Key question | How many voters are required to obtain a certain majority relation?
--- | ---
This paper | • Powerful SAT-based technique to solve the question of $k$-majority digraphs for arbitrary $k$
 | • Experimental perspective
 | • Seems like very few voters suffice in most cases

Related Work

- Known theoretical insight: Any digraph can be realized as the majority relation of a preference profile with
  - $O(n^3)$ voters (McGarvey, 1953)
  - $O(n \log n)$ voters (Erdős and Moser, 1964) (non-constructive)
  - $\leq n - \log n + 1$ voters (Fiol, 1992)
- Successful Applications of SAT in Social Choice Theory
  - Verification of well-known impossibilities (Tang and Lin, 2009)
  - Automated theorem search for ranking sets of objects (G. and Endriss, 2011)
  - (Im)possibility theorems for strategyproof majoritarian social choice functions (B. and G., 2014)
- Finding preference profiles of given Condorcet dimension (G., 2014)

Preliminaries

- Preference profiles $R = (R_1, R_2, \ldots, R_k)$
  - Finite set of $n$ alternatives, $k$ voters
  - Voters $i \in \{1, 2, \ldots, k\}$ with linear preference relations $R_i$ over alternatives
- Majority relation $\triangleright_R$
  - $a \triangleright_R b$ iff $|\{i : a R_i b\}| > |\{i : b R_i a\}|$
  - Can be represented by a digraph $G$ (we then say: $R$ induces $G$)
- Problem: Given a digraph $G$ and a positive integer $k$, is there a preference profile with $k$ voters that induces $G$? (We then say: $G$ is a $k$-majority digraph)
- Voter complexity of $G$: minimal $k$ such that $G$ is a $k$-majority digraph

Classical Approach: “Characterize and Conquer”

- Lemma. (B. et al., 2013)
  A digraph $(A, \triangleright)$ is a 3-majority digraph if and only if $\triangleright$ is complete and can be partitioned into $\triangleright_1 \cup \triangleright_2 = \triangleright$ such that
    - $(A, \triangleright_1)$ is a 2-majority digraph and
    - $\triangleright_2$ is acyclic
  - Whether $(A, \triangleright)$ is a 2-majority digraph can be checked efficiently (Yannakakis, 1982; Dushnik and Miller, 1941)

SAT-based Approach

- Encode any given problem instance into SAT (propositional logic)

Exhaustive Analysis

- Tournaments that are 3-inducible
  - All tournaments with $n \leq 7$
  (confirming a conjecture by Shepardson and Tovey, 2009)
- Tournaments that are 5-inducible
  - All tournaments with $n \leq 10$
  - All (semi-)regular tournaments with $n \leq 12$
  - Millions of instances of tournaments with sizes $10 \leq n \leq 100$
- Could not find a tournament that is not 5-inducible
  - Only aware of one concrete tournament with $\sim$600 million nodes
  - Existence of a 42-node tournament from pigeonhole principle

Empirical Analysis (PrefLib)

<table>
<thead>
<tr>
<th>Tournament (100% = 354)</th>
<th>Incomplete digraphs (100% = 185)</th>
</tr>
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<tbody>
<tr>
<td>Not feasible: 1</td>
<td>Not feasible: 99</td>
</tr>
<tr>
<td>3-inducible: 57</td>
<td>2-inducible: 25</td>
</tr>
<tr>
<td>5-inducible: 2</td>
<td>4-inducible: 3</td>
</tr>
<tr>
<td>Transitive: 294</td>
<td>6-inducible: 48</td>
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<td>8-inducible: 10</td>
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Stochastic Analysis

- Sampled majority digraphs (with 51 voters) according to 5 different stochastic models (average of 30 runs)

SAT-based Approach

- SAT-based implementation significantly outperforms classical approaches, e.g., running times for $k=3$ depending on $n$:

<table>
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<tr>
<th>Algorithm</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>20</th>
<th>50</th>
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<tr>
<td>SAT</td>
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<td>$&lt; 0.1s$</td>
</tr>
<tr>
<td>2-PARTITION</td>
<td>$&lt; 0.1s$</td>
<td>$&lt; 0.1s$</td>
<td>$2s$</td>
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