Majority Graphs of Assignment Problems and Properties of Popular Random Assignments

Felix Brandt, Johannes Hofbauer, Martin Suderland

Preliminaries

Assignment problem \((A,H,≥)\): \(n\) agents have linear preferences over \(n\) houses

Deterministic assignment or matching \(M\): \(n\) pairwise disjoint \((agent,house)\)-tuples

Random assignment \(p\): Probability distribution over deterministic assignments, \(p_h\) is the probability agent \(a\) receives for house \(h\)

Majority margin \(\phi(M,M')\): Number of agents that prefer \(M\) over \(M'\) minus the number of voters that prefer \(M'\) over \(M\)

Majority graph \(G\): Directed, weighted graph; one vertex for every matching and edge weights are equal to the majority margins

An assignment problem and the induced majority graph.

Popularity: A deterministic assignment \(M\) is popular if \(\phi(M,M') ≥ 0\) for all \(M'\). A random assignment is popular if \(\phi(p,p') = \sum_{M,M'} p(M)p'(M)\phi(M,M') ≥ 0\) for all \(p'\).

Stochastic Dominance: Agent \(a\) prefers \(p\) over \(p'\) if \(\sum_{h \in A} p_{a,h} ≥ \sum_{h' \in A} p'_{a,h'}\) for all \(h'\).

A random assignment satisfies envy-freeness if every agent weakly prefers his assignment to that of every other agent and weak envy-freeness if there is no other agent whose assignment is strictly preferred. An assignment rule satisfies strategyproofness if stating untrue preferences never gives a less or equally preferred assignment.

Decomposition of Assignment Problems

Decomposition: The decomposition of an assignment problem is the maximal partition of the set of houses such that all agents agree on the preferences over the partition’s subsets.

Rotation equivalence: Two decompositions are rotation equivalent if they agree on the agents’ preferences over all houses contained in the same subset and on agents’ preferences over the partition’s subsets modulo a circular rotation.

Theorem: Two assignment problems induce identical majority graphs if and only if their decompositions are rotation equivalent.

Uniqueness of Popular Random Assignments

Existence: Popular deterministic assignments do not have to exist [Gärdenfors, 1975]; existence of popular random assignments is guaranteed by the Minimax theorem [Kavitha et al., 2011].

→ When do we have a unique popular random assignment (PRA)?

1. Identical Preferences

Left shift: The left shift of a random assignment \(p\) is defined by \(L(p)_i ≝ p_i \mod n + 1\).

Theorem: A random assignment \(p\) is popular if and only if \(L(L(p)) = p\).

Corollary: Unique PRA if \(n\) is odd; infinitely many PRAs if \(n\) is even.

2. Experimental Results

Exact numbers for \(n ≤ 4\):

<table>
<thead>
<tr>
<th>(n)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Assignment problems</td>
<td>1</td>
<td>4</td>
<td>216</td>
<td>331 776</td>
</tr>
<tr>
<td>With unique PRA</td>
<td>1</td>
<td>2</td>
<td>54</td>
<td>35 904</td>
</tr>
<tr>
<td>Fraction</td>
<td>1</td>
<td>0.5</td>
<td>0.25</td>
<td>0.108</td>
</tr>
</tbody>
</table>

Computer experiments for \(n ≤ 7\):

Fraction of assignment problems admitting a unique PRA, 10 000 samples

<table>
<thead>
<tr>
<th>(n)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>IC</td>
<td>0.49</td>
<td>0.25</td>
<td>0.11</td>
<td>0.044</td>
<td>0.020</td>
<td>0.0086</td>
<td></td>
</tr>
<tr>
<td>Spatial</td>
<td>0.43</td>
<td>0.26</td>
<td>0.14</td>
<td>0.078</td>
<td>0.040</td>
<td>0.027</td>
<td></td>
</tr>
</tbody>
</table>

3. Selecting Popular Random Assignments

Experiments suggest that the fraction of assignment problems admitting a unique PRA decreases exponentially in \(n\). How can we narrow down the set of PRAs in a meaningful way?

- Minimize envy: Often leads to a unique PRA, however, still many cases with infinitely many PRAs.
- Minimize randomness: Maximizing randomness yields unique PRA while minimizing still admits infinitely many PRAs.
- Barycenter: Natural unique choice satisfying equal treatment of equals, may be infeasible to compute.

Envy-Freeness and Strategyproofness

Previous results: Popularity is incompatible with both envy-freeness and strategyproofness when \(n ≥ 3\) [Aziz et al., 2013].

Theorem: There exist assignment problems for which no PRA satisfies weak envy-freeness when \(n ≥ 5\).

Theorem: No PRA rule satisfies weak strategyproofness when \(n ≥ 7\).

Conclusion and Discussion

Uniqueness: Still many assignment problems without unique PRA.

Individual incentives: Popularity is incompatible with weak envy-freeness and weak strategyproofness, but fares well with respect to efficiency.

Open problem: Does the incomparability with weak strategyproofness also hold when weakening popularity to PC-efficiency?

References: