyproof. However, both PSCFs violate group-strategyproofness and satisfy the (significantly) weaker notion of ST-group-strategyproofness (Aziz et al., 2014).

**Remark 10 (Random assignment).** Within the special domain of random assignment (or house allocation), maximal lotteries are known as *popular mixed matchings* (Kavitha et al., 2011) or *popular random assignments* (Aziz et al., 2013b).

**Acknowledgements**

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**References**


APPENDIX

A. Preliminaries

Recall from Section 2 that $U$ is an infinite set of alternatives. For convenience we will assume that $\mathbb{N} \subseteq U$. For $n \in \mathbb{N}$, the notation $[n]$ is defined as $[n] = \{1, \ldots, n\}$. For two sets $X$ and $Y$ with $|X| = |Y|$, let $\Pi(X,Y)$ denote the set of all bijections from $X$ to $Y$. Then $\Pi(X) = \Pi(X,X)$ is the set of all bijections on $X$. We will frequently work with profiles in which alternatives are renamed according to some permutation of alternatives. For all $A,B \in \mathcal{F}(U)$, $\succ \in \mathcal{L}(A)$, and $\pi \in \Pi(A,B)$, let $\pi(\succ) = \{(\pi(x),\pi(y)) : (x,y) \in \succ\} \in \mathcal{L}(B)$ and, for $R \in \mathcal{R}|A,B$, $\pi(R) \in \mathcal{R}|B$ such that $(\pi(R))(\succ) = R(\pi^{-1}(\succ))$. A well-known symmetry condition for PSCFs is neutrality, which requires that all alternatives are treated equally. Formally, a PSCF is neutral if

$$\pi(f(R)) = f(\pi(R))$$

for all $A,B \in \mathcal{F}(U)$, $R \in \mathcal{R}|A$, and $\pi \in \Pi(A,B)$. (neutrality)

It can be shown that composition-consistency implies neutrality by replacing all alternatives with components of size two. This fact will be frequently used for the proof of our main theorem.

Lemma 1. Every composition-consistent PSCF satisfies neutrality.

Proof. Let $f$ be a composition-consistent PSCF, $A,B \in \mathcal{F}(U)$, $R \in \mathcal{R}|A$, and $\pi \in \Pi(A,B)$. We have to show that $\pi(f(R)) = f(\pi(R))$. To this end, let $p \in f(R)$. First, choose $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$ such that $b_i = \pi(a_i)$ for all $i \in [n]$. Since $U$ is infinite, there is $C = \{c_1, \ldots, c_n\} \in \mathcal{F}(U)$ such that $C \cap A = \emptyset$ and $C \cap B = \emptyset$. Now, let $R' \in \mathcal{R}|A \cup C$ such that $R'|A = R$ and $\{a_i,c_i\}$ is a component in $R'$ for all $i \in [n]$. Thus, we have that $p \in f(R'|A)$. Then, it follows from composition-consistency that $p' \in f(R'|C)$ where $p'_c = p_{a_i}$ for all $i \in [n]$. Let $R'' \in \mathcal{R}|B \cup C$ such that $R''|C = R'|C$ and $\{b_i,c_i\}$ is a component in $R''$ for all $i \in [n]$. Hence, we have that $f(R''|C) = f(R'|C)$. Again, by composition-consistency, we have that $p'' \in f(R''|B)$ where $p''_{b_i} = p'_{c_i}$ for all $i \in [n]$. Notice that $p'' = \pi(p)$ and $B = \pi(A)$. Since $R''|B = \pi(R)$ by construction of $R''$, we have $p'' \in f(\pi(R))$. Hence, $\pi(f(R)) \subseteq f(\pi(R))$. The fact that $f(\pi(R)) \subseteq \pi(f(R))$ follows from application of the above to $\pi(R)$ and $\pi^{-1}$.

The following notation is required for our proofs. For some set $X$, $\text{uni}(X)$ denotes the uniform distribution over $X$. In particular, for $A \in \mathcal{F}(U)$, $\text{uni}(A)$ is the uniform lottery over $A$, i.e., $\text{uni}(A) = 1/|A| \sum_{x \in A} x$. The support of a lottery $p$ is the set of all alternatives to which $p$ assigns positive probability, i.e., $\text{supp}(p) = \{x \in A : p_x > 0\}$. The 1-norm of $x \in \mathbb{Q}^n$ is denoted by $\|x\|$, i.e., $\|x\| = \sum_{i=1}^n |x_i|$. For $X \subseteq \mathbb{Q}^n$, the convex hull $\text{conv}(X)$ is the set of all convex combinations of elements of $X$, i.e.,

$$\text{conv}(X) = \{\lambda_1 a^1 + \cdots + \lambda_k a^k : a^i \in X, \lambda_i \in \mathbb{Q}_{\geq 0}, \sum_{i=1}^k \lambda_i = 1\}.$$
X is convex if $X = \text{conv}(X)$. The affine hull $\text{aff}(X)$ is the set of all affine combinations of elements of $X$, i.e.,

$$\text{aff}(X) = \{\lambda_1 a^1 + \cdots + \lambda_k a^k : a^i \in X, \lambda_i \in \mathbb{Q}, \sum_{i=1}^{k} \lambda_i = 1\}.$$ 

$X$ is an affine subspace if $X = \text{aff}(X)$. We say that $a^1, \ldots, a^k \in \mathbb{Q}^n$ are affinely independent if, for all $\lambda_1, \ldots, \lambda_k \in \mathbb{Q}$, $\sum_{i=1}^{k} \lambda_i a^i = 0$ and $\sum_{i=1}^{k} \lambda_i a^i = 0$ implies $\lambda_i = 0$ for all $i \in [k]$. The dimension of an affine subspace $X$, $\dim(X)$, is $k - 1$, where $k$ is the maximal number of affinely independent vectors in $X$. The dimension of a set $X$ is the dimension of $\text{aff}(X)$.

The linear hull $\text{lin}(X)$ is the set of all linear combinations of elements of $X$, i.e.,

$$\text{lin}(X) = \{\lambda_1 a^1 + \cdots + \lambda_k a^k : a_i \in X, \lambda_i \in \mathbb{Q}\}.$$ 

$B_\varepsilon(x) = \{y \in \mathbb{Q}^n : \|x - y\| < \varepsilon\}$ denotes the $\varepsilon$-ball around $x \in \mathbb{Q}^n$. The interior of $X \subseteq \mathbb{Q}^n$ in $Y \subseteq \mathbb{Q}^n$ is $\text{int}_Y(X) = \{x \in X : B_\varepsilon(x) \cap Y \subseteq X \text{ for some } \varepsilon > 0\}$. The relative interior of $X$ is the interior of $X$ in $\text{aff}(X)$, i.e., $\text{relint}(X) = \text{int}_{\text{aff}(X)}(X)$. The closure of $X \subseteq \mathbb{Q}^n$ in $Y \subseteq \mathbb{Q}^n$, $\overline{X}_Y$, is the set of all limit points of sequences in $X$ which converge in $Y$, i.e., $\overline{X}_Y = \{\lim_{n \to \infty} a^n : (a^n)_{n \in \mathbb{N}} \text{ converges in } Y \text{ and } a^n \in X \text{ for all } n \in \mathbb{N}\}$. $X$ is dense in $Y$ if $\overline{X}_Y = Y$. Alternatively, $X$ is dense at $y \in \mathbb{Q}^n$ if for every $\varepsilon > 0$ there is $x \in X$ such that $\|x - y\| < \varepsilon$. $X$ is dense in $Y$ if $X$ is dense at $y$ for every $y \in Y$.

B. Non-Probabilistic Social Choice

**Theorem 1.** There is no non-probabilistic SCF that satisfies population-consistency and composition-consistency.

**Proof.** Assume for contradiction that $f$ is a non-probabilistic SCF that satisfies population-consistency and composition-consistency. Let $A = \{a, b, c\}$ and consider the profiles $R^1, \ldots, R^6$ as depicted below.

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We claim that $\Delta(\{a, b\}) \subseteq f(R^i)$ for all $i \in \{1, \ldots, 6\}$. It follows from neutrality that $f(R^1) = \Delta(A)$. Again, by neutrality, $f(R^2|_{\{a,b\}}) = \Delta(\{a,b\})$ and $f(R^2|_{\{a,c\}}) = \Delta(\{a,c\})$. Notice that $\{a,b\}$ is a component in $R^2$. Hence, by composition-consistency,

$$f(R^2) = f(R^2|_{\{a,c\}}) \times_a f(R^2|_{\{a,b\}}) = \Delta(\{a,c\}) \times_a \Delta(\{a,b\}) = \Delta(A).$$
A similar argument yields $f(R^i) = \Delta(A)$ for $i = 3, 4$. Unanimity implies that $f(R^5|_{\{b,c\}}) = \{b\}$ and by neutrality, we have $f(R^5|_{\{a,b\}}) = \Delta(\{a,b\})$. Furthermore, $\{b,c\}$ is a component in $R^5$. Hence, by neutrality and composition-consistency,

$$f(R^5) = f(R^5|_{\{a,b\}}) \times_b f(R^5|_{\{b,c\}}) = \Delta(\{a,b\}) \times_b \{b\} = \Delta(\{a,b\}).$$

Similarly, $f(R^6) = \Delta(\{a,b\})$.

Every profile $R^i$ is a vector in the 5-dimensional unit simplex $\mathcal{R}|_A$ in $\mathbb{Q}^6$. The corresponding vectors are depicted below.

$$
\begin{pmatrix}
R^1 \\
R^2 \\
R^3 \\
R^4 \\
R^5 \\
R^6
\end{pmatrix} =
\begin{pmatrix}
1/3 & 1/3 & 1/3 & 0 & 0 & 0 \\
1/2 & 0 & 0 & 0 & 1/2 & 0 \\
0 & 1/2 & 0 & 1/2 & 0 & 0 \\
0 & 0 & 1/2 & 0 & 0 & 1/2 \\
1/2 & 1/2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1/2 & 0 & 1/2
\end{pmatrix}
$$

It can be checked that $R^1, \ldots, R^6$ are affinely independent, i.e., $\dim(\langle R^1, \ldots, R^6 \rangle) = 5$. It follows from population-consistency that $\Delta(\{a,b\}) \subseteq f(R)$ for every $R \in \text{conv}(\langle R^1, \ldots, R^6 \rangle)$. Hence, $\{R \in \mathcal{R}|_A : |f(R)| = 1\}$ is not dense in $\mathcal{R}|_A$ at $1/6 R^1 + \cdots + 1/6 R^6$, which contradicts decisiveness of $f$. \qed

C. Probabilistic Social Choice

In this section we prove that every PSCF that satisfies population-consistency and composition-consistency has to return maximal lotteries. The high-level structure of the proof is described after Theorem 2 in Section 5.

C.1. $ML$ satisfies population-consistency and composition-consistency

We first show that $ML$ is a PSCF that satisfies population-consistency and composition-consistency. This statement is split into two lemmas.

Lemma 2. $ML$ is a PSCF.

Proof. $ML$ is continuous. Let $A \in \mathcal{J}(U)$, $R \in \mathcal{R}|_A$, and $p \in ML(R)$. Let $R^k \in \mathcal{R}|_A$ and $p^k \in ML(R^k)$ for all $k \in \mathbb{N}$ such that $(R^k)_{k \in \mathbb{N}}$ converges to $R$ and $(p^k)_{k \in \mathbb{N}}$ converges to $p$. We have to show that $p \in f(R)$. To this end, for every $k \in \mathbb{N}$, let $M$ and $M^k$ be the zero-sum games associated with $R$ and $R^k$, respectively. By assumption, $p^k$ is a maximin strategy in $M^k$, i.e., $(p^k)^T M^k q \geq 0$ for all $q \in \Delta(A)$. We have to show that $p$ is a maximin strategy in $M$. Assume otherwise, i.e., there is $q \in \Delta(A)$ such that $p^T M q < 0$. Then,

$$(p^k)^T M^k q = (p^k)^T (M^k - M) q + (p^k)^T M q = (p^k)^T (M^k - M) q + (p^k - p)^T M q + p^T M q < 0,$$

21
for some large enough \( k \), since \( \|M^k - M\| \to 0 \) and \( \|p^k - p\| \to 0 \) for \( k \to \infty \). This contradicts that \( p^k \) is a maximin strategy in \( M^k \).

The fact that \( f(R) \) is convex for every \( R \in \mathcal{R} \) follows from convexity of the set of maximin strategies for all (symmetric) zero-sum games.

\( ML \) obviously satisfies unanimity by definition.

\( ML \) satisfies decisiveness. Let \( A \in \mathcal{F}(U) \) and \( R \in \mathcal{R}_A \). It is easy to see that, for every \( \varepsilon > 0 \), we can find \( R' \in B_\varepsilon(R) \cap \mathcal{R}_A \) and \( k \in \mathbb{N} \) such that \( kR'(x,y) \) is an odd integer for all \( x,y \in A \). It follows from Laffond et al. (1997) that \( |f(R')| = 1 \). Hence, \( f \) is decisive. \( \Box \)

The set of symmetric zero-sum games with a unique maximin strategy inherits openness from the set of all zero-sum games with a unique maximin strategy (Bohnenblust et al., 1950, pp. 56–58). Hence, the set of profiles with unique maximal lottery is open and dense in the set of all profiles and the set of profiles with multiple maximal lotteries is nowhere dense.

**Lemma 3.** \( ML \) satisfies population-consistency and composition-consistency.

**Proof.** To simplify notation, for every \( v \in \mathbb{Q}^n \) and \( X \subseteq [n] \), we denote by \( v_X \) the restriction of \( v \) to indices in \( X \), i.e., \( v_X = (v_i)_{i \in X} \).

\( ML \) satisfies population-consistency. Let \( A \in \mathcal{F}(U) \), \( R', R'' \in \mathcal{R}_A \), and \( p \in ML(R') \cap ML(R'') \). Let \( M' \) and \( M'' \) be the zero-sum games associated with \( R' \) and \( R'' \), respectively. Then, by definition of \( ML \), \( p^TM'q \geq 0 \) and \( p^TM''q \geq 0 \) for all \( q \in \Delta(A) \). Hence, for all \( \lambda \in [0,1] \),

\[
p^T (\lambda M' + (1-\lambda)M'')q = \lambda p^TM'q + (1-\lambda)p^TM''q, \quad \geq 0 \geq 0
\]

for all \( q \in \Delta(A) \), which implies \( p \in ML(\lambda R' + (1-\lambda)R'') \).

\( ML \) satisfies composition-consistency. Let \( A', B \in \mathcal{F}(U) \) such that \( A' \cap B = \{b\} \), \( A = A' \cup B \), and \( R \in \mathcal{R}_A \) such that \( B \) is a component in \( R \). To simplify notation, let \( C = A \setminus B \). By \( M, M_{A'}, M_B, \) and \( M_C \) we denote the zero-sum games associated with \( R, R|_{A'}, R|_B, \) and \( R|_C \), respectively. Notice first that \( M \) and \( M_{A'} \) take the following form for some \( v \in \mathbb{Q}^{A \setminus B} \):

\[
M = \begin{pmatrix}
M_C & | & v & \ldots & v \\
-(-v^T) & | & - & \ldots & - \\
\vdots & & & & & \\
-(-v^T) & & & & & \\
\end{pmatrix}
\quad M_{A'} = \begin{pmatrix}
M_C & | & v \\
-(-v^T) & | & 0 \\
\end{pmatrix}.
\]

Let \( p \in ML(R|_{A'}) \times_b ML(R|_B) \). Then, there are \( p^{A'} \in ML(R|_{A'}) \) and \( p^B \in ML(R|_B) \).
such that \( p = p^A \times_b p^B \). Let \( q \in \Delta(A) \). Then,
\[
p^T M q = p^T C M_C q_C + \|p_B\|(-v)^T q_C + p^T C_L v \|q_B\| + p^T B M_B q_B \\
= (p^T C, \|p_B\|) M_{A'}(q_C, \|q_B\|)^T + p^T B M_B q_B \\
= (p^A)^T M_{A'}(q_C, \|q_B\|) + \|p_B\| (p^B)^T M_B q_B \geq 0,
\]
since \( p^A \in ML(R|A') \) and \( p^B \in ML(R|B) \), respectively. Hence, \( p \in ML(R) \).

For the other direction, let \( p \in ML(R) \). We have to show that there are \( p^A' \in ML(R|A') \) and \( p^B \in ML(R|B) \) such that \( p = p^A' \times_b p^B \). First, if \( \|p_B\| = 0 \) we may choose \( p^A' = p_A \) and an arbitrary \( p^B \in ML(R|B) \), i.e., \( (p^B)^T M_B \geq 0 \) for all \( q \in \Delta(B) \). Otherwise, let \( p^A' = (p_C, \|p_B\|) \) and \( p^B = p_B/\|p_B\| \).

Let \( q \in \Delta(A') \). Then,
\[
(p^A')^T M_{A'} q = p^T C M_C q_C + \|p_B\|(-v)^T q_C + p^T C_L v \|q_B\| \\
= p^T C M_C q_C + \|p_B\|(-v)^T q_C + p^T C_L v \|q_B\| + \|q_B\| \|p_B\| M_B = 0 \\
= p^T M(q_C, \|q_B\|) p_B^T \geq 0,
\]
since \( p \in ML(R) \) and hence \( p^A' \in ML(R|A') \).

Let \( q \in \Delta(B) \). If \( \|p_B\| = 0 \), it follows from the definition of \( p^B \) that \( (p^B)^T M_B q \geq 0 \). Otherwise,
\[
\|p_B\|^2 (p^B)^T M_B q = \|p_B\| p_B^T M_B q \\
= \|p_B\| p_B^T M_B q + p_C^T C_M C p_C + \|p_B\|(-v)^T p_C + \|p_B\| p_C^T v = 0 \\
= (p_C, p_B)^T M(p_C, \|p_B\|) q = p^T M(p_C, \|p_B\|) q \geq 0.
\]

Hence, \( p^B \in ML(R|B) \).

\[\square\]

**C.2. Binary Choice**

As the basis of our characterization of \( ML \) we consider agendas of size two. The following lemma states that, on two alternatives, whenever a composition-consistent PSCF returns a non-degenerate lottery, it has to return all lotteries. Interestingly, the proof uses composition-consistency on three-element agendas, even though the statement only concerns agendas of size two. In order to simplify notation, define
\[
p^A = \lambda a + (1 - \lambda) b.
\]
Lemma 4. Let $A = \{a,b\}$ and $f$ be a PSCF that satisfies composition-consistency. Then, for all $R \in \mathcal{R}_A$ and $\lambda \in (0,1)$, $p^\lambda \in f(R)$ implies $f(R) = \Delta(A)$.

Proof. Assume $p^\lambda \in f(R)$ for some $\lambda \in (0,1)$. Define $R' \in \mathcal{R}_{\{a,b,c\}}$ as depicted below.

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</table>

Notice that $R'|_A = R$. By definition of $R'$, it follows that $p^\lambda \in f(R'|_A)$. Neutrality implies that $\lambda a + (1 - \lambda)c \in f(R'|_{\{a,c\}})$. Since $A$ is also a component in $R'$, we have $\lambda p^\lambda + (1 - \lambda)c \in f(R'|_{\{a,c\}}) \times_a f(R'|_A) = f(R')$. Since $\{b,c\}$ is a component in $R'$, we have $p^\lambda \in f(R'|_A) = f(R)$.

Applying this argument repeatedly, we get $p^{\lambda 2^k} \in f(R)$ for all $k \in \mathbb{N}$. Since $\lambda 2^k \to 0$ for $k \to \infty$ and $f$ is continuous, we get $p^0 = b \in f(R)$. Similarly, it follows that $p^1 = a \in f(R)$. The fact that $f(R)$ is convex implies that $f(R) = \Delta(A)$. \hfill \Box

The characterization of ML for agendas of size two proceeds along the following lines. By unanimity and neutrality, we know which lotteries have to be returned by every composition-consistent PSCF for three particular profiles. Then population-consistency implies that every such PSCF has to return all maximal lotteries. Lastly, we use Lemma 4 to show that the function is not decisive if it additionally returns lotteries which are not maximal.

Lemma 5. Let $A = \{a,b\}$ and $f$ be a PSCF that satisfies population-consistency and composition-consistency. Then $f(R) = ML(R)$ for every $R \in \mathcal{R}_A$.

Proof. First, note that $R \in \mathcal{R}_A$ is fully determined by $R(a,b)$. Let $R \in \mathcal{R}_A$ be the profile such that $R(a,b) = 1/2$. Since, by definition, $f(R) \neq \emptyset$ there is $\lambda \in [0,1]$ such that $p^\lambda \in f(R)$. Neutrality implies that $p^{1-\lambda} \in f(R)$ and hence, by convexity of $f(R)$, $p^{1/2} = 1/2 (p^\lambda + p^{1-\lambda}) \in f(R)$. It follows from Lemma 4 that $f(R) = \Delta(A)$.

Now, let $R \in \mathcal{R}_A$ be a profile such that $R(a,b) = 1$. Unanimity implies that $a \in f(R)$. By population-consistency, we get $a \in f(R')$ for all $R' \in \mathcal{R}_A$ with $R'(a,b) \in [1/2,1]$. Similarly, $b \in f(R')$ for all $R' \in \mathcal{R}_A$ with $R'(b,a) \in [1/2,1]$. This already shows that $ML(R) \subseteq f(R)$ for every $R \in \mathcal{R}_A$.

Finally, let $R \in \mathcal{R}_A$ be a profile such that $R(a,b) = r > 1/2$. If $f(R) \neq \{a\}$, there is $\lambda \in [0,1]$ such that $p^\lambda \in f(R)$. Population-consistency implies that $p^\lambda \in f(R')$ for every $R' \in \mathcal{R}_A$ with $R'(a,b) \in [1/2,r]$. But then $\{R \in \mathcal{R}_A : R(a,b) \in [1/2,r]\} \subseteq \{R \in \mathcal{R}_A : |f(R)| > 1\}$ is not dense in $\mathcal{R}_A$, which contradicts decisiveness of $f$. An analogous argument shows that $f(R) = \{b\}$ whenever $R(a,b) < 1/2$. 

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In summary, we have that \( f(R) = \{a\} \) if \( R(a, b) \in (1/2, 1] \), \( f(R) = \{b\} \) if \( R(a, b) \in [0, 1/2) \), and \( f(R) = \Delta(A) \) if \( R(a, b) = 1/2 \). Thus, \( f = ML \) (as depicted in Figure 1(a)).

### C.3. \( f \subseteq ML \)

The first lemma in this section shows that every PSCF that satisfies population-consistency and composition-consistency is Condorcet-consistent for profiles that are close to the uniform profile \( \text{uni}(\mathcal{L}(A)) \), i.e., the profile in which every preference relation is assigned the same fraction of voters. We prove this statement by induction on the number of alternatives. Every profile close to the uniform profile that admits a Condorcet winner can be written as a convex combination of profiles that have a component. For these profiles we know from the induction hypothesis that the Condorcet winner has to be chosen.

**Lemma 6.** Let \( f \) be a PSCF that satisfies population-consistency and composition-consistency and \( A \in \mathcal{F}(U) \). Then, \( f \) satisfies Condorcet-consistency in a neighborhood of the uniform profile \( \text{uni}(\mathcal{L}(A)) \).

**Proof.** Let \( f \) be a PSCF that satisfies population-consistency and composition-consistency and \( A \in \mathcal{F}(U) \) with \( |A| = n \). Let furthermore \( R \in \mathcal{R}|_A \) be such that \( \|R - \text{uni}(\mathcal{L}(A))\| \leq \varepsilon_n = (4^n \pi^{n^2/2})^{-1} \) and \( a \in A \) is a Condorcet winner in \( R \). We show that \( a \in f(R) \) by induction over \( n \). An example that illustrates the idea for \( n = 3 \) is given at the end of the proof. For \( n = 2 \), the claim follows directly from Lemma 5.

For \( n > 2 \), fix \( b \in A \setminus \{a\} \). First, we introduce some notation. For \( \succ \in \mathcal{L}(A) \) we denote by \( \succ^{-1} \) the preference relation that reverses all pairwise comparisons, i.e., \( x \succ^{-1} y \) iff \( y \succ x \) for all \( x, y \in A \). By \( \succ^{b \rightarrow a} \) we denote the preference relation that is identical to \( \succ \) except that \( b \) is moved upwards or downwards (depending on whether \( a \succ b \) or \( b \succ a \)) until it is next to \( a \) in the preference relation. Formally, if \( a \succ b \), let \( X = \{x \in A : a \succ x \succ b\} \) and

\[
\succ^{b \rightarrow a} = \succ \setminus (X \times \{b\}) \cup (\{b\} \times X).
\]

If \( b \succ a \), let \( X = \{x \in A : b \succ x \succ a\} \) and

\[
\succ^{b \rightarrow a} = \succ \setminus (\{b\} \times X) \cup (X \times \{b\}).
\]

Notice that for every \( \succ' \in \mathcal{L}(A) \), \( \succ' = \succ^{b \rightarrow a} \) for at most \( n - 1 \) distinct preference relations \( \succ \). Furthermore, we say that \( \{a, b\} \) is a *component* in \( \succ \) if \( X = \emptyset \).

We first show that composition-consistency implies Condorcet-consistency for a particular type of profiles. For \( \succ \in \mathcal{L}(A) \), let \( S \in \mathcal{R}|_A \) such that \( S(\succ) + S(\succ)^{b \rightarrow a} = S(\succ^{-1}) = 1/2 \). We have that \( S(a, x) = 1/2 \) for all \( b \in A \setminus \{a\} \) and hence, \( a \) is a Condorcet winner in \( S \). We prove that \( a \in f(S) \) by induction over \( n \). For \( n = 2 \), this follows from Lemma 5. For \( n > 2 \), let \( x \in A \setminus \{b\} \) such that \( x \succ y \) for all \( y \in A \) or \( y \succ x \) for all \( y \in A \). This is always possible since \( n > 2 \). Notice that \( A \setminus \{x\} \) is a component in \( S \) and \( S(x, y) = 1/2 \) for all \( y \in A \setminus \{x\} \). If \( x = a \), it follows from composition-consistency and Lemma 5 that \( a \in f(S) \). If \( x \neq a \), it follows from the induction hypothesis that \( a \in f(S|_{A \setminus \{x\}}) \). Lemma 5 implies
that \( a \in f(S_{\langle a,x \rangle}) \) as \( S(a,x) = \frac{1}{2} \). Then, it follows from composition-consistency that \( a \in f(S_{\langle a,x \rangle}) \times_a f(S_{A\setminus\{x\}}) = f(S) \).

Now, for every \( \succ \in \mathcal{L}(A) \) such that \( \{a,b\} \) is not a component in \( R \) and \( 0 < R(\succ) \leq R(\succ^{-1}) \), let \( S^\succ \in \mathcal{R}_{A} \) such that

\[
S^\succ(\succ) + S^\succ(\succ^{b\rightarrow a}) = S^\succ(\succ^{-1}) = \frac{1}{2} \quad \text{and} \quad S^\succ(\succ)/S^\succ(\succ^{-1}) = R(\succ)/R(\succ^{-1}).
\]

From what we have shown before, it follows that \( a \in f(S^\succ) \) for all \( \succ \in \mathcal{L}(A) \).

The rest of the proof proceeds as follows. We show that \( R \) can be written as a convex combination of profiles of the type \( S^\succ \) and a profile \( R' \) in which \( \{a,b\} \) is a component and \( a \) is a Condorcet winner. Since \( R \) is close to the uniform profile, \( R(\succ') \) is almost identical for all preference relations \( \succ' \) and hence \( S^\succ(\succ') \) is close to 0 for all preference relations \( \succ' \) in which \( \{a,b\} \) is a component. As a consequence, \( R'(\succ') \) is almost identical for all preference relations \( \succ' \) in which \( \{a,b\} \) is a component and \( R'|_{A\setminus\{b\}} \) is close to the uniform profile for \( n-1 \) alternatives, i.e., \( \text{uni}(\mathcal{L}(A \setminus \{b\})) \). By the induction hypothesis, \( a \in f(R'|_{A\setminus\{b\}}) \). Since \( \{a,b\} \) is a component in \( R' \), it follows from composition-consistency that \( a \in f(R') \).

We define

\[
S = 2 \sum_{\succ} R(\succ^{-1})S^\succ,
\]

where the sum is taken over all \( \succ \) such that \( \{a,b\} \) is not a component in \( \succ \) and \( 0 < R(\succ) \leq R(\succ^{-1}) \) (in case \( R(\succ) = R(\succ^{-1}) \) we pick one of \( \succ \) and \( \succ^{-1} \) arbitrarily). Now, let \( R' \in \mathcal{R}_{A} \) such that

\[
R = (1 - \|S\|)R' + S.
\]

Note that, by definition of \( S \), \( R'(\succ) = 0 \) for all \( \succ \in \mathcal{L}(A) \) such that \( \{a,b\} \) is not a component in \( \succ \). Hence, \( \{a,b\} \) is a component in \( R' \). By the choice of \( R' \), we have that

\[
\|S\| = \sum_{\succ \in \mathcal{L}(A)} S(\succ) \leq \frac{n! - 2(n-1)!}{n!} + \varepsilon_n = 1 - \frac{2}{n} + \varepsilon_n.
\]

Using this fact, a simple calculation shows that

\[
R'(\succ) \leq \frac{R(\succ) - S(\succ)}{\frac{n}{2} - \varepsilon_n} \leq \frac{1}{\frac{n}{2} - \varepsilon_n} \leq \frac{1}{2(n-1)!} + \frac{\varepsilon_{n-1}}{4(n-1)!}.
\]

Since, for every preference relation \( \succ \) where \( \{a,b\} \) is a component, there is exactly one other preference relation identical to \( \succ \) except that \( a \) and \( b \) are swapped, we have that

\[
R'(\succ) \leq \frac{1}{(n-1)!} + \frac{\varepsilon_{n-1}}{2(n-1)!},
\]

for every \( \succ \in \mathcal{L}(A \setminus \{b\}) \). Now, consider the profile \( R'|_{A\setminus\{b\}} \). By the above calculation, we have that

\[
\|R'|_{A\setminus\{b\}} - \text{uni}(\mathcal{L}(A \setminus \{b\}))\| \leq \varepsilon_{n-1}.
\]
Since \( S^\succsim(a, x) = \frac{1}{2} \) for all \( x \in A \setminus \{a\} \) and \( \succ \in \mathcal{L}(A) \), we have that \( R'(a, x) \geq \frac{1}{2} \) for all \( x \in A \setminus \{a\} \). Thus, \( a \) is a Condorcet winner in \( R'|_{A \setminus \{b\}} \). From the induction hypothesis it follows that \( a \in f(R'|_{A \setminus \{a\}}) \). Using the fact that \( R'(a, b) \geq \frac{1}{2} \), Lemma 5 implies that \( a \in f(R'|_{\{a,b\}}) \). Finally, composition-consistency entails \( a \in f(R'|_{\{a,b\}}) \times_a f(R'|_{\{a,b\}}) = f(R') \).

In summary, \( a \) is a Condorcet winner in \( R'|_{A \setminus \{b\}} \). From the induction hypothesis it follows that \( a \in f(R'|_{A \setminus \{a\}}) \). Using the fact that \( R'(a, b) \geq \frac{1}{2} \), Lemma 5 implies that \( a \in f(R'|_{\{a,b\}}) \). Using the fact that \( R' \) is a convex combination of profiles of the type \( S^\succsim \) and \( R' \), it follows from population-consistency that \( a \in f(R) \).

We now give an example for \( A = \{a, b, c\} \) which illustrates the proof idea. Consider the following profile \( R \), where \( 0 \leq \varepsilon \leq \varepsilon_3 \).

\[
\begin{array}{cccccc}
(1+2\varepsilon)/6 & 1/6 & 1/6 & (1-\varepsilon)/6 & (1-\varepsilon)/6 & 1/6 \\
\hline
a & a & b & b & c & c \\
b & c & a & c & a & b \\
c & b & c & a & b & a \\
\end{array}
\]

\( R \)

Then, we have that \( \| R - \text{uni}(\mathcal{L}(A)) \| \leq \varepsilon_3 \). Now consider \( \succ \) with \( b \succ c \succ a \), which yields \( S^\succsim \) as depicted below.

\[
\begin{array}{ccc}
1/2 & (1-\varepsilon)/2 & \varepsilon/2 \\
\hline
a & b & c \\
c & c & b \\
b & a & a \\
\end{array}
\]

\( S^\succsim \)

Here, \( \{b, c\} \) is a component in \( S^\succsim \) and \( a \) is a Condorcet-winner in \( S^\succsim \). Hence, it follows from Lemma 5 that \( a \in f(S^\succsim) \). No other profiles of this type need to be considered, as \( \succ \) and \( \succ^{-1} \) are the only preference relations in which \( \{a, b\} \) is not a component. Then, we have \( R' \) and \( R'|_{\{a,c\}} \) as follows.

\[
\begin{array}{cccc}
(1+2\varepsilon)/4 & 1/4 & (1-\varepsilon)/4 & (1-\varepsilon)/4 \\
\hline
a & b & c & c \\
b & a & a & b \\
c & c & b & a \\
\end{array}
\]

\( R' \)

\[
\begin{array}{cc}
(1+\varepsilon)/2 & (1-\varepsilon)/2 \\
\hline
a & c \\
c & a \\
\end{array}
\]

\( R'|_{\{a,c\}} \)

\[
\begin{array}{cc}
(2+\varepsilon)/4 & (2-\varepsilon)/4 \\
\hline
a & b \\
b & a \\
\end{array}
\]

\( R'|_{\{a,b\}} \)

It follows from Lemma 5 that \( a \in f(R'|_{\{a,c\}}) \) and \( a \in f(R'|_{\{a,b\}}) \). Then, composition-consistency implies that \( a \in f(R') = f(R'|_{\{a,c\}}) \times_a f(R'|_{\{a,b\}}) \).

In summary, we have that \( R = \frac{2}{3} R' + \frac{1}{3} S^\succsim \), \( a \in f(R') \) and \( a \in f(S^\succsim) \). Thus, population-consistency implies that \( a \in f(R) \).  

\( \square \)
**Lemma 7.** Every PSCF that satisfies population-consistency and composition-consistency returns the uniform lottery over all Condorcet winners for all profiles in a neighborhood of the uniform profile \( \text{uni}(L(A)) \).

**Proof.** Let \( f \) be a PSCF that satisfies population-consistency and composition-consistency and \( A \in \mathcal{F}(U) \). Moreover, let \( R \in \mathcal{R}_A \) such that \( \| R - \text{uni}(L(A)) \| \) is small and \( A' \subseteq A \) the set of Condorcet winners in \( R \). We actually prove a stronger statement, namely that \( \Delta(A') \subseteq f(R) \). Every alternative in \( A' \) is a Condorcet winner in \( R \). It follows from Lemma 6 that \( x \in f(R) \) for every \( x \in A' \). Since \( f(R) \) is convex, \( \Delta(A') \subseteq f(R) \) follows.

For the remainder of the proof, we need to define two classes of profiles that are based on regularity conditions imposed on the corresponding majority margins. Let \( A \in \mathcal{F}(U) \) and \( A' \subseteq A \). A profile \( R \in \mathcal{R}_A \) is

- regular on \( A' \) if \( \sum_{y \in A} g_R(x, y) = 0 \) for all \( x \in A \), and
- strongly regular on \( A' \) if \( g_R(x, y) = 0 \) for all \( x, y \in A \).

In the following five lemmas we show that, for every \( A' \subseteq A \), every profile on \( A \) can be affinely decomposed into profiles of three different types: profiles that are strongly regular on \( A' \), certain regular profiles, and profiles that admit a strict Condorcet winner in \( A' \).

**Lemma 8.** Let \( A' \subseteq A \in \mathcal{F}(U) \). Then, \( \dim(\{ R \in \mathcal{R}_A : R \text{ is strongly regular on } A' \}) = |A|! - \binom{|A'|}{2} - 1 \).

**Proof.** For ease of notation, let \( X = \{ R \in \mathcal{R}_A : R \text{ is strongly regular on } A' \} \). We will characterize \( X \) using a set of linear constraints. By definition, \( X = \{ R \in \mathcal{R}_A : g_R(x, y) = 0 \text{ for all } x, y \in A' \} \). Recall that \( g_R(x, y) = \sum_{\gamma : x \succ y} R(\gamma) - \sum_{\gamma : y \succ x} R(\gamma) \). Since \( g_R(x, x) = 0 \) for all \( R \in \mathcal{R}_A \) and \( x \in A \), \( X \) can be characterized by \( \binom{|A'|}{2} \) homogeneous linear constraints in the \( (|A|! - 1) \)-dimensional space \( \mathcal{R}_A \), which implies that \( \dim(X) \geq |A|! - \binom{|A'|}{2} - 1 \). Equality holds but is not required for the following arguments. We therefore omit the proof.

Secondly, we determine the dimension of the space of all skew-symmetric \( n \times n \) matrices that correspond to regular profiles and vanish outside their upper left \( n' \times n' \) sub-matrix, i.e.,

\[ M_{n'} = \{ M \in \mathbb{Q}^{n \times n} : M = -M^T, \sum_{j=1}^{n} M_{ij} = 0 \text{ for all } i \in [n] \text{ and } M_{ij} = 0 \text{ if } \{i, j\} \not\subseteq [n'] \}. \]

\(^{17}\) Similar decompositions of majority margin matrices have been explored by Zwicker (1991) and Saari (1995).
In Lemma 10, we then proceed to show that every matrix of this type can be decomposed into matrices induced by a subset of regular profiles for which we know that every PSCF has to return the uniform lottery over the first \( n' \) alternatives (possibly among other lotteries).

**Lemma 9.** \( \dim(M_{n'}) = \left( \frac{n'}{2} \right) - (n' - 1) \).

**Proof.** First note that the space of all \( n \times n \) matrices has dimension \( n^2 \). We show that \( M_{n'} \) can be characterized by a set of \( (n^2 - (n')^2) + \left( \frac{n'}{2} \right) + n' \) homogenous linear constraints. Let \( M \in \mathbb{Q}^{n \times n} \) and observe that \( (n^2 - (n')^2) \) constraints are needed to ensure that \( M \) vanishes outside of \([n'] \times [n']\), \( \left( \frac{n'}{2} \right) \) + \( n' \) constraints are needed to ensure skew-symmetry of \( M \) on \([n'] \times [n']\), i.e., \( M_{ij} = -M_{ji} \) for all \( i, j \in [n'] \), \( j \geq i \), and \( (n' - 1) \) constraints are needed to ensure that the first \( n' \) rows (and hence the columns) of \( M' \) sum up to 0, i.e., \( \sum_{j=1}^{n} M_{ij} = 0 \) for all \( i \in [n' - 1] \). It follows from skew-symmetry and the latter \( n' - 1 \) constraints that the \( n' \)th row of \( M \) sums up to 0, since

\[
\sum_{j=1}^{n} M_{n'j} = \sum_{i,j=1}^{n} M_{ij} - \sum_{i=1}^{n-1} \sum_{j=1}^{n} M_{ij} = 0.
\]

Hence, \( \dim(M_{n'}) \geq (n')^2 - \left( \frac{n'^2}{2} \right) - (n' - 1) = \left( \frac{n'^2}{2} \right) - (n' - 1) \). The last \( n - n' \) rows of \( M \) trivially sum up to 0. Equality holds but is not required for the following arguments. We therefore omit the proof.

Let \( M_{n'}^H \) be the space of all matrices in \( M_{n'} \) induced by a Hamiltonian cycle on \([n']\), i.e.,

\[
M_{n'}^H = \left\{ M \in M_{n'} : M_{s_is_j} = \begin{cases} 1 & \text{if } j \equiv i + 1 \pmod{n'}, \\ -1 & \text{if } i \equiv j + 1 \pmod{n'}, \\ 0 & \text{otherwise,} \end{cases} \right\}
\]

with the convention that \( M_{2}^H = \{0\} \). We now show that the linear hull of \( M_{n'}^H \) is \( M_{n'} \).

**Lemma 10.** \( \lin(M_{n'}^H) = M_{n'} \).

**Proof.** The idea underlying the proof is as follows: every matrix \( M \in M_{n'} \) corresponds to a weighted directed graph with vertex set \([n]\) where the weight of the edge from \( i \) to \( j \) is \( M_{ij} \). If \( M \neq 0 \), there exists a cycle in the subgraph induced by \([n]\) along edges with positive weight of length at least three. For this cycle we can find two matrices \( M^1, M^2 \in M_{n'}^H \), i.e., matrices that correspond to Hamiltonian cycles in this subgraph, such that subtracting \( M^1 \) and \( M^2 \) from \( M \) yields a matrix with smaller norm than \( M \). Iterating this process yields a decomposition of \( M \) into matrices in \( M_{n'}^H \). Figure 2 illustrates this idea.

Let \( M \in M_{n'} \) and \( \kappa \in \mathbb{Q} \setminus \{0\} \) such that \( \kappa M \in \mathbb{N}^{n \times n} \). We prove that \( M \in \lin(M_{n'}^H) \). More specifically, we show, by induction over \( \kappa \|M\| \), that \( M = \sum_{i=1}^{\kappa \|M\|} \lambda_i(M^{2i-1} + M^{2i}) \) with \( \lambda_i \in \mathbb{Q} \) and \( M^{2i-1}, M^{2i} \in M_{n'}^H \) for all \( i \in [\kappa \|M\|] \). If \( \kappa \|M\| = 0 \) then \( M = 0 \). Hence, the induction hypothesis is trivial.
If $\kappa\|M\| \neq 0$, i.e., $M \neq 0$, we can find $k \in \{n\} \setminus \{2\}$ and $\{s_1, \ldots, s_{n'}\} = \{n\}$ such that $M_{s_is_{i+1}+1} > 0$ for all $i \in [k-1]$ and $M_{s_ks_1} > 0$. Note that $\{s_1, \ldots, s_k\}$ defines a cycle of length at least three in the graph that corresponds to $M$. We define $M^1, M^2 \in M_{n'}^H$ by letting

$$M^1_{s_is_j} = \begin{cases} 1 & \text{if } j \equiv i + 1 \pmod{n'}, \\ -1 & \text{if } i \equiv j + 1 \pmod{n'}, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$M^2_{s_is_j} = \begin{cases} 1 & \text{if } j \equiv i + 1 \pmod{n'} \text{ and } i \in [k-2] \text{ or } i = k-1 \text{ and } j = n', \\ -1 & \text{if } i \equiv j + 1 \pmod{n'} \text{ and } j \in [k-2] \text{ or } j = k-1 \text{ and } i = n', \\ 1 & \text{if } i \equiv j + 1 \pmod{n'} \text{ and } i \in [n'] \setminus [k] \text{ or } i = k \text{ and } j = 1, \\ -1 & \text{if } j \equiv i + 1 \pmod{n'} \text{ and } j \in [n'] \setminus [k] \text{ or } j = k \text{ and } i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\lambda = \frac{1}{2} \min\{M_{ij} : i, j \in [n] \text{ and } M_{ij} > 0\}$ and $M' = M - \lambda(M^1 + M^2)$. We claim that $\kappa\|M'\| = \kappa\|M\| - 1 - l$ with $l \in \{0, 1, 2\}$ (depending on whether $M_{s_{k-1}s_k}$ and $M_{s_ks_1}$ are greater than $\lambda$ or less than $-\lambda$). By construction, we have that $M' = M$ except for the following: $M'_{s_is_{i+1}} = M_{s_is_{i+1}} - 2\lambda$ for $i \in [k-2]$, $M'_{s_{k-1}s_k} = M_{s_{k-1}s_k} - \lambda$, $M'_{s_ks_1} = M_{s_ks_1} - \lambda$, and $M'_{s_1s_n} = M_{s_1s_n} - \lambda$ (and the corresponding entries due to skew-symmetry). Recall that $k \geq 3$. Hence,

$$\|M'\| = \|M\| - 2(k-2)\lambda - 2\lambda + (2 - 2l)\lambda = \|M\| - 2(k-2 + l)\lambda,$$

for some $l \in \{0, 1, 2\}$. This implies that $\kappa M' = \kappa M - 2\kappa(k-2+l)\lambda \in \mathbb{N}^{n \times n}$. Furthermore, we have $\kappa\|M'\| \leq \kappa\|M\| - 1$, since $k \geq 3$. From the induction hypothesis and our previous calculation we know that $M' = \sum_{i=2}^{k\|M\|} \lambda_i(M^{2i-1} + M^{2i})$ with $\lambda_i \in \mathbb{Q}$ and $M^i \in M_{n'}^H$ for all $i$. By construction of $M'$, we have that $M = \sum_{i=1}^{\kappa\|M\|} \lambda_i(M^{2i-1} + M^{2i})$ with $\lambda_1 = \lambda$.

Lemma 11 leverages Lemmas 7, 8, 9, and Lemma 10 to show two statements. First, it identifies the dimension of the space of all profiles that are regular on $A' \subseteq A$. Secondly, it proves that there is a full-dimensional subset of the space of all profiles that are regular on $A'$ for which every PSCF that satisfies population-consistency and composition-consistency returns the uniform lottery over $A'$.

**Lemma 11.** Let $f$ be a PSCF that satisfies population-consistency and composition-consistency and $A' \subseteq A \in \mathcal{F}(U)$. Then, there is $X \subseteq \mathcal{R}|_A$ of dimension $|A|! - |A'|$ such that, for every $R \in X$, $R$ is regular on $A'$ and $\text{uni}(A') \in f(R)$.

**Proof.** To simplify notation, we assume without loss of generality that $A = [n]$ and $A' = [n']$. For $M \in \mathbb{Q}^{n \times n}$ and $\pi \in \Pi(A)$, let $\pi(M)$ be the matrix that results from $M$ by permuting the rows and columns of $M$ according to $\pi$, i.e., $(\pi(M))_{ij} = M_{\pi(i)\pi(j)}$. 

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From Lemma 8 we know that we can find a set \( T = \{ R_1, \ldots, R^{n!(n')^2} \} \) of affinely independent profiles that are strongly regular (and thus regular) on \( A' \). Since \( T \) can be chosen such that every \( R \in T \) is close to uni(\( \mathcal{L}(A) \)), it follows from Lemma 7 that uni(\( A' \)) \( \mathcal{F}(R) \) for all \( R \in T \) and \( T \cup S \) is a set of affinely independent profiles.

For every \( \pi \in \Pi([n]) \) with \( \pi(i) \in [n'] \) for all \( i \in [n'] \), let \( S^\pi \) be defined as follows: \( S^\pi(\varnothing) = \frac{1}{2n'} \) if

\[
\begin{aligned}
\pi(k + 1) \succ \cdots \succ \pi(n') \succ \pi(1) \succ \cdots \succ \pi(k) \succ n' + 1 \succ \cdots \succ n \\
\text{or} \\
n \succ \cdots \succ n' + 1 \succ \pi(k + 1) \succ \cdots \succ \pi(n') \succ \pi(1) \succ \cdots \succ \pi(k),
\end{aligned}
\]

for some \( k \in [n'] \). Note that the fractional collective preferences of \( S^\pi \) are equal to \( M^\pi \) \( \mathbb{Q}^{n \times n} \) where

\[
(\pi(M^\pi))_{ij} = \begin{cases} 
\lambda & \text{if } j \equiv i + 1 \pmod{n'} \\
-\lambda & \text{if } i \equiv j + 1 \pmod{n'} \\
0 & \text{otherwise},
\end{cases}
\]

for some \( \lambda > 0 \). Moreover, by the choice of \( \pi \), \( A' \) and \( A \setminus A' \) are components in \( S^\pi \). By construction of \( S^\pi \), alternative 1 is a Condorcet winner in \( S^\pi|\{1, n'+1\} \). Hence, by Lemma 5, \( 1 \in f(S^\pi|\{1, n'+1\}) \). Moreover, \( \sigma^k(S^\pi) = S^\pi \) for all \( \sigma^k \in \Pi(A) \) such that there is \( k \in [n'] \) with \( \sigma^k(\pi(i)) = \pi(i+k) \) for all \( i \in A' \) and \( \sigma^k(i) = i \) for all \( i \in A \setminus A' \). Now, let \( p \in f(S^\pi|A') \).

Then, by neutrality and convexity of \( f(S^\pi|A') \), uni(\( A' \)) = \( \frac{1}{n'} \sum_{k=1}^{n'} \sigma^k(p) \in f(S^\pi|A') \). As \( 1 \in f(S^\pi|\{1, n'+1\}) \), it follows from composition-consistency that uni(\( A' \)) \( \mathcal{F}(p) \).
By construction, for every $M \in \mathcal{M}_H^R$, there is $\pi \in \Pi(A)$ such that $\lambda M = M^\pi$. Hence, $\mathcal{M}_H^R = \{M^\pi : \pi \in \Pi(A')\}$ spans $\mathcal{M}_H^R$, and, by Lemma 10, $\mathcal{M}_H^R$ also spans $\mathcal{M}_R$, i.e., $\text{lin}(\mathcal{M}_H^R) = \mathcal{M}_R$. By Lemma 9, $\dim(\mathcal{M}_H^R) = (n' - 1)$ and $\dim(S^\pi = \{S^\pi : \pi \in \Pi(A')\}) \geq (n' - 1)$.

Let $\{M^1, \ldots, M^{(n'-1)}\}$ be a basis of $\mathcal{M}_R$ and $S = \{S^1, \ldots, S^{(n'-1)}\}$ be the corresponding profiles, i.e., $M^i$ represents the fractional collective preferences of $S^i$. We claim that $\mathcal{T} \cup S$ is a set of affinely independent profiles. Let $R_1, \ldots, R_l \in \mathcal{T}$ and $S_1, \ldots, S_m \in S$ be pairwise disjoint. Assume for contradiction that $\sum_i \lambda_i R_i + \sum_i \mu_i S_i = 0$ with $\lambda_i, \mu_i \in \mathbb{Q}$ such that $\sum_i \lambda_i + \sum_i \mu_i = 0$. This implies that $\sum_i \mu_i M^i = 0$, which in turn implies $\mu_i = 0$ for all $i \in [m]$, since the $M_i$’s are linearly independent. Hence, $\sum_i \lambda_i R_i = 0$ and $\sum_i \lambda_i = 0$.

Finally, for every PSCF that returns some lottery which is not maximal, there is a set of profiles with a strict Condorcet winner for which the function returns a uniform lottery over a subset of alternatives. Furthermore, none of these profiles is regular and hence, this set of profiles is affinely independent from the profiles obtained in Lemma 11.

**Lemma 12.** Let $f$ be a PSCF that satisfies population-consistency and composition-consistency. If $f \not\subseteq \mathcal{ML}$, there are $A' \subseteq A \in \mathcal{F}(U)$ and $Y \subseteq \mathcal{R}_A$ of dimension $|A'| - 1$ such that $\text{uni}(A') \in f(R)$ for every $R \in Y$ and no $R \in \text{aff}(Y)$ is regular on $A'$.

**Proof.** If $f \not\subseteq \mathcal{ML}$, there are $A \in \mathcal{F}(U)$, $R \in \mathcal{R}_A$, and $p \in f(R)$ such that $p \notin \mathcal{ML}(R)$. Since $p$ is not a maximal lottery, by definition, there is $x \in A$ such that $g(x, p) > 0$. We first use composition-consistency to “blow up” alternatives such the resulting lottery is uniform. Let $\kappa$ be the greatest common divisor of $\{p_y : y \in A\}$, i.e., $\kappa = \max\{s \in \mathbb{Q} : \exists y \in A : p_y/s \in \mathbb{N}\}$. For every $y \in A$, let $A_y \in \mathcal{F}(U)$ such that $|A_y| = \max\{1, p_y/\kappa\}$, $A_y \cap A = \{y\}$, and all $A_y$ are pairwise disjoint. Moreover, let $A^u = \bigcup_{y \in A} A_y$. Now, choose $R^u \in \mathcal{R}_A$ such that $R^u|_A = R$, $A_y$ is a component in $R^u$ for every $y \in A$, and $R^u|_{A_y} = \text{uni}(\mathcal{L}(A_y))$ for every $y \in A_y$. Hence, $\text{uni}(A_y) \in f(R^u|_{A_y})$ for every $y \in A$, as $f$ is neutral and $f(R^u|_{A_y})$ is convex. To simplify notation, let $A^p = \bigcup_{y \in \text{supp}(p)} A_y$. By composition-consistency, it follows that $p' = \text{uni}(A^p) \in f(R^p)$. Observe that

$$g_{R^u}(x, p') = \sum_{y \in \text{supp}(p) \setminus \{x\}} \frac{|A_y|}{|A^u|} g_{R^u}(x, y) = \sum_{y \in A \setminus \{x\}} p_y g_{R^u}(x, y) > 0.$$

We now construct a profile $R' \in \mathcal{R}_A$ such that $x$ is a strict Condorcet winner in $R'$ and $\text{uni}(A^p) \in f(R')$. To this end, let $R'' \in \mathcal{R}_A$ be the uniform mixture of all profiles that arise from $R^u$ by permuting all alternatives in $A^p \setminus \{x\}$, i.e.,

$$R' = \frac{1}{|A^p \setminus \{x\}|!} \sum_{\pi \in \Pi(A^p) : \pi(y) = x \text{ for all } y \in A^p \setminus \{x\}} \pi(R^u).$$
Then, \( g_R(x, y) = g_R(x, z) > 0 \) for all \( y, z \in A^p \setminus \{x\} \).

Let \( R^{\text{uni}} = \text{uni}(\mathcal{L}(A^u)) \) and define, for \( \lambda \in [0, 1],
\[
R^{\lambda} = \lambda R'' + (1 - \lambda) R^{\text{uni}}.
\]

It follows from Lemma 6 that \( y \in f(R^{\text{uni}}) \) for all \( y \in A^u \). Convexity of \( f(R^{\text{uni}}) \) implies that \( f(R^{\text{uni}}) = \Delta(A^u) \). Hence, by population-consistency, \( p' \in f(R^{\lambda}) \) for all \( \lambda \in [0, 1] \).

Now, let \( S \in \mathcal{R}_{A^u} \) such that \( g_S(y, z) = 0 \) for all \( y, z \in A^p \cup \{x\} \) and \( g_S(y, z) = 1 \) for all \( y \in A^p \cup \{x\}, z \in A^u \setminus (A^p \cup \{x\}) \). For \( \lambda \in [0, 1], \)
\[
S^{\lambda} = \lambda S + (1 - \lambda) R^{\text{uni}}.
\]

Note that every \( y \in A^P \cup \{x\} \) is a Condorcet winner in \( S^{\lambda} \). It follows from population-consistency and Lemma 6 that, for small \( \lambda > 0 \), \( y \in f(S^{\lambda}) \) for all \( y \in A^P \cup \{x\} \) and, by convexity of \( \Delta(A^P \cup \{x\}) \subseteq f(S^{\lambda}) \). In particular, \( p' \in f(S^{\lambda}) \) for small \( \lambda > 0 \).

Finally, let
\[
R' = \frac{1}{3} R^{\lambda} + \frac{2}{3} S^{\lambda},
\]
for some small \( \lambda > 0 \). Population-consistency implies that \( p' \in f(R') \). Moreover, \( g_R(x, y) > 0 \) for all \( y \in A^u \setminus \{x\} \), i.e., \( x \) is a strict Condorcet winner in \( R' \), and hence, it follows from Lemma 6 that \( x \in f(R') \).

If \( p_x > 0 \) then, by construction, \( p' = \text{uni}(A^P \cup \{x\}) \in f(R') \). If \( p_x = 0 \) then \( p' = \text{uni}(A^P) \in f(R') \). In this case it follows from convexity of \( f(R') \) that \( \text{uni}(A^P \cup \{x\}) = \frac{1}{(|A^P| + 1)} x + \frac{|A^P|}{(|A^P| + 1)} \text{uni}(A^P \setminus \{x\}) \in f(R') \).

Hence, in either case, we get a profile \( R' \) such that \( \text{uni}(A^P \cup \{x\}) \in f(R') \) and the corresponding matrix \( M^x \) of fractional collective preferences restricted to \( A^P \cup \{x\} \) takes the form

\[
M^x = \lambda \cdot 
\begin{pmatrix}
0 & \ldots & 0 & -1 & 0 & \ldots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \ldots & 0 & -1 & 0 & \ldots & 0 \\
1 & \ldots & 1 & 0 & 1 & \ldots & 1 \\
0 & \ldots & 0 & -1 & 0 & \ldots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \ldots & 0 & -1 & 0 & \ldots & 0
\end{pmatrix}
\]

for some \( \lambda > 0 \) where all entries except the \( x \)th row and column are zero. Let \( n' = |A^P \cup \{x\}| \).

For every \( y \in A^p \), let \( M^y \in \mathbb{Q}^{n' \times n'} \) be such that \( M^y_{yz} = -M^y_{zy} = \lambda \) for all \( z \neq y \) and 0 otherwise. Let \( \pi^y \in \Pi(A^u) \) be such that \( \pi^y(x) = y \) and \( \pi^y(z) = z \) for all \( z \in A^u \setminus (A^p \cup \{x\}) \) and \( R^y = \pi^y(R') \). Then, for every \( y \in A^P \cup \{x\}, R^y \) induces the fractional collective preferences \( M^y \) and, by neutrality, \( \text{uni}(A^P \cup \{x\}) \in f(R^y) \).

Now, let \( Y = \{R^y: y \in A^P \setminus \{x\}\} \). Furthermore, \( \dim(Y) = |A^P \cup \{x\}| - 1 \) since \( Y \) is a set of linearly independent vectors. Assume there is \( R \in \mathcal{R}_{A^u} \) that is regular on \( A^P \cup \{x\} \)

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and \( R \in \text{aff}(Y) \), i.e.,
\[
R = \sum_{y \in \mathcal{A}^p \setminus \{x\}} \lambda_y R^y,
\]
and \( \sum_{y \in \mathcal{A}^p \setminus \{x\}} \lambda_y = 1 \). Let \( N \in \mathbb{Q}^{n' \times n'} \) denote the fractional collective preferences of \( R|_{\mathcal{A}^p \cup \{x\}} \). Since \( R \) is regular on \( \mathcal{A}^p \cup \{x\} \), \( \sum_{z \in \mathcal{A}^p \cup \{x\}} N_{yz} = 0 \) for all \( y \in \mathcal{A}^p \cup \{x\} \). By assumption, we have \( N = \sum_{y \in \mathcal{A}^p \setminus \{x\}} \lambda_y M^y \) which implies that
\[
0 = \sum_{y \in \mathcal{A}^p \setminus \{x\}} \sum_{z \in \mathcal{A}^p \cup \{x\}} \lambda_y M^y_{xz} = \sum_{y \in \mathcal{A}^p \setminus \{x\}} \lambda_y M^y_{xy} = -\lambda \sum_{y \in \mathcal{A}^p \setminus \{x\}} \lambda_y = -\lambda,
\]
which is a contradiction to \( \lambda > 0 \). Hence, \( R \not\in \text{aff}(Y) \). Choosing \( A' = \mathcal{A}^p \cup \{x\} \) yields the desired result. \( \square \)

In Lemma 13 we finally show that every PSCF that satisfies population-consistency and composition-consistency has to yield maximal lotteries. The structure of the proof is as follows. We assume for contradiction that a PSCF satisfies population-consistency and composition-consistency, but returns a lottery that is not maximal. Then we can find a set of profiles with full dimension for which the uniform lottery over a subset of alternatives is returned and the uniform profile is in its interior. Thus, this set contains a profile with a strict Condorcet winner that is close to the uniform profile. For every profile in an \( \varepsilon \)-ball around this strict Condorcet profile, the function has to return the uniform lottery over a subset of alternatives and the lottery with probability one on the Condorcet winner, which contradicts decisiveness.

**Lemma 13.** Every PSCF \( f \) that satisfies population-consistency and composition-consistency has to yield maximal lotteries, i.e., \( f \subseteq \text{ML} \).

**Proof.** Let \( f \) be a PSCF that satisfies population-consistency and composition-consistency and \( A \in \mathcal{F}(U) \). For \( |A| = 2 \) the statement follows from Lemma 5. For \( |A| > 2 \), assume for contradiction that \( f \not\subseteq \text{ML} \). By Lemma 12, there is \( A' \subseteq A \) and \( Y \subseteq R|_A \) of dimension \( |A'|-1 \) such that \( \text{uni}(A') \in f(R) \) for every \( R \in Y \) and \( R \) is not regular on \( A' \) if \( R \in \text{aff}(Y) \). By Lemma 11, there is a set \( X \subseteq R|_A \) of dimension \( |A|! - |A'|! \) such that \( \text{uni}(A') \in f(R) \) and \( R \) is regular on \( A' \) for every \( R \in X \).

Every affine combination of profiles that are regular on \( A' \) is a profile that is regular on \( A' \). Hence, every \( R \in \text{aff}(X) \) is regular on \( A' \). From Lemma 12 it follows that no \( R \in \text{aff}(Y) \) is regular on \( A' \). Therefore, \( \text{aff}(X) \cap \text{aff}(Y) = \emptyset \) which implies that \( \dim(X \cup Y) = \dim(X) + \dim(Y) = |A|! - |A'|! + |A'|-1 = |A|! - 1 \). Furthermore, it follows from population-consistency that \( \text{uni}(A') \in f(R) \) for every \( R \in \text{conv}(X \cup Y) \). Since \( \text{uni}(\mathcal{L}(A)) \) is in the interior of \( \text{conv}(X \cup Y) \), there are \( x \in A' \) and \( R^C \in \text{int}(Y) \) such that \( x \) is a strict Condorcet winner in \( R^C \). Hence, there is \( \varepsilon > 0 \) such that, for every \( R' \in B_\varepsilon(R^C) \cap R|_A \), \( R' \in \text{conv}(X \cup Y) \) and \( x \) is a strict Condorcet winner in \( R' \). Then, we get that \( x \in f(R') \) and \( \text{uni}(A') \in f(R') \) for every \( R' \in B_\varepsilon(R^C) \cap R|_A \). Thus, \( \{ R' \in R|_A : |f(R', A)| = 1 \} \) is not dense in \( R|_A \) at \( R^C \). This contradicts decisiveness of \( f \). \( \square \)
C.4. \( ML \subseteq f \)

In this section we show that every PSCF \( f \) that only yields maximal lotteries in fact has to yield all maximal lotteries. To this end, we first prove an auxiliary lemma. It was shown by McGarvey (1953) that every complete and anti-symmetric relation is the majority relation of some profile with a bounded number of voters. We show an analogous statement for fractional collective preferences and fractional preference profiles.

**Lemma 14.** Let \( M \in \mathbb{Q}^{n \times n} \) be a skew-symmetric matrix. Then, there are \( R \in \mathbb{R}|[n] \) and \( \lambda \in \mathbb{Q}_{>0} \) such that \( R(i,j) = \lambda M_{ij} \) for all \( i, j \in [n] \). Furthermore, if there is \( \pi \in \Pi([n]) \) such that \( M_{ij} = M_{\pi(i)\pi(j)} \) for all \( i, j \in [n] \), then \( R = \pi(R) \).

**Proof.** For all \( i, j \in [n] \) with \( i \neq j \), let \( R_{ij} \in \mathbb{R}|[n] \) be the profile such that \( R_{ij}(\succeq) = 1/(n-1)! \) if \( i \succ j \) and \( \{i, j\} \) is a component in \( R \) and \( R_{ij}(\succeq) = 0 \) otherwise. By construction, we have that \( R_{ij}(i, j) = 1 \) and \( R_{ij}(x, y) = 0 \) for all \( \{x, y\} \neq \{i, j\} \). Let \( \lambda = 1/\sum_{i,j: M_{ij} > 0} M_{ij} \) and \( R = \lambda \sum_{i,j: M_{ij} > 0} M_{ij} R_{ij} \). Then, we have that \( R(i,j) = \lambda M_{ij} \) for all \( i, j \in [n] \). The second part of the lemma follows from the symmetry of the construction.

For profiles which admit a unique maximal lottery it trivially follows that \( f = ML \). It turns out that every maximal lottery that is a vertex of the set of maximal lotteries in one of the remaining profiles is the limit point of a sequence of maximal lotteries of a sequence of profiles with a unique maximal lottery converging to the original profile. The proof of Lemma 15 heavily relies on continuity of \( f \).

**Lemma 15.** Let \( f \) be a PSCF. Then, \( f \subseteq ML \) implies \( f = ML \).

**Proof.** Let \( f \subseteq ML \) be a PSCF, \( A \in \mathcal{F}(U) \), and \( R \in \mathbb{R}|A \). By neutrality, we can assume without loss of generality that \( A = [n] \). We want to show that \( f(R) = ML(R) \). If \( ML(R) \) is a singleton, this is trivial, since \( f(R) \) is non-empty by definition of a PSCF. Hence, consider the case where \( ML(R) \) is not a singleton. Let \( p \in ML(R) \) and assume with loss of generality that \( \text{supp}(p) = [k] \). By \( M \) we denote the fractional collective preferences of \( R \).

We first consider the case where \( k \) is odd. Let \( N \in \mathbb{Q}^{n \times n} \) be defined as follows.

\[
N = \begin{pmatrix}
0 & -\frac{1}{p_1 p_2} & 0 & \ldots & 0 & \frac{1}{p_k p_1} & 1 & \ldots & 1 \\
\frac{1}{p_1 p_2} & 0 & \ddots & \ddots & \ddots & \vdots & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
-\frac{1}{p_k p_1} & 0 & \ldots & 0 & \frac{1}{p_{k-1} p_k} & 0 & 1 & \ldots & 1 \\
-1 & \ldots & -1 & 0 & \ldots & 0 & \vdots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
-1 & \ldots & -1 & 0 & \ldots & 0 & \vdots & \ddots & \vdots \\
\end{pmatrix}
\]
Intuitively, \( N \) defines a weighted cycle on \([k]\). Note that \((p^T N)_i = 0\) for all \(i \in \text{supp}(p)\) and \((p^T N)_i > 0\) for all \(i \in A \setminus \text{supp}(p)\), i.e., \(p\) is a quasi-strict maximin strategy of \(N\) in the sense of Harsanyi (1973b). By Lemma 14, there are \(S \in \mathcal{R}_A\) and \(c \in \mathbb{Q}_{>0}\) such that \(cN\) is the zero-sum game associated with \(S\). Since \(p\) is a maximin strategy in \(N\), it follows that \(p \in ML(S)\). For \(\varepsilon > 0\), we define \(R^\varepsilon = (1 - \varepsilon)R + \varepsilon S\). Population-consistency implies that \(p \in ML(R^\varepsilon)\). Moreover,

\[
M^\varepsilon = (1 - \varepsilon)M + \varepsilon cN
\]

is the zero-sum game associated with \(R^\varepsilon\). Observe that \(p\) is a quasi-strict maximin strategy of \(M^\varepsilon\) for every \(\varepsilon > 0\). Hence, for every maximin strategy \(q \in \Delta(A)\) of \(M^\varepsilon\), it follows that \((q^T M^\varepsilon)_i = 0\) for every \(i \in [k]\) and \(q_i = 0\) for every \(i \notin [k]\). It follows from a result by Cayley (1847) that

\[
\det((N_{ij})_{i,j \in [k-1]}) = \prod_{l=1}^{k-2} \left( \frac{1}{p_l p_{l+1}} \right)^2 \neq 0,
\]

and hence, \((N_{ij})_{i,j \in [k]}\) has rank at least \(k-1\). More precisely, \((N_{ij})_{i,j \in [k]}\) has rank \(k-1\) as it is skew-symmetric and \(k\) is odd, which implies that it does not have full rank. Furthermore, \(\det((M_{ij}^\varepsilon)_{i,j \in [k-1]})\) is a polynomial in \(\varepsilon\) of order at most \(k - 1\) and hence, has a most \(k - 1\) zeros. Thus, we can find a sequence \((\varepsilon_l)_{l \in \mathbb{N}}\) which converges to zero such that \((M_{ij}^\varepsilon)_{i,j \in [k]}\) has rank \(k - 1\) for all \(l \in \mathbb{N}\). In particular, if \((q^T M^\varepsilon)_i = 0\) for all \(i \in [k]\), then \(q = p\). This implies that \(p\) is the unique maximin strategy of \(M^\varepsilon\) for all \(l \in \mathbb{N}\) and hence, \(\{p\} = ML(R^\varepsilon) = f(R^\varepsilon)\) for all \(l \in \mathbb{N}\). It follows from continuity of \(f\) that \(p \in f(R)\).

Now we consider the case where \(k\) is even. \(ML(R)\) is a polytope because it is the solution space of a linear feasibility program. Assume that \(p\) is a vertex of \(ML(R)\). We first show that \(p\) is not quasi-strict. Assume for contradiction that \(p\) is quasi-strict, i.e., \((p^T M)_i > 0\) for all \(i \notin [k]\). Then, \(\text{supp}(q) \subseteq [k]\) for every maximin strategy \(q\) of \(M\). But then \((1 + \lambda)p - \lambda q\) is also a maximin strategy of \(M\) for small \(\lambda > 0\) as \(p\) is a quasi-strict maximin strategy of \(M\). This contradicts the assumption that \(p\) is a vertex of \(ML(R)\).

Hence, we may assume without loss of generality that \((p^T M)_{k+1} = 0\). Let \(e_1 = M_{k+1,1}/p_2\) and \(e_i = (M_{k+1,i} + p_{i-1}e_{i-1})/p_{i+1}\) for \(i \in \{2, \ldots, k - 1\}\) and define \(N \in \mathbb{Q}^{n \times n}\) as follows.

\[
N = \begin{pmatrix}
0 & e_1 & 0 & \ldots & 0 & 0 & 1 & \ldots & 1 \\
-1 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \ldots & 0 & -e_{k-1} & 0 & 0 & \ddots & \ddots & \ddots & \ddots \\
0 & \ldots & 0 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
-1 & \ldots & -1 & 0 & \ldots & 0 \\
\end{pmatrix}
\]

\(^{18}\)The proof of this statement does not make use of the fact that \(k\) is even and therefore also holds (but is not needed) for odd \(k\).
Note that $N_{1,k} = N_{k,1} = 0$. By Lemma 14, there are $S \in \mathcal{R}|_A$ and $c \in \mathbb{Q}_{>0}$ such that $cN$ is the zero-sum game associated with $S$. For $\varepsilon > 0$, let $R^\varepsilon = (1 - \varepsilon)R + \varepsilon S$. Then,

$$M^\varepsilon = (1 - \varepsilon)M + \varepsilon cN$$

is the zero-sum game associated with $R^\varepsilon$. We claim that $p^\varepsilon$ defined as follows is a maximin strategy in $M^\varepsilon$. To this end, let $s_\varepsilon = \frac{\varepsilon c}{1 - \varepsilon + \varepsilon c}$.

$$p^\varepsilon_i = \begin{cases} (1 - s_\varepsilon)p_i & \text{if } i \in [k], \\ s_\varepsilon & \text{if } i = k + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $(pT N)_1 = -p_2 e_1 = -M_{k+1,1}$ and, for $i \in \{2, \ldots, k - 1\}$,

$$(pT N)_i = p_{i-1} e_i - p_{i+1} e_i = p_i e_i - (M_{k+1,i} + p_{i-1} e_{i-1}) = -M_{k+1,i}.$$ To determine $(pT N)_k$, we first prove inductively that $p_i e_i = \frac{1}{p_{i+1}} \sum_{j=1}^{i} M_{k+1,j} p_j$ for all $i \in [k - 1]$. For $i = 1$ this follows from the definition of $e_1$. Now, let $i \in \{2, \ldots, k - 1\}$. Then,

$$p_i e_i = \frac{p_i}{p_{i+1}} (M_{k+1,i} + p_{i-1} e_{i-1}) = \frac{p_i}{p_{i+1}} (M_{k+1,i} + \frac{1}{p_i} \sum_{j=1}^{i-1} M_{k+1,j} p_j) = \frac{1}{p_{i+1}} \sum_{j=1}^{i} M_{k+1,j} p_j,$$

where the second equality follows from the induction hypothesis. Now,

$$(pT N)_k = p_{k-1} e_{k-1} = \frac{1}{p_k} \sum_{j=1}^{k-1} M_{k+1,j} p_j = -\frac{1}{p_k} M_{k+1,k} p_k = -M_{k+1,k},$$

where the third equality follows from the fact that $(pT M)_{k+1} = 0$. Furthermore, for all $i \in [k]$, $(p^\varepsilon M)_i = s_\varepsilon M_{k+1,i}$.

Then, for $i \in [k]$,

$$((p^\varepsilon)T M^\varepsilon)_i = (1 - \varepsilon) s_\varepsilon M_{k+1,i} + \varepsilon c (1 - s_\varepsilon)(-M_{k+1,i}) = 0.$$ Furthermore, it follows from $(pT M)_{k+1} = 0$ that $((p^\varepsilon)T M^\varepsilon)_{k+1} = 0$ as $M_{k+1,k+1} = 0$, and, for $i \in A \setminus [k + 1]$,

$$((p^\varepsilon)T M^\varepsilon)_i \geq (1 - \varepsilon) s_\varepsilon M_{k+1,i} + \varepsilon c \geq -(1 - \varepsilon) s_\varepsilon + \varepsilon c > 0.$$ This shows that $p^\varepsilon$ is a maximin strategy of $M^\varepsilon$ and hence, $p^\varepsilon \in ML(R^\varepsilon)$. Since $|\text{supp}(p^\varepsilon)|$ is odd, it follows from the first case that $p^\varepsilon \in f(R^\varepsilon)$. Note that $s_\varepsilon$ goes to 0 as $\varepsilon$ goes to 0. Hence, $p^\varepsilon$ goes to $p$ as $\varepsilon$ goes to 0. It now follows from continuity of $f$ that $p \in f(R)$.

Together, we have that $p \in f(R)$ for every vertex $p$ of $ML(R)$. Since every lottery in $ML(R)$ can be written as a convex combination of vertices, convexity of $f(R)$ implies $f(R) = ML(R).$ \qed
Theorem 2 then follows directly from Lemmas 13 and 15.

**Theorem 2.** A PSCF $f$ satisfies population-consistency and composition-consistency if and only if $f = ML$. 