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Majority Graphs of Assignment Problems and Properties of Popular Random Assignments

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Preliminaries

Assignment problem (A,H,$\succsim$): n agents have linear preferences over n houses

Deterministic assignment or **matching M:** n pairwise disjoint ($agent$,$house$)-tuples

Random assignment p: Probability distribution over deterministic assignments. $p_{a,h}$ is the probability agent a receives for house h

Majority margin $\phi(M,M')$: Number of agents that prefer M over M' minus the number of voters that prefer M' over M

Majority graph G: Directed, weighted graph; one vertex for every matching and edge weights are equal to the majority margins

An assignment problem and the induced majority graph.

Popularity: A deterministic assignment M is popular if $\phi(M,M') \geq 0$ for all M'. A random assignment is popular if $\phi(p,p') = \sum_{(M,M')} p(M)\, p'(M')\, \phi(M,M') \geq 0$ for all p'.

Stochastic Dominance: Agent a prefers p over p' if $\sum_{h \succsim_a h'} p_{a,h} \geq \sum_{h \succsim_a h'} p'_{a,h}$ for all h'.

A random assignment satisfies envy-freeness if every agent weakly prefers his assignment to that of every other agent and weak envy-freeness if there is no other agent whose assignment is strictly preferred.

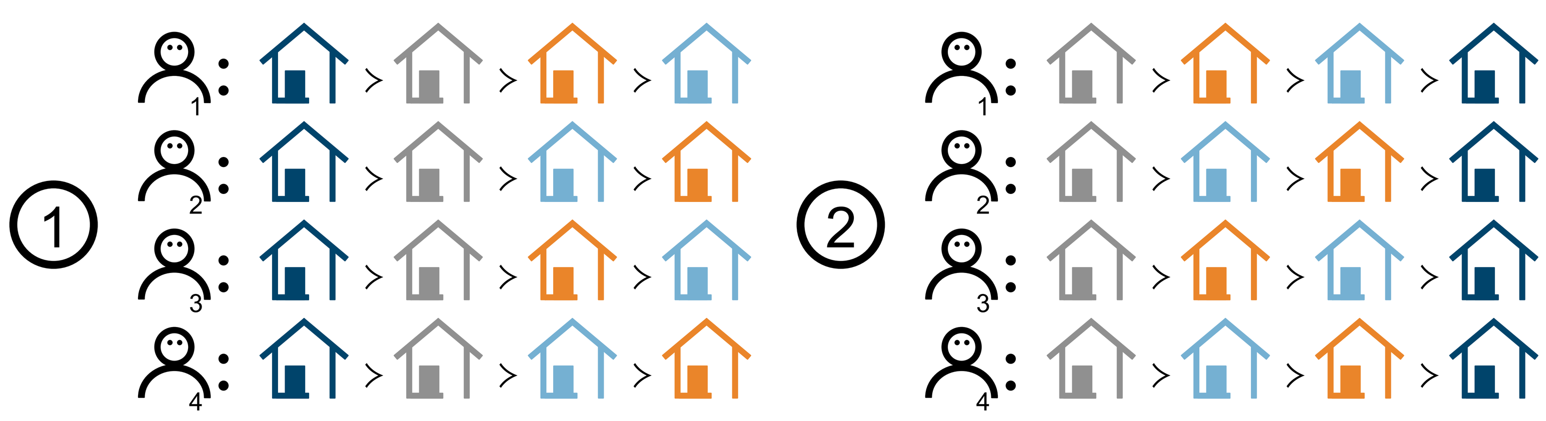
An assignment rule satisfies strategyproofness if stating untrue preferences always gives a less or equally preferred assignment. It satisfies weak strategyproofness if stating untrue preferences never gives a more preferred assignment.

Decomposition of Assignment Problems

Decomposition: The decomposition of an assignment problem is the maximal partition of the set of houses such that all agents agree on the preferences over the partition's subsets.

Rotation equivalence: Two decompositions are rotation equivalent if they agree on the agents' preferences over all houses contained in the same subset and on agents' preferences over the partition's subsets modulo a circular rotation.

Theorem: Two assignment problems induce identical majority graphs if and only if their decompositions are rotation equivalent.



Three assignment problems with identical decompositions. Assignment problems 1 and 2 are rotation-equivalent and thus induce the same majority graph.

Uniqueness of Popular Random Assignments

Existence: Popular deterministic assignments do not have to exist [Gärdenfors, 1975]; existence of popular random assignments is guranteed by the Minimax theorem [Kavitha et al., 2011].

→ When do we have a unique popular random assignment (PRA)?

1. Identical Preferences

Left shift: The left shift of a random assignment p is defined by $L(p)_{i,j} = p_{i,j \bmod n+1}$.

Theorem: A random assignment p is popular if and only if $L(L(p)) = p$.

Corollary: Unique PRA if n is odd; infinitely many PRAs if n is even.

2. Experimental Results

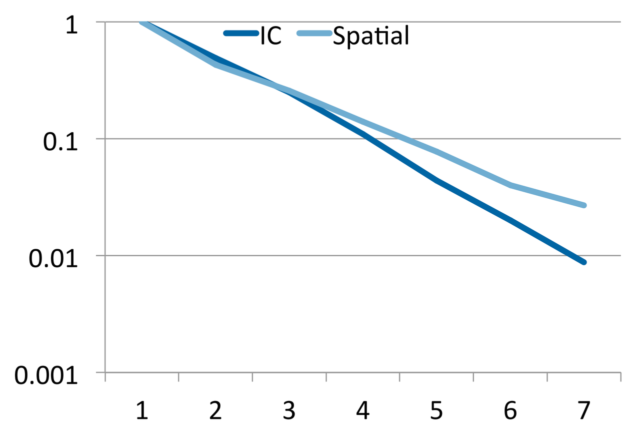
Exact numbers for $n \leq 4$:

n	1	2	3	4
Assignment problems	1	4	216	331 776
With unique PRA	1	2	54	35 904
Fraction	1	0.5	0.25	0.108

Computer experiments for $n \leq 7$:

Fraction of assignment problems admitting a unique PRA, 10 000 samples

n	1	2	3	4	5	6	7
IC	1	0.49	0.25	0.11	0.044	0.020	0.0088
Spatial	1	0.43	0.26	0.14	0.078	0.040	0.027



3. Selecting Popular Random Assignments

Experiments suggest that the fraction of assignment problems admitting a unique PRA decreases exponentially in n. How can we narrow down the set of PRAs in a meaningful way?

- **Minimize envy:** Often leads to a unique PRA, however, still many cases with infinitely many PRAs.
- **Minimize randomness:** Maximizing randomness yields unique PRA while minimizing still admits infinitely many PRAs.
- **Barycenter:** Natural unique choice satisfying equal treatment of equals, may be infeasible to compute.

Envy-Freeness and Strategyproofness

Previous results: Popularity is incompatible with both envy-freeness and strategyproofness when $n \geq 3$ [Aziz et al., 2013].

Theorem: There exist assignment problems for which no PRA satisfies weak envy-freeness when $n \geq 5$.

Theorem: No PRA rule satisfies weak strategyproofness when $n \geq 7$.

Conclusion and Discussion

Uniqueness: Still many assignment problems without unique PRA.

Individual incentives: Popularity is incompatible with weak envy-freeness and weak strategyproofness, but fares well with respect to efficiency.

Open problem: Does the incompatibility with weak strategyproofness also hold when weakening popularity to PC-efficiency?

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