Analyzing the Practical Relevance of Voting Paradoxes via Ehrhart Theory, Computer Simulations, and Empirical Data

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ABSTRACT
More and more results from social choice theory are used to argue about collective decision making in computational multiagent systems. A large part of the social choice literature studies voting paradoxes in which seemingly mild properties are violated by common voting rules. In this paper, we investigate the likelihood of the Condorcet Loser Paradox (CLP) and the Agenda Contraction Paradox (ACP) using Ehrhart theory, computer simulations, and empirical data. We present the first analytical results for the CLP on four alternatives and show that our experimental results, which go well beyond four alternatives, are in almost perfect congruence with the analytical results. It turns out that the CLP—which is often cited as a major flaw of some Condorcet extensions such as Dodgson’s rule, Young’s rule, and MaxiMin—is of no practical relevance. The ACP, on the other hand, frequently occurs under various distributional assumptions about the voters’ preferences. The extent to which it is real threat, however, strongly depends on the voting rule, the underlying distribution of preferences, and, surprisingly, the parity of the number of voters.

Categories and Subject Descriptors
[Computing Methodologies]: Modeling and Simulation;
[Applied Computing]: Economics

Keywords
Social choice theory; voting; Ehrhart theory

1. INTRODUCTION
More and more results from social choice theory are used to argue about collective decision making in computational multiagent systems (see, e.g., [10, 7, 27, 8]). A large part of the social choice literature studies voting paradoxes in which seemingly mild properties are violated by common voting rules. There are a number of sweeping impossibilities, which entail that there exists no “optimal” voting rule that avoids all paradoxes. As a consequence, much of the research in social choice theory is concerned with whether a paradox can appear for a given voting rule or not. However, it turns out that some paradoxes—while possible in principle—will almost never occur in practice.

An extreme example of this phenomenon was recently revealed for the voting rule TEQ. Due to its unwieldy recursive definition, it was unknown for more than 20 years whether TEQ satisfies any of a number of very basic desirable properties. In 2011, Brandt et al. [6] have shown that TEQ violates all of these properties. However, their proof is non-constructive and only shows the existence of astronomically large counterexamples requiring about 10^{136} alternatives. Despite the existence of smaller computer-generated counterexamples, computer experiments have shown that these counterexamples are extremely rare and that TEQ satisfies the desirable properties for all practical purposes [5]. These findings motivated us to provide analytical, experimental, and empirical justifications for such statements.

In this paper, we study two voting paradoxes. The first is the well-known Condorcet loser paradox (CLP), which occurs when a voting rule selects the Condorcet loser, an alternative that loses against every other alternative in pairwise majority contests. Perhaps surprisingly, this paradox affects some Condorcet extensions, i.e., voting rules that are guaranteed to select an alternative that wins against every other alternative in pairwise majority contests. Common affected Condorcet extensions are Dodgson’s rule, Young’s rule, and MaxiMin [16]. The second paradox, called agenda contraction paradox (ACP), occurs when removing losing alternatives changes the set of winners. There are only few voting rules that do not suffer from this paradox, one of them being the essential set. In fact, all common voting rules that violate the CLP also violate the ACP.

In principle, quantitative results on voting paradoxes can be obtained via three different approaches. The analytical approach uses theoretical models to quantify paradoxes based on certain assumptions about the voters’ preferences. Analytical results usually tend to be quite hard to obtain and are limited to simple—and often unrealistic—assumptions. The experimental approach uses computer simulations based on underlying stochastic models of how the preference profiles are distributed. Experimental results have less general validity than analytical results, but can be obtained for arbitrary distributions of preferences. Finally, the empirical approach is based on evaluating real-world data to analyze how frequently paradoxes actually occur or how frequently they would have occurred if certain voting rules had been used for the given preferences. Unfortunately, only very limited real-world data for elections is available.

Our main results are as follows.
Using Ehrhart theory, we compute upper bounds for the CLP as well as the exact probabilities under which the CLP occurs for MaxiMin when there are four alternatives and preferences are distributed according to the Impartial Anonymous Culture (IAC) distribution. This approach also yields the exact limit probabilities (for CLP and ACP) when the number of voters goes to infinity. To the best of our knowledge, these are the first analytical results for the CLP on four alternatives.

For both the CLP and the ACP, we thoroughly analyze a variety of other settings with more alternatives and other stochastic preference models using computer simulations. For those settings in which the analytical approach is also feasible, our results are in almost perfect congruence with the analytical results. This is strong evidence for the accuracy of our simulation results.

It turns out that the CLP—which is often cited as a major flaw of some Condorcet extensions—is of no practical relevance. The maximum probability under all preference models we studied is 2.2% (for MaxiMin, three voters, four alternatives, and IAC). In more realistic settings, it is much lower. For Dodgson’s rule, it never exceeds 0.01%. We did not find any occurrence of the paradox in real-world data, neither in the PrefLib library [23] nor in millions of elections based on data from the Netflix Prize [3].

The ACP, on the other hand, frequently occurs under various distributional assumptions about the voters’ preferences. The extent to which it is a real threat, however, strongly depends on the voting rule, the underlying distribution of preferences, and, surprisingly, the parity of the number of voters. If the number of voters is much larger than the number of alternatives, less discriminating voting rules seem to fare better than more discriminating ones. For example, when there are 1,000 voters and four alternatives, the probability for the ACP under Copeland’s rule and IAC is 9% while it occurs with a probability of 33% for Borda’s rule. When there are less voters, the parity of the number of voters plays a surprisingly strong role. For example, if there are 6, 26% of voters, but only 26% for 51 voters. These results are in line with the empirical data we analyzed.

2. RELATED WORK

There is a huge body of research on the quantitative study of voting paradoxes. Gehrlein [17] focuses on Condorcet’s paradox, the event that no Condorcet winner exists in an election, which is, presumably, the most studied voting paradox; Gehrlein and Lepelley [18], on the other hand, provide an overview of many paradoxes and, in particular, analyze the influence of group coherence. In addition, Gehrlein and Lepelley [18] survey different tools and techniques that have been applied over the years for the quantitative study of voting paradoxes.

The analytical study of voting paradoxes under the assumption of IAC is most effectively done via Ehrhart theory, which goes back to the year 1962 and the French mathematician Eugène Ehrhart [14]. Interestingly, parts of these results have been reinvented (in the context of social choice) by Huang and Chua [20] in 2000, before Ehrhart’s original work was independently rediscovered for social choice by Wilson and Pritchard [30] and Lepelley et al. [21] about eight years ago.

Current research on the probability of voting paradoxes under IAC is based on algorithms that build upon Ehrhart’s results, such as the algorithm developed by Barvinok [2]. For many years, these approaches were limited to cases with three or less alternatives. Recent advances in software tools and mathematical modeling enabled the study of elections with four alternatives. (Bruns and Söger [9] and Schürmann [28] provide such results for Condorcet’s paradox, the Condorcet efficiency of plurality and the similarity between plurality and plurality with runoff.) Schürmann [28] shows how symmetries in the formulation of the paradoxes can be exploited to further facilitate the corresponding computations.

For the CLP (sometimes also referred to as “Borda’s paradox”) many quantitative results are known [18], which are, however, limited to simple voting rules and scoring rules in particular. These results also include some empirical evidence for the paradox under plurality ([18], p.15) and suggest that it is an unlikely yet possible problem in practice. Interestingly, the CLP for Condorcet extensions has—to the best of our knowledge—only been considered by Plassmann and Tideman [25]. However, they restrict their analysis to the three-alternative case and find that the CLP never occurs, which is unsurprising since provably four alternatives are required for the Condorcet extensions they considered.

The ACP appears to have received less attention in the quantitative literature on voting paradoxes. Some limit probabilities for scoring rules were obtained by Gehrlein and Fishburn [18], p. 282-284). Fishburn [15] experimentally studied a variant of this paradox called “winner turns loser paradox” for Borda’s rule under Impartial Culture. For Condorcet extensions, Plassmann and Tideman [25] considered another variant of the ACP under a spatial model, but again limit their experiments to three alternatives. These few results already seem to indicate that the ACP might occur even under realistic assumptions. However, there are no results for more than three alternatives, Condorcet extensions, and the ACP in its full generality.

The preference models we consider (such as IC, IAC, and the Mallows-φ model) have also found widespread acceptance for the experimental analysis of voting rules within the multiagent systems and AI community (see, e.g., [1, 4, 19]).

3. MODELS AND DEFINITIONS

Let \( A \) be a set of \( m \) alternatives and \( N = \{1, \ldots, n\} \) a set of voters. Each voter is equipped with a

\[ x \succ_i y \text{ as voter } i \text{ (strictly) preferring alternative } x \text{ to alternative } y. \]

A (preference) profile (or an election) is an \( n \)-tuple of preference relations and will be denoted by \( R := (\succ_1, \ldots, \succ_n) \).

We will sometimes consider the restriction of \( \succ_i \) to a subset of alternatives \( B \subseteq A \), called an agenda. Such a restriction will be denoted by \( R|_B := (\succ_1|_B, \ldots, \succ_n|_B) \).

3.1 Stochastic Preference Models

In this paper we consider five of the most common stochastic preference models. These models vary in their degree of realism. Impartial culture (IC) and impartial anonymous

\[ x \succ_i y \iff x \nless y \forall y \in A. \quad \text{One may alternatively define } \succ_i \text{ as the irreflexive component of a complete, antisymmetric, and} \]

transitive relation \( \preceq \).

\[ A \]
culture (IAC), for example, are usually considered as rather unrealistic. However, the simplicity of these models enables the use of analytical tools that cannot be applied to the other models. IC and IAC typically yield higher probabilities for paradoxes than other preference models and can therefore be seen as worst-case estimates (see, e.g., [26]). We only give informal definitions here; for more extensive treatments see, e.g., Critchlow et al. [12] and Marden [22].

**Impartial culture** The most widely-studied distribution is the so-called *impartial culture (IC)*, under which every possible preference relation has the same probability of 1.

Thus, every preference profile is equally likely to occur.

**Impartial anonymous culture** In contrast to IC the *impartial anonymous culture (IAC)* is not based on the probabilities of individual preferences but on the probabilities of whole profiles. Under IAC one assumes that each possible anonymous preference profile on n voters is equally likely to occur. A more formal definition is given in Section 4.1.

**Mallows-φ model** In Mallows-φ model, the distance to a reference ranking (or ground truth) is measured by means of the Kendall-tau distance and a parameter φ is used to indicate the dispersion. The case φ = 1 means absolute dispersion and coincides with IC, the case φ = 0 corresponds to no dispersion and every voter always picks the “true” ranking. We chose φ = 0.8 to simulate voters with relatively bad estimates, which leads to situations in which paradoxes are more likely to occur.

**Pólya-Eggenberger urn model** In the Pólya-Eggenberger urn model, each possible preference relation is represented by a ball in an urn from which individual preferences are drawn. After each draw, the chosen ball is put back and α ∈ N: new balls of the same kind are added to the urn. While the urn model subsumes both impartial culture (α = 0) and impartial anonymous culture (α = 1), we set α = 10 to obtain a reasonably realistic interdependence of individual preferences.

**Spatial model** In the spatial model alternatives and agents are placed in a multi-dimensional space uniformly at random and the agents’ preferences are then determined by the Euclidean distances to the alternatives (closer alternatives are preferred to more distant ones). The spatial model is considered particularly realistic in political science where the dimensions are interpreted as different aspects of the alternatives. We chose the simple case of two dimensions for our analysis.²

### 3.2 Voting Rules

A voting rule is a function f that maps a preference profile to a non-empty set of winners.

For a preference profile R, let \( g_{xy} := |\{i \in N : x \succ_i y\} - |\{i \in N : y \succ_i x\}| \) denote the *majority margin* of x against y. A very influential concept in social choice is the notion of a Condorcet winner, an alternative that wins against any other alternative in a pairwise majority contest. Alternative x is a *Condorcet winner (CW)* of a profile R if \( g_{xy} > 0 \) for all \( y \in A \setminus \{x\} \). Conversely, alternative x is a Condorcet loser (CL) if \( g_{yx} > 0 \) for all \( y \in A \setminus \{x\} \). Neither Condorcet winners nor Condorcet losers necessarily exist, but whenever they do they are unique. A voting rule f is called a *Condorcet extension* if \( f(R) = \{x\} \) whenever x is the Condorcet winner in R.

In the following paragraphs we briefly introduce the voting rules considered in this paper.

**Borda’s Rule** Under Borda’s rule each alternative receives from 0 to \( |A| - 1 \) points from each voter (depending on the position the alternative is ranked in). The alternatives with highest accumulated score win.

**Maximin** The Maximin rule is only concerned with the highest defeat of each alternative in a pairwise majority contest. It yields all alternatives as winners which have the maximal value of \( \min_{y \in A} g_{xy} \).

**Young’s Rule** Young’s rule yields all alternatives that can be made a Condorcet winner by removing a minimal number of voters.

**Dodgson’s Rule** Dodgson’s rule selects all alternatives that can be made a Condorcet winner by a minimal number of pairwise swaps of adjacent alternatives in the individual preference relations.

**Essential Set** Consider the symmetric two-player zero-sum game G given by the skew-symmetric matrix with entries \( g_{xy} \) for all pairs of alternatives x, y. The *essential set* is the set of all alternatives that are played with positive probability in some mixed Nash equilibrium of G.⁴

Except for Borda’s rule, all presented voting rules are in fact Condorcet extensions. While Borda’s rule, Maximin, and the essential set can be computed efficiently, Young’s rule and Dodgson’s rule have been shown to be complete for parallel access to NP. The essential set is one of the few voting rules that do suffer from neither the CLP nor the ACP, and is merely included as a reference. For more formal definitions and computational properties of these rules, we refer to Brandt et al. [8].

### 3.3 Voting Paradoxes

In this paper we focus on two voting paradoxes whose occurrence can be determined given a voting rule f and a preference profile R.

Let f be a voting rule. Formally, a (voting) paradox is a characteristic function that maps a preference profile to 0 or 1. In the latter case, we say the paradox occurs for voting rule f at profile R.

The *Condorcet Loser Paradox (CLP)* occurs when a voting rule selects the Condorcet loser as a winner.

**Definition 1**. Given a voting rule f the Condorcet loser paradox CLPf is defined as

\[
\text{CLP}_f(R) = \begin{cases} 
1 & \text{if } f(R) \text{ contains a CL} \\
0 & \text{otherwise.}
\end{cases}
\]

The agenda contraction paradox (ACP) occurs when reducing the set of alternatives, by eliminating unchosen alternatives, influences the outcome of an election.

²These mixed equilibria are also known as maximal lotteries in probabilistic social choice.
4. QUANTIFYING VOTING PARADOXES

In this section we present the three general approaches for quantifying voting paradoxes: the analytical approach via Ehrhart theory, the experimental approach via computer simulations, and the empirical approach via real-world data.

4.1 Exact Analysis via Ehrhart Theory

Anonymous preference profiles only count the number of voters for each possible ranking on m alternatives. For m alternatives there are m! rankings. An anonymous preference profile can be seen as an integer point in an m!-dimensional space. There is one dimension for each ranking and so the point contains full information how many voters have each ranking. The set $S_{m,n}$ of anonymous preference profiles on m alternatives with n voters can be seen as the set of all integer points $z = (z_1, \ldots, z_{m!}) \in \mathbb{Z}^{m!}$ which satisfy

(i) $z_i \geq 0$ for all $i \in \{1, \ldots, m\}$, and

(ii) $\sum_{i=1}^{m!} z_i = n$.

Under IAC each anonymous preference profile is assumed to be equally likely to occur. In order to determine the probability of a paradox under IAC it is enough to compute the number of points belonging to preference profiles in which the paradox occurs and compare them to the total number of points in $S$. Many paradoxes can be described with the help of linear constraints, i.e., the set of points belonging to the event can be described with the help of (in)equalities, a polytope.

For the set $S_{m,n}$ of all anonymous preference profiles on m alternatives with n voters it is known that:

$$|S_{m,n}| = \binom{m! + n - 1}{m! - 1}.$$

To determine the probability of a paradox for a fixed number of alternatives and an arbitrary number of voters one needs the concept of dilations. For an integer parameter $n \in \mathbb{N}$, the dilation of a polytope $P$ by n is the polytope $nP = \{nx : x \in P\}$. The function $L(P, n)$, in the variable n, is defined as $L(P, n) = |nP \cap \mathbb{Z}^n|$, i.e., $L(P, n)$ describes the number of integer points lying inside the dilation $nP$. Let m be fixed. If $P_n$ is the polytope defined by the linear constraints describing an anonymous profile $X_n$ on n voters. We know that the probability of $X_n$ under IAC is given by:

$$P(X_n) = \frac{L(P_n, n)}{|S_{m,n}|}.$$

To be able to determine the probability of (many) voting paradoxes under IAC one only needs to evaluate $L(P, n)$. This can be done with the help of Ehrhart theory. Ehrhart [14] was the first to show how $L(P, n)$ can be evaluated. He showed that $L(P, n)$ can be described by a special kind of function, called quasi- or Ehrhart-polynomials. A function $f : \mathbb{Z} \to Q$ is a quasi-polynomial of degree $d$ and period $q$ if there exists a list of q polynomials $f_i : \mathbb{Z} \to Q$ ($0 \leq i < q$) of degree $d$ such that $f(n) = f_i(n)$ if $n \equiv i \mod q$. $q$ is called the period of the quasi-polynomial.

Quasi-polynomials can be determined with the help of computer programs such as LATTÉ (De Loera et al. [13]) or NORMALIZ (Bruns and Söger [9]). Note that the computation of quasi-polynomials is computationally very demanding, especially because the dimension of the polytopes grows factorial in the number of alternatives. This limits analytical results under IAC to rather small numbers of alternatives. The only program which we know of which is able to compute polytopes connected to elections with up to four alternatives is NORMALIZ. Even NORMALIZ does not always compute the whole quasi-polynomial, but sometimes only the leading coefficients of the polynomial, which fortunately is enough to compute the limit probability of a paradox when the number of voters goes to infinity.

**Finding a Quasi-polynomial for MaxiMin**

As an example for the method just described, we consider the CLP${}_\text{MaxiMin}$ in four-alternative elections under IAC, the probabilities of which can be computed from a quasi-polynomial. The polynomial is of degree 23 and has a period of 5,040.

In order to determine the full polynomial, we first need to describe the corresponding polytope with equalities and inequalities. Recall the definition of MaxiMin from Section 3.2:

$$f_{\text{MaxiMin}}(R) := \arg \max_{x \in A} \min_{y \in A} g_{yx}.$$  

For $f_{\text{MaxiMin}}(R) = 1$ the Condorcet loser of $R$ has to have the lowest highest defeat. Formally, there is $x \in A$ such that for all $y \in A \setminus \{x\}$

(i) $g_{yx} > 0$, and

(ii) $\min_{x \in A \setminus \{x\}} g_{xz} \leq \min_{z \in A \setminus \{y\}} g_{yz}$.

Note that the second statement is equivalent to $\max_{x \in A \setminus \{x\}} g_{xz} \leq \max_{x \in A \setminus \{y\}} g_{yx}$. Let now $A = \{a, b, c, d\}$ and assume $x = d$. We then have that

$$g_{ad} > 0, \quad g_{bd} > 0, \quad \text{and} \quad g_{cd} > 0,$$

which implies that $\max_{x \in A \setminus \{d\}} g_{cd} > 0$. Furthermore,

$$\max_{x \in A \setminus \{y\}} g_{xy} > 0$$

for all $y \in \{a, b, c\}$, which implies that either $g_{ab}, g_{bc}, g_{ca} > 0$ or $g_{ab}, g_{cb}, g_{ac} > 0$. On both cases there will be a majority cycle between a, b, and c. Due to symmetry we can choose one direction of the cycle arbitrarily. Let us assume $g_{ab}, g_{bc}, g_{ca} > 0$. Then,

$$\max_{x \in A \setminus \{a\}} g_{xa} = g_{ca}, \quad \max_{x \in A \setminus \{b\}} g_{xb} = g_{ab}, \quad \text{and} \quad \max_{x \in A \setminus \{c\}} g_{xc} = g_{bc}.$$

Condition (i) is already represented in the form of linear inequalities. In order to model condition (ii) we have to determine $\max_{x \in A \setminus \{d\}} g_{ad}$ and need to make a case distinction for the seven possible outcomes. The inequalities for the case $\max_{x \in A \setminus \{d\}} g_{ad} = \{g_{ad}\}$ are

$$g_{ad} - g_{bd} > 0 \quad \text{and} \quad g_{ad} - g_{cd} > 0.$$  

Condition (ii) furthermore yields

$$g_{ca} - g_{ad} \geq 0, \quad g_{ab} - g_{ad} \geq 0, \quad \text{and} \quad g_{bc} - g_{ad} \geq 0.$$  

Each case belongs to a different polytope and the polytopes are pairwise distinct, so we can compute each quasi-polynomial separately and later combine them to one. To
get the final polynomial we have to multiply by eight for the four different possible choices of a Condorcet loser and the two possible directions of the pairwise majority cycle. This then enables us to efficiently evaluate the exact probabilities for any number of voters. The results are depicted in Figure 2. The leading coefficient of the quasi-polynomial can also be used to determine the limit probability which is given by

\[
P(\text{CLP}_\text{Maximin} = 1 \mid m = 4, n \to \infty) = 8 \cdot \frac{485052253637930099}{6443662124777472000000} \approx 0.06%.
\]

4.2 Experimental Analysis

As we will see, simulating elections with the help of computers is a viable way of achieving very good approximations for the probabilities we are looking for. It even turns out that the results of our simulations are almost indistinguishable from the theoretical result obtained via Ehrhart theory (with the exception of the limit case, which cannot be realized via simulations).

More specifically, the experimental approach works as follows: a profile source creates random preference profiles according to a specific preference model. The profiles are then used to compute the winner(s) according to a given voting rule and to determine if the paradox occurs. For instance, to determine the occurrence of the CLP only a single winner determination has to be performed. The ACP, on the other hand, requires the evaluation of the voting rule for multiple modified profiles. Any such experiment is repeated frequently and carried out for each pair of \(n\) and \(m\). In many cases in which we covered a wide range of voters, we did not consider every possible value of \(n\) but, more economically, only simulated the values: 1–30, 49–51, 99–101, 199–201, 499–501, 999–1,001.

In contrast to many other studies, we are concerned about the statistical significance of our experimental results. Thus, we also computed 99%-confidence intervals for each data point we generated. To this end, we used the binofit function in MATLAB which is based on the standard approach by Clopper and Pearson [11]. It shows that, based on our sampling rate of \(10^5\) and \(10^6\), respectively, the 99%-confidence intervals are pleasantly small. Hence, even though they are depicted in all of the figures throughout this paper, sometimes it can be difficult to recognize them.

4.3 Empirical Analysis

The most valuable quantification of voting paradoxes would be their actual frequency in real-world elections. As mentioned before, real-world election data about is generally relatively sparse, incomplete, and inaccurate. This makes empirical research on this topic relatively difficult. Otherwise, the empirical approach strongly resembles the experimental approach. Since the given data is often not in the desired form of preference profiles, it is preprocessed and profiles are generated. Then, usually also with the help of computers, it is determined if a voting paradox occurred in a situation or would have occurred if a different voting rule had been used.

Empirical studies differ from the other two approaches in the way that researchers have no control over the number of alternatives and voters, and instead have to use existing data.

For this paper we used two sources of empirical data. First, we used the 314 profiles with strict order preferences from the PrefLib library [23]. Second, we had access to the 54,650 preference profiles over four alternatives without a Condorcet loser which belong to the roughly 11 million four-alternative elections which Mattei et al. [24] derived from the Netflix Prize data [3]. The second set of profiles we could use to analyze the two paradoxes for the Condorcet extensions, since they would not occur in profiles with a Condorcet winner anyway.

5. CONDORCET LOSER PARADOX

In this section we present our findings on the CLP. We conclude that—even though the CLP is possible in principle—it is so unlikely that it cannot be used as an argument against any of the Condorcet extensions we considered.

5.1 An Upper Bound

Before analyzing the CLP for concrete voting rules, we discuss an upper bound valid for all Condorcet extensions. For a Condorcet extension to choose the Condorcet loser a profile obviously has to satisfy two conditions. First, there has to exist a Condorcet loser in the profile. Second, no Condorcet winner may exist in the profile. In the case of four-alternative elections—which is the first interesting case—we can compute the quasi-polynomial via Ehrhart theory and so know the exact probabilities for each number of voters. The derivation and presentation of the quasi-polynomial is omitted due to space constraints. It has degree 23 and contains 24 polynomials. The resulting probabilities for up to 1,000 voters—and a comparison with the results of an experimental analysis—can be obtained from Figure 1. The value of the limit probability is approximately 8%.

![Figure 1: Probability of the event that a Condorcet loser paradox could choose a Condorcet loser in four-alternative elections under IAC.](image)

Especially for small even numbers of voters where the probability is around 20% the upper bound is too high to
discard the CLP for Condorcet extensions altogether, and even the limit probability of 8% is relatively large.

Also, for an increasing number of alternatives this problem does not vanish (for elections with 50 and 51 one voters and up to 100 alternatives the probabilities range between 5% and 25%).

It should be noted that, these upper bounds turn out to be relatively independent from the underlying preference distribution (among the models we considered, cf. Section 5.3).

5.2 Results under IAC

As indicated already, for concrete Condorcet extensions the picture is quite clear: even under IAC, the risk of the considered Condorcet extensions selecting the Condorcet loser is very low, as shown in Figure 2 and Figure 3 for four-alternative elections. The highest probability was found for CLP\text{MaxiMin} with 2.2% for three voters (CLP\text{Young} with about 0.9%). The limit probability of CLP\text{MaxiMin}, with 0.06% is so low that for sufficiently large electorates it would occur in only one out of 10,000 elections. The same seems to hold for the limit probability for CLP\text{Young}. The probability of CLP\text{Dodgson} is even significantly lower, with a maximum of about 0.01% in elections with 9,999 voters. We could determine the limit probability of 0.01% only for an approximation of Dodgson’s rule by Tideman [29], which seems to be close to that for Dodgson’s rule, based on our experimental data.

When increasing the number of alternatives the probabilities drop even further. For elections with more than ten alternatives they reach a negligibly small level of less than 0.005% for all considered rules and in no simulations with twelve or more alternatives we could find any occurrence of the paradox.

5.3 Results under Other Preference Models

Figure 4, as one would expect, shows that under more realistic assumptions the probability of the CLP decreases further in four-alternative elections with 50/51 voters, with the highest probability occurring under the unrealistic assumption of IC and the lowest probability under what may be the most realistic model in many settings, the spatial model. Actually, in our experiments, Dodgson’s rule never selected a Condorcet loser in the spatial model.

Similarly, we could not find any occurrence of the CLP in real-world data, which may be considered the strongest evidence that the CLP virtually never materializes in practice.\textsuperscript{5}

6. AGENDA CONTRACTION PARADOX

Recall that the agenda contraction paradox (ACP) occurs when a reduced set of alternatives (created by the unavailability of losing alternatives) influences the outcome of an election. For many cases, it may be considered a generalization of the CLP as the following argument shows. Suppose the Condorcet loser \(x\) is uniquely selected by a voting rule \(f\) which implements majority rule on two-alternative choice sets. Then restricting the alternatives to \(\{x, y\}\) for some other alternative \(y\) (for which then \(g_{xy} > 0\)) leads to the new winner \(y\).

As we will see, the ACP is much more of a practical problem than the CLP. The picture, however, is not black and white. Whether or not it is a serious threat depends on the voting rule, the underlying preference distribution, and on the parity of the number of voters.

\textsuperscript{5}We tested 314 preference profiles with strict orders from the PrefLib library as well as the roughly 11 million four-alternative elections which Mattei et al. [24] derived from the Netflix Prize data. While about 54,000 of those elections were susceptible to the CLP, it never occurred under the rules we considered in this paper. In contrast, under plurality it already occurred in twelve out of the 314 PrefLib-instances.
6.1 Varying Voting Rules

The ACP probability strongly varies for different voting rules (see Figure 5). Borda’s rule generally exhibits the worst behavior of the rules studied, with probabilities of up to 56% and 34% for large electorates with 1,000 voters. In contrast, while, for small even numbers of voters, Copeland’s rule also frequently fails agenda contraction, for large electorates, it is quite robust to the ACP (with only about 8% occurrence probability).

The reason for this gap appears to be two-fold: First, Condorcet extensions are safe from this paradox as long as a Condorcet winner exists; Borda’s rule, by contrast, is not. Second, the discriminatory power of voting rules (i.e., their ability to select small winning sets) strongly supports the paradox. As soon as a single majority-dominated alternative is selected, the ACP has to occur. For large numbers of voters, this is in line with Copeland’s rule being least discriminating among those evaluated. The essential set among the most discriminating known voting rules immune to the ACP, but presumably less discriminating than Copeland’s rule.

The behavior of Maximin is almost identical to that of Young’s and Dodgson’s rule. Confirming our approximate “limit” results of 1,000 voters, we were able to analytically compute the limit probability for Maximin with \( \frac{331}{2048} \approx 16\% \). This is in perfect congruence with the (rounded) values for Maximin, Young’s rule, and Dodgson’s rule.

It should also be noted that with less than 100 voters, the parity of the number of voters plays a major role. For even numbers, significantly higher probabilities arise (which is particularly true for Copeland’s rule, see above) at least part of this can be explained by a reduced probability for Condorcet winners in these cases.

For more alternatives (see the right-hand side of Figure 5), the relative behavior remains vastly unchanged with probabilities further increasing to values larger than 40% to 80% (mostly since the likelihood of a Condorcet winner decreases roughly at the same rate).

6.2 Varying Preference Models

Figure 6 extends the analysis of the previous section by additionally considering preference models beyond IAC. The overall picture regarding the different rules remains the same. For large electorates Copeland’s rule outperforms the other rules, whereas Borda’s rule performs worst.

Regarding the different preference models, three classes emerge from Figure 6.

First, for Mallows-\( \phi \) we observe probabilities that are vanishing with increased numbers of voters. Under the spatial model this is true as well, with the surprising exception of Borda’s rule, for which the picture looks completely different and the probability does not go below 20% in the spatial model. Presumably, this can be explained via Borda’s inability to select the Condorcet winner in this setting, a hypothesis that deserves further study, however. On the contrary, the other rules appear to be benefitting from the fact that the existence of a Condorcet winner becomes very likely under models with high voter interdependence.

Second, as expected, the assumption of IC serves as an upper bound for all other preference models. The results for IAC are not much lower, fostering the impression that IAC could only be an unrealistic upper bound. That this is not the case already follows from the data for the urn model.

Third, the urn model yields much lower values compared to IAC and IC. The absolute numbers, however, are still beyond any acceptable levels (with 23% to 4% for 1,000 voters).

The findings in the empirical data corroborate our experimental findings. In PrefLib the ACP occurs 17 times for Borda, three times for Copeland and exactly once for Maximin as well as Young’s and Dodgson’s rule. In the Netflix data set, where the number of voters is at least 350, Copeland performs much better than the other Condorcet extensions (4,400 compared to 18,470 occurrences for the other Condorcet extensions). Borda’s rule virtually always suffers from the ACP on this data set: there are 54,620 instances of ACPs already when considering profiles that do not have a Condorcet winner (there are 54,650 of such).

7. CONCLUSION

We investigated the likelihood of the CLP and the ACP using Ehrhart theory, computer simulations, and empirical data. The CLP is often cited as a major flaw of some Condorcet extensions such as Dodgson’s rule, Young’s rule, and Maximin. For example, Fishburn regards Condorcet extensions that suffer from the CLP (specifically referring to the three rules mentioned above) as “‘dubious’ extensions of the basic Condorcet criterion” ([16], p. 480). While this is intelligible from a theoretical point of view, our results have shown that the CLP is of virtually no practical concern. The ACP, on the other hand, frequently occurs under various distributional assumptions about the voters’ preferences. The extent to which it is real threat, however, strongly depends on the voting rule, the underlying distribution of preferences, and, surprisingly, the parity of the number of voters. Our main quantitative results for the worst case are summarized in Table 1. Potential future work includes the analysis of other voting paradoxes (such as monotonicity failures or the no-show paradox) and other rules (such as Nanson’s rule or Black’s rule).
Figure 5: Comparison between ACP probabilities for different voting rules under IAC.

Figure 6: Comparison between ACP\textsubscript{Borda}, ACP\textsubscript{Maximin} and ACP\textsubscript{Copeland} for varying preference models in four-alternative elections. The values of ACP\textsubscript{Young} and ACP\textsubscript{Dodgson} are omitted since they strongly resemble the ones of ACP\textsubscript{Maximin}.

<table>
<thead>
<tr>
<th>Paradox</th>
<th>Condorcet loser paradox</th>
<th>Agenda contraction paradox</th>
</tr>
</thead>
<tbody>
<tr>
<td>Model</td>
<td>IAC</td>
<td>IAC</td>
</tr>
<tr>
<td>( n ) ( m )</td>
<td>{1,...,1000}</td>
<td>{50,51}</td>
</tr>
<tr>
<td>Essential set</td>
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<td>0%</td>
</tr>
<tr>
<td>Borda</td>
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<td>0%</td>
</tr>
<tr>
<td>Copeland</td>
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<td>0%</td>
</tr>
<tr>
<td>Dodgson</td>
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<td>Young</td>
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</tr>
<tr>
<td>MaxiMin</td>
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<td>0.15%</td>
</tr>
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</table>

Table 1: Rounded maximal CLP and ACP probabilities which occurred during our simulations.
REFERENCES


