Majority Graphs of Assignment Problems and Properties of Popular Random Assignments

Felix Brandt Johannes Hofbauer Martin Suderland

Abstract

Randomized mechanisms for assigning objects to individual agents have received increasing attention by computer scientists as well as economists. In this paper, we study a property of random assignments, called *popularity*, which corresponds to the well-known notion of Condorcet-consistency in social choice theory. Our contribution is threefold. First, we define a simple condition, which can be checked in polynomial time, that characterizes whether two assignment problems induce the same majority graph. Secondly, we analytically and experimentally investigate the uniqueness of popular random assignments. Finally, we prove that popularity is incompatible with very weak notions of both strategyproofness and envy-freeness. This settles two open problems by Aziz et al. (2013) and reveals an interesting tradeoff between social and individual goals in random assignment.

1 Introduction

Assigning objects to individual agents is a fundamental problem that has received considerable attention by computer scientists as well as economists. In its simplest form, the problem is known as the *assignment problem*, the *house allocation problem*, or *two-sided matching with one-sided preferences*. Formally, an assignment problem concerns a set of agents $\mathcal{A}$, a set of houses $\mathcal{H}$, and the agents' (ordinal) preferences over the houses $\succsim$. For simplicity, it is often assumed that $\mathcal{A}$ and $\mathcal{H}$ are of equal size. The central question is how to assign exactly one house to each agent. An important assumption in this setting is that monetary transfers between the agents are not permitted.[1] Possible applications include assigning dormitories to students, jobs to applicants, rooms to housemates, processor time slots to jobs, parking spaces to employees, offices to workers, kidneys to patients, etc.

In this paper, we focus on the notion of popularity due to Gärdenfors (1975). An assignment is popular if there is no other assignment that is preferred by a majority of the agents. Popular assignments thus correspond to the well-studied notion of (weak) Condorcet winners in social choice theory. Unpopular assignments are unstable in the sense that a proposal to move to another assignment would be supported by a majority of the agents. Unfortunately, the assignment setting is not immune to the Condorcet paradox and there are assignment problems that do not admit a popular assignment (Gärdenfors, 1975). However, Kavitha et al. (2011) have shown that existence *can* be guaranteed when allowing randomization and appropriately extending the definition of popular assignments to popular *random* assignments. A random assignment is popular if there does not exist another random assignment that is preferred by an *expected* majority of agents. Randomization is a natural and widespread technique to establish *ex ante* fairness in assignment. It is, for example, easily seen that every deterministic assignment violates 'equal treatment of equals' when all agents have identical preferences. As Hofstee (1990) notes, " [...] if scarcity arises, lottery is the only just procedure (barring [...] the dividing of goods or their denial

[1] Monetary transfers may be impossible or highly undesirable, as is the case if houses are public facilities provided to low-income people. There are a number of settings such as voting, kidney exchange, or school choice in which money cannot be used as compensation due to practical, ethical, or legal constraints (Klaus et al., 2016; Roth, 2015; Manlove, 2013).

to everyone, neither of which is appropriate in the present context) [...]".

Popular random assignments satisfy a particularly strong notion of economic efficiency called PC-efficiency, unmatched by other common assignment rules, and can be efficiently computed via linear programming. The formulation as a linear program allows one to easily accommodate for additional constraints (such as equal treatments of equals or assignment quotas) (Aziz et al., 2013). As popularity only takes into account how many agents prefer one assignment to another, it suffices to consider the pairwise majority comparisons between all possible assignments in order to determine popular assignments and popular random assignments. This information can easily be captured by a weighted graph, the *majority graph*, where the set of vertices equals the set of possible assignments and edge weights are determined via majority comparisons. Such graphs are routinely studied in social choice theory. In fact, as pointed out by Aziz et al. (2013), random assignment is 'merely' a special case of the general social choice setting and popular random assignments correspond to so-called maximal lotteries in general social choice (see, also, Brandl et al., 2016b).

In social choice theory, it is well-known that all weighted majority graphs can be induced by some configuration of preferences (McGarvey, 1953; Debord, 1987). Majority graphs induced by assignment problems, on the other hand, constitute only a small subclass of all possible majority graphs. For example, it is easily seen that the number of vertices—i.e., the number of possible assignments—is always $n!$ where n is the number of agents and houses. On top of that, assignment problems impose certain structural restrictions on the corresponding majority graphs.

Contributions. The contribution of this paper is threefold. First, we investigate the relationship between assignment problems and majority graphs. More precisely, we define a natural *decomposition* of assignment problems and show that two assignment problems induce identical majority graphs if and only if their decompositions are *rotation equivalent*. Our proof is constructive in the sense that it is possible to check whether a given majority graph can be induced by an assignment problem. If this the case, we can provide all assignment problems inducing the given majority graph.

We then study the uniqueness of popular random assignments. We prove that if all agents share the same preferences, the resulting popular random assignment is unique if there is an odd number of agents and there are infinitely many popular random assignments if the number of agents is even. Moreover, we provide a sufficient condition for the existence of a unique popular random assignment under unrestricted preferences. Here, computer experiments suggest that the number of assignment problems giving rise to a unique popular random assignment decreases exponentially with the number of agents. This is in contrast to the general social choice setting where maximal lotteries, a generalization of popular random assignments, are known to be almost always unique (see, e.g., Brandl et al., 2016b).

Finally, we are able to answer two open questions posed by Aziz et al. (2013). Aziz et al. show that popularity is incompatible with strong notions of strategyproofness and envy-freeness. We prove that theses impossibilities still hold when considering the significantly weaker notions of weak strategyproofness and weak envy-freeness when the number of agents is at least five and seven, respectively.

Related work. In the context of deterministic assignments, popularity was first considered by Gärdenfors (1975) who also showed that popular assignments need not always exist. Mahdian (2006) proved an interesting threshold for the existence of popular assignments: if there are n agents and the number of houses exceeds αn with $\alpha \approx 1.42$, then the probability that there is a popular assignment converges to 1 as n goes to infinity. Abraham et al. (2007) proposed a polynomial-time algorithm that can both verify whether a popular assignment exists and find a popular assignment of maximal cardinality if it exists.

A closely related line of research considers popularity in marriage markets, i.e., two-sided matching with two-sided preferences. In this setting, every stable matching is also popular. Kavitha and Nasre (2009) further reduced the set of popular assignments by considering "optimal" popular assignments. Biró et al. (2010) defined a strong variant of popularity and provided algorithmic results for marriage markets and the more general roommate markets. Huang and Kavitha (2011) studied marriage markets with possible inacceptabilities and the problem of finding popular matchings of maximal cardinality. The tradeoff between popularity and cardinality of a matching was investigated by Kavitha (2014) who also provided bounds on the size of popular matchings. Cseh et al. (2015) considered the complexity of finding popular matchings if one side is allowed to express indifferences in its preferences.

Finally, popular *random* assignments were introduced by Kavitha et al. (2011). Aziz et al. (2013) initiated the study of axiomatic properties such as efficiency, fairness, and strategyproofness of popular random assignments.

The two most-studied random assignment rules in the literature are *random serial dictatorship (RSD)* and the *probabilistic serial rule (PS)* (see, e.g., Bogomolnaia and Moulin, 2001), both of which may result in unpopular outcomes (Aziz et al., 2013). See Section 6 for a more detailed discussion of RSD and PS.

2 Preliminaries

An *assignment problem* is a triple $(\mathcal{A}, \mathcal{H}, \succsim)$ consisting of a set of agents $\mathcal{A}$, a set of houses $\mathcal{H}$, $|\mathcal{A}| = |\mathcal{H}| = n$, and a preference profile $\succsim = (\succsim_1, \ldots, \succsim_n)$ containing preferences $\succsim_a \subseteq \mathcal{H} \times \mathcal{H}$ for all $a \in \mathcal{A}$. We assume individual preferences $\succsim_a$ to be antisymmetric, complete and transitive. $\succsim_a$ is denoted as a comma-separated list, i.e., $a\colon h_1, h_2, h_3$ means $h_1 \succsim_a h_2 \succsim_a h_3$. $\succ_a$ represents the strict part of $\succsim_a$, i.e., $h \succ_a h'$ if $h \succsim_a h'$ but not $h' \succsim_a h$. When comparing sets of houses H, H', we write $H \succsim_a H'$ if $h \succsim_a h'$ for all $h \in H, h' \in H'$.

A *deterministic assignment* (or *matching*) M is a subset of $\mathcal{A} \times \mathcal{H}$ such that $|M| = n$ and all tuples in M are pairwise disjoint. We write $M(a) = h$ and $M(h) = a$ if $(a, h) \in M$. Let $\mathcal{M}(n)$ denote the set of all matchings of size n.

Denote by $[k]$ the set of all natural numbers up to k, i.e., $[k] = \{1, \ldots, k\}$. A *random assignment* is a matrix $p \in \mathbb{R}^{n \times n}$ with $p_{i,j} \geq 0$ for all $i, j \in [n]$, $\sum_{i \in [n]} p_{i,j} = 1$ for all $j \in [n]$, and $\sum_{j \in [n]} p_{i,j} = 1$ for all $i \in [n]$. We interpret $p_{i,j}$ as the probability with which agent a_i receives house h_j. Denote by $\mathcal{R}(n)$ the set of all random assignments of size n and by $p_{[i]}$ the vector $(p_{i,1}, \ldots, p_{i,n})$. Note that by the Birkhoff-von Neumann theorem, we have that every probability distribution over deterministic assignments induces a unique random assignment while every random assignment can be written as a probability distribution over deterministic assignments (see, e.g., Kavitha et al., 2011).

A *random assignment rule* f is a function that returns a random assignment p for all assignment problems $(\mathcal{A}, \mathcal{H}, \succsim)$.

For two deterministic assignments M, M' and an agent a with preferences $\succsim_a$ we define

$$\phi_{\succsim_a}(M(a), M'(a)) = \begin{cases} 1 & \text{if } M(a) \succ_a M'(a), \\ -1 & \text{if } M'(a) \succ_a M(a), \\ 0 & \text{otherwise.} \end{cases}$$

With slight abuse of notation, we also use $\phi_{\succsim_a}(M, M') = \phi_{\succsim_a}(M(a), M'(a))$ whenever suitable. For an assignment problem $(\mathcal{A}, \mathcal{H}, \succsim)$, denote by $\phi_{\succsim}(M, M')$ the natural extension of ϕ to all agents in $\mathcal{A}$, $\phi_{\succsim}(M, M') = \sum_{a \in \mathcal{A}} \phi_{\succsim_a}(M, M')$.

When considering random assignments, we define

$$\phi_{\succsim}(p, p') = \sum_{i \in [n]} \sum_{j, j' \in [n]} p_{i,j} p'_{i,j'} \phi_{\succsim_{a_i}}(h_j, h_{j'}).$$

Observe that ϕ is skew-symmetric, i.e., $\phi_{\succsim}(M, M') = -\phi_{\succsim}(M', M)$ as well as $\phi_{\succsim}(p, p') = -\phi_{\succsim}(p', p)$.

In the following, we formally introduce the concepts of popularity, majority graphs, stochastic dominance, envy-freeness, and strategyproofness.

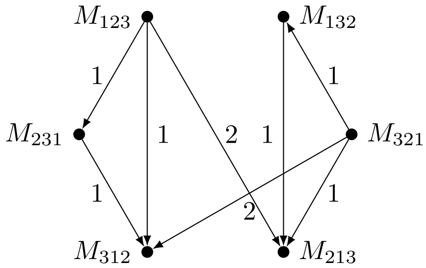
Popularity. Let $(\mathcal{A}, \mathcal{H}, \succsim)$ be an assignment problem. Then, a deterministic assignment M is *popular* if $\phi_{\succsim}(M, M') \geq 0$ for all $M' \in \mathcal{M}(n)$. Correspondingly, a random assignment p is *popular* if $\phi_{\succsim}(p, p') \geq 0$ for all $p' \in \mathcal{R}(n)$. Popular deterministic assignments need not always exist but the Minimax Theorem implies that every assignment problem admits at least one popular *random* assignment (Kavitha et al., 2011).

Majority Graph. For a given assignment problem $(\mathcal{A}, \mathcal{H}, \succsim)$ we define the corresponding *majority graph* $G = (V, E, w)$ by letting the set of vertices be the set of all possible deterministic assignments and setting the edge weights according to the agents' preferences over these assignments, i.e., $V = \mathcal{M}(n)$, $E = \mathcal{M}(n) \times \mathcal{M}(n)$, and $w(M, M') = \phi_{\succsim}(M, M')$. We consequently have that $|V| = n!$ and $|E| = 1/2\, n!(n! - 1)$.

Different assignment problems may induce identical majority graphs. Consider for instance $(\mathcal{A}, \mathcal{H}, \succsim)$ and $(\mathcal{A}, \mathcal{H}, \succsim')$ with $\mathcal{A} = \{a_1, a_2, a_3\}$, $\mathcal{H} = \{h_1, h_2, h_3\}$, and $\succsim$ and $\succsim'$ as given below.

$$\succsim \; = \begin{array}{ll} a_1: & h_1, h_2, h_3 \\ a_2: & h_2, h_1, h_3 \\ a_3: & h_1, h_2, h_3 \end{array} \qquad \succsim' = \begin{array}{ll} a_1: & h_3, h_1, h_2 \\ a_2: & h_3, h_2, h_1 \\ a_3: & h_3, h_1, h_2 \end{array}$$

For both assignment problems we obtain identical majority graphs $G = (V, E, w)$ with $V = \mathcal{M}(3)$, $E = V \times V$ and $w(M, M') = \phi_{\succsim}(M, M') = \phi_{\succsim'}(M, M')$ for all $M, M' \in V$:



$$\begin{aligned} M_{123} &= \{(a_1, h_1), (a_2, h_2), (a_3, h_3)\} \\ M_{231} &= \{(a_1, h_2), (a_2, h_3), (a_3, h_1)\} \\ M_{312} &= \{(a_1, h_3), (a_2, h_1), (a_3, h_2)\} \\ M_{132} &= \{(a_1, h_1), (a_2, h_3), (a_3, h_2)\} \\ M_{321} &= \{(a_1, h_3), (a_2, h_2), (a_3, h_1)\} \\ M_{213} &= \{(a_1, h_2), (a_2, h_1), (a_3, h_3)\} \end{aligned}$$

Recall that $w(M, M') = \phi_{\succsim}(M, M') = -\phi_{\succsim}(M', M) = -w(M', M)$. Edges with negative weight as well as edges with weight zero are omitted for the sake of clarity.

Note that in order to determine which random assignments are popular for a given assignment problem $(\mathcal{A}, \mathcal{H}, \succsim)$, it suffices to consider the corresponding majority graph G. All information relevant for the computation—$\mathcal{M}(n)$ and $\phi_{\succsim}(M, M') = w(M, M')$ for all $M, M' \in \mathcal{M}(n)$—can be obtained from G. For the majority graph given above, and thereby for assignment problems $(\mathcal{A}, \mathcal{H}, \succsim)$ and $(\mathcal{A}, \mathcal{H}, \succsim')$, M_{123} and M_{321} are the only popular matchings. Any randomization between these matchings constitutes a popular random assignment.

Stochastic Dominance. So far, agents are solely endowed with an ordinal preference relation that allows the comparison of deterministic assignments, but not that of random assignments. We therefore propose to extend preferences over houses to preferences over probability distributions based on *stochastic dominance (SD)*. We have that $p_{[i]} \succsim^{\mathrm{SD}}_{a_i} p'_{[i]}$ if $\sum_{h_j \in \mathcal{H}, h_j \succsim_{a_i} h} p_{i,j} \geq \sum_{h_j \in \mathcal{H}, h_j \succsim_{a_i} h} p'_{i,j}$ for all $h \in \mathcal{H}$. In this case, we say that a_i weakly SD-prefers p to p'. With slight abuse of notation, we sometimes also write $p \succsim^{\mathrm{SD}}_{a_i} p'$.

This preference extension is of special importance as $p \succsim^{\mathrm{SD}}_{a_i} p'$ if and only if p yields more expected utility than p' with respect to all von Neumann-Morgenstern utility functions consistent with a_i's ordinal preferences $\succsim_{a_i}$ (see, e.g., Bogomolnaia and Moulin, 2001; Katta and Sethuraman, 2006).

Given an assignment problem $(\mathcal{A}, \mathcal{H}, \succsim)$, a random assignment $p \in \mathcal{R}(n)$ is *SD-efficient* if there is no $p' \in \mathcal{R}(n)$ such that $p' \succsim^{\mathrm{SD}}_{a} p$ for all $a \in \mathcal{A}$ and $p' \succ^{\mathrm{SD}}_{a'} p$ for some $a' \in \mathcal{A}$.

While stochastic dominance is the most common preference extension, there are also other natural preference extensions that can be used to define variants of efficiency, strategyproofness, and envy-freeness. In particular, there is a weakening of stochastic dominance called bilinear dominance (BD) and a strengthening of SD called pairwise comparison (PC). We refer to Aziz et al. (2014, 2015) for more details.

Strategyproofness and Envy-freeness. Strategyproofness requires that stating one's true preferences is always at least as good as misrepresenting them, while envy-freeness requires that every agent weakly prefers his allocation to that of all others.

Formally, an assignment rule f is *strategyproof* if for all $(\mathcal{A}, \mathcal{H}, \succsim)$, $a \in \mathcal{A}$, and $(\mathcal{A}, \mathcal{H}, \succsim')$ such that $\succsim_{a'} = \succsim'_{a'}$ for all $a' \in \mathcal{A} \setminus \{a\}$ we have that $f(\mathcal{A}, \mathcal{H}, \succsim) \succsim^{\mathrm{SD}}_{a} f(\mathcal{A}, \mathcal{H}, \succsim')$. Since the SD preference extension only yields an incomplete preference relation over lotteries, one can also define a weaker notion of strategyproofness that merely requires that no agent benefits by misrepresenting his preferences. (In other words, a manipulation only counts as a manipulation if it leads to more expected utility *for all* expected utility representations of the agent's ordinal preferences.) f satisfies *weak strategyproofness* if for all $(\mathcal{A}, \mathcal{H}, \succsim)$ and $a \in \mathcal{A}$ there is no $(\mathcal{A}, \mathcal{H}, \succsim')$ with $\succsim_{a'} = \succsim'_{a'}$ for all $a' \in \mathcal{A} \setminus \{a\}$ such that $f(\mathcal{A}, \mathcal{H}, \succsim') \succ^{\mathrm{SD}}_{a} f(\mathcal{A}, \mathcal{H}, \succsim)$. Note that what we call strategyproofness and weak strategyproofness are sometimes also referred to as *strong strategyproofness* and *strategyproofness* in the literature.

A random assignment $p \in \mathcal{R}(n)$ satisfies *envy-freeness* if $p_{[i]} \succsim^{\mathrm{SD}}_{a_i} p_{[j]}$ for all agents $a_i \in \mathcal{A}$ and $j \in [n] \setminus \{i\}$. Similarly as above, one can define a weaker notion of envy-freeness. p satisfies *weak envy-freeness* if there is no agent a_i such that $p_{[j]} \succ^{\mathrm{SD}}_{a_i} p_{[i]}$ for some $j \in [n]$.

3 Decomposition of Assignment Problems

This section focuses on the question under which conditions two assignment problems induce the same majority graph. Recall that the set of popular random assignments depends on the majority graph only, i.e., identical induced majority graphs imply identical sets of popular random assignments. We will provide an easily verifiable condition that holds if and only if two assignment problems have the same majority graphs. Furthermore, given a majority graph, it is possible to determine all assignment problems that induce this graph.

Given an assignment problem $(\mathcal{A}, \mathcal{H}, \succsim)$, we say that $(\mathcal{A}, H_k, \succsim^k)_{k \in [m]}$ is a *decomposition* of $(\mathcal{A}, \mathcal{H}, \succsim)$, if (*i*)–(*iii*) hold and there is no $m' > m$ for which (*i*)–(*iii*) can also be satisfied:

(*i*) $\dot{\bigcup}_{k \in [m]} H_k = \mathcal{H}$ with $H_k \neq \emptyset$ for all $k \in [m]$,

(*ii*) $h \succsim_a h'$ implies $h \succsim^k_a h'$ for all $k \in [m], h, h' \in H_k, a \in \mathcal{A}$, and

(*iii*) $h \succ_a h'$ for all $h \in H_k, h' \in H_{k'}, 1 \leq k < k' \leq m, a \in \mathcal{A}$.

By decomposing $(\mathcal{A}, \mathcal{H}, \succsim)$, we thus partition $\mathcal{H}$ into nonempty subsets such that all agents prefer houses contained in H_k to houses in $H_{k'}$ if and only if $k < k'$. At the same time, agents' preferences over houses contained in the same H_k stay the same. It is easy to see that for every assignment problem there exists a unique decomposition. Note that if for this decomposition it holds that $m = 1$, we will name it the trivial decomposition.

Let $(\mathcal{A}, \mathcal{H}, \succsim)$ and $(\mathcal{A}, \mathcal{H}, \succsim')$ be two assignment problems and $(\mathcal{A}, H_k, \succsim^k)_{k\in[m]}$ and $(\mathcal{A}, H_{k'}, \succsim'^{k'})_{k'\in[m']}$ their corresponding decompositions. We say that $(\mathcal{A}, H_k, \succsim^k)_{k\in[m]}$ and $(\mathcal{A}, H_{k'}, \succsim'^{k'})_{k'\in[m']}$ are *rotation equivalent* if there exists $d \in [m]$ such that $\succsim^k = \succsim'^{((k+d-1) \mod m)+1}$ for all $k \in [m]$. For the sake of readability, we hereafter use mod_1 defined by $k \mod_1 k' = ((k-1) \mod k') + 1$. Rotation equivalence can thus be rewritten as $\succsim^k = \succsim'^{(k+d) \mod_1 m}$. Intuitively, two decompositions are rotation equivalent if they agree on the partitioning of $\mathcal{H}$, agents' preferences within the partition's subsets and the ordering of those subsets *modulo m*.

For better illustration of the concept consider the following brief example with four agents $\mathcal{A} = \{a_1, \dots, a_4\}$, four houses $\mathcal{H} = \{h_1, \dots, h_4\}$ and preference profiles $\succsim$, $\succsim'$, and $\succsim''$.

$$\succsim = \begin{array}{l} a_1\colon\ h_1,\ h_2,\ h_3,\ h_4 \\ a_2\colon\ h_1,\ h_2,\ h_4,\ h_3 \\ a_3\colon\ h_1,\ h_2,\ h_3,\ h_4 \\ a_4\colon\ h_1,\ h_2,\ h_4,\ h_3 \end{array} \qquad \succsim' = \begin{array}{l} a_1\colon\ h_2,\ h_3,\ h_4,\ h_1 \\ a_2\colon\ h_2,\ h_4,\ h_3,\ h_1 \\ a_3\colon\ h_2,\ h_3,\ h_4,\ h_1 \\ a_4\colon\ h_2,\ h_4,\ h_3,\ h_1 \end{array} \qquad \succsim'' = \begin{array}{l} a_1\colon\ h_1,\ h_3,\ h_4,\ h_2 \\ a_2\colon\ h_1,\ h_4,\ h_3,\ h_2 \\ a_3\colon\ h_1,\ h_3,\ h_4,\ h_2 \\ a_4\colon\ h_1,\ h_4,\ h_3,\ h_2 \end{array}$$

We see that $\mathcal{H}$ is partitioned into the sets $\{h_1\}, \{h_2\}, \{h_3, h_4\}$ in all three decompositions with agents' preferences over the houses within those sets being identical in all cases. For better exposition, dotted lines are added in between the components. However, only the decompositions of $(\mathcal{A}, \mathcal{H}, \succsim)$ and $(\mathcal{A}, \mathcal{H}, \succsim')$ are rotation equivalent.

Our first theorem links rotation equivalent decompositions to identical majority graphs.

Theorem 1. *Let $(\mathcal{A}, \mathcal{H}, \succsim)$ and $(\mathcal{A}, \mathcal{H}, \succsim')$ be two assignment problems that induce majority graphs G and G', respectively. Then, $G = G'$ if and only if the decompositions of $(\mathcal{A}, \mathcal{H}, \succsim)$ and $(\mathcal{A}, \mathcal{H}, \succsim')$ are rotation equivalent.*

Due to space constraints, the proof of Theorem 1 is given in the Appendix. Since the proof of the direction from left to right is constructive, it is easy to develop an algorithm that, given a majority graph, finds all assignment problems that induce this graph. This algorithm can also answer the question whether a given graph is induced by an assignment problem.

Two consequences of Theorem 1 are that the set of popular (random) assignments is invariant under component-wise rotation and that preference profiles that only admit the trivial decomposition induce a unique majority graph.

We conclude this section with an observation regarding the number of different majority graphs that can be induced by assignment problems of size n. Directly counting the number of majority graphs is not possible because we lack a suitable characterization thereof. Still, by Theorem 1, we know that two assignment problems induce identical majority graphs if and only if their decompositions are rotation equivalent. We make use of this correspondence and actually sum up the number of assignment problems of size n which do not have rotation equivalent decompositions. This number can be computed to be

$$N(n) = \sum_{i\in[n]} \frac{(-1)^{i+1}}{i} \left(\sum_{\substack{x_0,\dots,x_i\in\mathbb{N}_0 \\ 0=x_0<\dots<x_i=n}} \prod_{j\in[i]} \binom{n - x_{j-1}}{x_j - x_{j-1}} \cdot ((x_j - x_{j-1})!)^n \right).$$

n	$N(n)$	$n!^n$	$N(n)/n!^n$
1	1	1	1
2	3	4	0.75
3	194	216	0.898
4	329 898	331 776	0.994
5	24 841 082 904	24 883 200 000	0.998

Table 1: Number of inducible majority graphs relative to the number of assignment problems of size n.

It turns out that $N(n)$ is roughly equivalent to $n!^n$. See Table 1 for the exact values of $N(n)$ and $n!^n$ up to $n = 5$. Note that the total number of assignment problems of size n is exactly $n!^n$, which implies that a nontrivial decomposition is impossible for a vast majority of profiles.

As most assignment problems hence induce different majority graphs, the question remains which ratio of the possible majority graphs may be induced. Regarding majority graphs in the context of social choice, observe that the total number of directed, weighted graphs (V, E, w) with edge weights $|w(e)| \leq n$ for all $e \in E$ is $(2n+1)^{1/2\ n!(n!-1)}$. The fraction of those graphs that can indeed be induced by an assignment problem of size n is comparatively small, it can easily be upper-bounded by $n!^n/n^{n!}$. Given the many interdependencies of edge weights due to the fact that agents only have preferences over n houses but we have $n!$ vertices, this result confirms the naive intuition that most majority graphs cannot be induced by an assignment problem.

In this context, it is worth noting that the empty graph, i.e., the majority graph with $w(e) = 0$ for all $e \in E$ cannot be induced by any assignment problem of size $n > 2$. This can easily be seen when considering two matchings $M, M' \in \mathcal{M}(n)$ where $M(h) = M'(h')$, $M(h') = M'(h'')$, $M(h'') = M'(h)$, and $M(h''') = M'(h''')$ for all $h''' \in \mathcal{H} \setminus \{h, h', h''\}$. Here, $\phi_{\succsim}(M, M') \in \{3, 1, -1, -3\}$. We consequently obtain that whenever $n > 2$, it is impossible that all random assignments are popular, or, put differently, popularity always imposes a restriction on the set of random assignments.

4 Uniqueness of Popular Random Assignments

We now want to have a closer look at popularity. As already briefly discussed before, popular deterministic assignments need not always exist (Gärdenfors, 1975). When considering random assignments instead, Kavitha et al. (2011) have shown that there always is at least one popular random assignment. It is easy to show that the set of popular random assignments is convex, i.e., if there are at least two different popular random assignments, there are infinitely many.

Hence, a natural question is which preference profiles admit a unique popular random assignment. In other words, under which circumstances does popularity already restrict the set of desirable random assignments to a singleton?

In order to tackle this question, we first focus on the setting where all agents have identical preferences, and completely characterize the set of popular random assignments for arbitrary n. Such situations are not particularly unlikely, for instance if objects are consistently evaluated by size or monetary value (see, also, Bogomolnaia and Moulin, 2002).

For this restricted case we are able to show that there is a unique popular random assignment if n is odd and infinitely many if n is even. We will use this result to provide a sufficient (yet not necessary) condition for uniqueness in the case of general preferences. To get a better idea about the frequency of unique popular random assignments, we will conclude the section by presenting the results of computer experiments.

4.1 Identical Preferences

In this subsection we consider assignment problems $(\mathcal{A}, \mathcal{H}, \succsim)$ where all n agents have identical preferences. Without loss of generality let us assume agents always prefer houses with a lower index, i.e., $h_k \succ_a h_{k'}$ for all $h_k,\ h_{k'} \in \mathcal{H}, 1 \leq k < k' \leq n$, and $a \in \mathcal{A}$. As the preferences $\succsim$ only depend on the number of agents in this subsection, we simplify notation by writing $\phi(p, p') = \phi_{\succsim}(p, p')$.

The upcoming theorem builds on a *left shift* of probabilities. The left shift function $L(p)$ maps the probability an agent a receives for house h_k to the probability he receives for the next less preferred house h_{k+1}. We define the function $L\colon \mathcal{R}(n) \to \mathcal{R}(n)$, $(L(p))_{i,j} = p_{i,(j \mod n)+1}$.

It holds that the set of all popular random assignments consists of exactly those random assignments, that are invariant under double application of L.

Theorem 2. *Let $(\mathcal{A}, \mathcal{H}, \succsim)$ be an assignment problem where all agents have identical preferences. Then, a random assignment $p \in \mathcal{R}(n)$ is popular if and only if $L(L(p)) = p$.*

The complete proof can be found in the Appendix. The following corollary precisely characterizes the set of popular random assignments for the case of identical preferences.

Corollary 1. *Let $(\mathcal{A}, \mathcal{H}, \succsim)$ be a random assignment problem where all agents have identical preferences. If n is odd, there exists a unique popular random assignment p, namely $p_{i,j} = 1/n$ for all $i, j \in [n]$. If n is even, there exist multiple popular random assignments, namely* $\mathrm{conv}(E(n))$ *with*

$$\begin{aligned} E(n) &= \{e^I \in \mathcal{R}(n) \colon I \subseteq [n], |I| = n/2\} \\ e^I_{i,j} &= \begin{cases} 2/n & \text{if either } i \in I \text{ and } j \text{ odd, or } i \notin I \text{ and } j \text{ even,} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

For illustration consider for instance a situation where six agents have identical preferences over six houses. By Theorem 2 we know that a random assignment $p \in \mathcal{R}(6)$ is popular if and only if $L(L(p)) = p$. A possible representative satisfying this condition is p as given below.

$$\begin{aligned} a_1&\colon\ h_1, h_2, h_3, h_4, h_5, h_6 \\ a_2&\colon\ h_1, h_2, h_3, h_4, h_5, h_6 \\ a_3&\colon\ h_1, h_2, h_3, h_4, h_5, h_6 \\ a_4&\colon\ h_1, h_2, h_3, h_4, h_5, h_6 \\ a_5&\colon\ h_1, h_2, h_3, h_4, h_5, h_6 \\ a_6&\colon\ h_1, h_2, h_3, h_4, h_5, h_6 \end{aligned} \qquad p = \begin{pmatrix} 1/3 & 0 & 1/3 & 0 & 1/3 & 0 \\ 1/4 & 1/12 & 1/4 & 1/12 & 1/4 & 1/12 \\ 0 & 1/3 & 0 & 1/3 & 0 & 1/3 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 1/12 & 1/4 & 1/12 & 1/4 & 1/12 & 1/4 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \end{pmatrix}$$

With respect to Corollary 1, we have that $p = 1/2\, e^{\{1,2,6\}} + 1/4\, e^{\{1,2,4\}} + 1/4\, e^{\{1,4,5\}}$. Also note that $|E(6)| = 20$ and every convex combination over those random assignments is popular.

4.2 Separable Assignment Problems

While we are unable to obtain a characterization of popular random assignments similar to the one above for the case of general preferences, we can give a sufficient condition for assignment problems, which, if satisfied, guarantees uniqueness of popular random assignments.

We say that an assignment problem $(\mathcal{A}, \mathcal{H}, \succsim)$ is *separable* if we can partition the set of agents $\mathcal{A} = A_1 \dot\cup \ldots \dot\cup A_m$ and the set of houses $\mathcal{H} = H_1 \dot\cup \ldots \dot\cup H_m$ such that $|A_k| = |H_k|$ for all $k \in [m]$ and moreover $H_k \succ_a \mathcal{H} \backslash H_k$ for all $k \in [m], a \in A_k$. Given a separable assignment problem $(\mathcal{A}, \mathcal{H}, \succsim)$ we call $(A_k, H_k, \succsim^k)$, $k \in [m]$, its *reduced assignment problems.* Note that $\succsim^k$ is defined as $\succsim$ restricted to A_k and H_k, or, more formally, $h \succsim_a^k h'$ if $h, h' \in H_k, a \in A_k, h \succsim_a h'$. We give an example of a separable assignment problem below.

SD-efficiency of popular random assignments implies that a random assignment p is popular with respect to a separable assignment problem $(\mathcal{A}, \mathcal{H}, \succsim)$ if and only if for all $k \in [m]$, p restricted to A_k and H_k is popular with respect to $(A_k, H_k, \succsim^k)$. Regarding uniqueness, we thus obtain that a separable assignment problem has a unique popular random assignment if and only if all its reduced assignment problems have unique popular random assignments.

To better illustrate this statement, first observe that if all agents have different top choices, the corresponding assignment problem is obviously separable, $\mathcal{A}$ and $\mathcal{H}$ are simply partitioned into singletons. In this case, it is not surprising that there is a unique popular random assignment, namely to assign to every agent his most-preferred house.

Taking cue from that observation, we now allow some agents to share their first choices, hereby obtaining combinations of identical preferences and preferences with disjoint top choices. For every such combination, we can repeatedly apply Corollary 1 to calculate all popular random assignments. We remark that this yields a significant increase in the number of settings where Theorem 2 may be applied. Consider for example an instance where a certain group of agents consistently ranks houses of small size according to a different criterion—e.g., price—while a second group is focused on medium size and yet another one desires large houses.

To this end, assume an instance with eight families—the agents $\mathcal{A} = \{a_1, \ldots, a_8\}$—who are interested in buying a house. There are eight houses available, $\mathcal{H} = \{h_1, \ldots, h_8\}$, that differ in both the number of rooms and the price. Let houses h_1, h_2, h_3 have five rooms, h_4, h_5, h_6 six rooms and h_7, h_8 seven rooms. Due to the number of their children, families a_1, a_2, a_3 need five rooms, families a_4, a_5, a_6 need six, and a_7, a_8 even require seven. For simplicity, assume that houses with a lower index are cheaper in price and preferred to more expensive ones. Based on this assumption, we can easily determine all families' top preferences; they are given below on the left side.

We see that $(\mathcal{A}, \mathcal{H}, \succsim)$ is separable and together with Corollary 1 this precisely gives all popular random assignments. It holds that a random assignment $p \in \mathcal{R}(8)$ is popular if it is of the form depicted below with $0 \leq \lambda \leq 1$.

$a_1\colon\ h_1, h_2, h_3, \ldots$
$a_2\colon\ h_1, h_2, h_3, \ldots$
$a_3\colon\ h_1, h_2, h_3, \ldots$
$a_4\colon\ h_4, h_5, h_6, \ldots$
$a_5\colon\ h_4, h_5, h_6, \ldots$
$a_6\colon\ h_4, h_5, h_6, \ldots$
$a_7\colon\ h_7, h_8, \ldots$
$a_8\colon\ h_7, h_8, \ldots$

$$p = \begin{pmatrix} 1/3 & 1/3 & 1/3 & 0 & 0 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 1/3 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 1/3 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 1/3 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 1-\lambda \\ 0 & 0 & 0 & 0 & 0 & 0 & 1-\lambda & \lambda \end{pmatrix}$$

4.3 Experimental Results

So far, we have identified some criteria which imply that $(\mathcal{A}, \mathcal{H}, \succsim)$ admits a unique popular random assignment. However, there are plenty of preference profiles that admit a unique popular random assignment, although Theorem 2 cannot be applied, even when including our observation about separable assignment problems. Consider for instance the assignment problem $(\mathcal{A}, \mathcal{H}, \succsim)$ with $n = 5$ and $\succsim$ as depicted below. We see that neither do all agents share identical preferences nor is $(\mathcal{A}, \mathcal{H}, \succsim)$ separable. Nevertheless, there is a unique popular random assignment $p \in \mathcal{R}(5)$.

$$\succsim = \begin{array}{rl} a_1: & h_2, h_5, h_4, h_3, h_1 \\ a_2: & h_5, h_2, h_4, h_3, h_1 \\ a_3: & h_2, h_1, h_4, h_3, h_5 \\ a_4: & h_2, h_1, h_3, h_5, h_4 \\ a_5: & h_2, h_5, h_1, h_3, h_4 \end{array} \qquad p = \begin{pmatrix} 0 & 1/7 & 0 & 5/7 & 1/7 \\ 0 & 0 & 1/7 & 1/7 & 5/7 \\ 2/7 & 3/7 & 1/7 & 1/7 & 0 \\ 3/7 & 2/7 & 2/7 & 0 & 0 \\ 2/7 & 1/7 & 3/7 & 0 & 1/7 \end{pmatrix}$$

Our next goal is to determine the fraction of profiles that admit a unique popular random assignment, depending on n. We can compute this number exactly as long as n is relatively small. However, as the number of preference profiles is $(n!)^n$, exact computation quickly becomes infeasible, even when exploiting symmetries with respect to both agents and houses. Note that for instance for $n = 6$, we already have more than $1.3 \cdot 10^{17}$ different profiles. The exact number of profiles admitting a unique popular random assignment for $n \leq 4$ is given in Table 2.

n	1	2	3	4
Unique	1	2	54	35 904
Total	1	4	216	331 776
Fraction	1	0.5	0.22	0.094

Table 2: Number of profiles that admit a unique popular random assignment and the total number of profiles for $n \leq 4$.

To overcome the intractability of computing the exact fraction of profiles admitting a unique popular random assignment but still obtain a quantitative insight, we automatically sample preference profiles and verify whether they admit multiple popular random assignments. For the sampling process, we focus on two common parameter-free stochastic models. First, we choose each agent's preferences uniformly at random, which is known as the *impartial culture (IC)* model.

In the *spatial model*, we sample a point in the unit square for every $a \in \mathcal{A}$ and $h \in \mathcal{H}$ and determine agents' preferences by their proximity to each house, i.e., the Euclidian distance between the corresponding points (see, e.g., Ordeshook, 1993; Austen-Smith and Banks, 2000). For a more profound discussion of stochastic preference models please see for instance Critchlow et al. (1991) and Marden (1995).

For both models, Table 3 summarizes the results for 10 000 samples each. Figure 1 provides a visualization where the probability that a randomly picked assignment problem admits a unique popular random assignment is plotted on a logarithmic scale. We see that this probability decreases exponentially in n, where the decreases are slightly more distinctive when going from an odd n to an even one compared to from an even n to an odd one. A possible explanation might be related to Theorem 2.

n	1	2	3	4	5	6	7
IC	1	0.49	0.25	0.11	0.044	0.020	0.0088
Spatial	1	0.43	0.26	0.14	0.078	0.040	0.027

Table 3: Fraction of preference profiles admitting a unique popular random assignment when preferences are sampled according to either IC or the spatial model; 10 000 samples for each n.

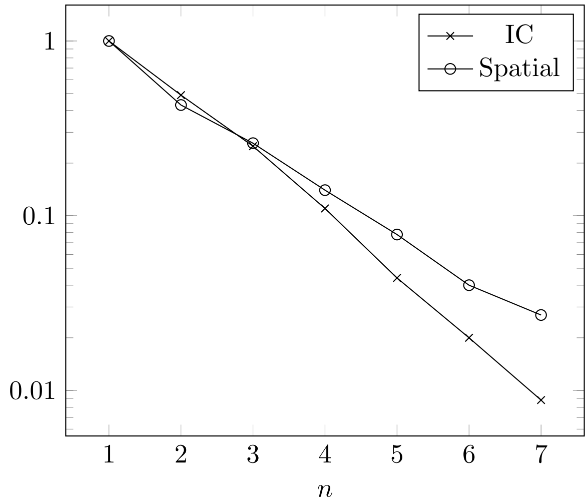


Figure 1: Probability that a randomly selected assignment problem of size n admits a unique popular random assignment. Preferences are sampled according to either IC or the spatial model.

The observation that the fraction of assignment problems admitting a unique popular random assignment compared to the total number of assignment problems decreases exponentially in n stands in sharp contrast to results obtained in the social choice setting. Recall that popular random assignments correspond directly to maximal lotteries. Maximal lotteries are unique in many cases (Laffond et al., 1997; Le Breton, 2005) and the set of preference profiles admitting a unique maximal lottery is open and dense (Brandl et al., 2016b). The set of profiles that admit multiple maximal lotteries is therefore nowhere dense and thus negligible.

5 Envy-freeness and Strategyproofness

In this section, we investigate to which extent popularity is compatible with envy-freeness and strategyproofness. Put differently, we want to know whether for every random assignment problem there exists a popular random assignment that satisfies envy-freeness and whether there exists a random assignment rule that satisfies both popularity and strategyproofness. Prior research in this direction by Aziz et al. (2013) has established the following results. First, it was shown that there exists a profile with $n = 3$ for which no popular assignment satisfies envy-freeness. Secondly, popularity was proven to be incompatible with strategyproofness when $n \geq 3$. Whether both results also hold for *weak* envy-freeness and *weak* strategyproofness, respectively, was left as an open problem.

We are able to answer this question in the affirmative: We provide a profile with $n = 5$ for which no popular random assignment satisfies weak envy-freeness and that can easily be extended to $n \geq 5$. In addition, we show that no random assignment rule can satisfy popularity and weak strategyproofness simultaneously whenever $n \geq 7$.

Theorem 3. *There exist assignment problems for which no popular random assignment satisfies weak envy-freeness when* $n \geq 5$.

Theorem 4. *No popular random assignment rule satisfies weak strategyproofness when* $n \geq 7$.

Due to space constraints, the proofs for both theorems can be found in the Appendix.

The results presented in this section do not only hold for the SD-extension, but similarly for *bilinear dominance*, which leads to much weaker notions of strategyproofness and envy-freeness (see Section 2).

6 Conclusion and Discussion

We have analyzed the structure of majority graphs induced by assignment problems and investigated the uniqueness, envy-freeness, and strategyproofness of popular random assignments and popular random assignment rules, respectively. It has turned out that most assignment problems admit more than one popular random assignment and that popularity does not align well with individual incentives as popularity is incompatible with weak envy-freeness and also with weak strategyproofness. On the other hand, it is known that popular random assignments satisfy a very strong notion of efficiency (PC-efficiency) and even maximize social welfare according to the canonical skew-symmetric bilinear (SSB) utility functions induced by the agents' preferences (see Brandl et al., 2015). This hints at an interesting tradeoff between social goals (such as efficiency and popularity) and individual goals (such as envy-freeness and strategyproofness) in random assignment. For comparison, the two most-studied assignment rules RSD and PS fail to satisfy PC-efficiency (and thus popularity). In fact, RSD does not even satisfy SD-efficiency. On the other hand, these rules fare better in terms of individual incentives of agents. RSD satisfies strategyproofness and PS satisfies envy-freeness. This tradeoff has been observed before. For example, Bogomolnaia and Moulin (2001) have shown that SD-efficiency and strategyproofness are incompatible. When allowing ties in individual preferences, Katta and Sethuraman (2006) proved that no random assignment rule simultaneously satisfies SD-efficiency, weak strategyproofness, and weak envy-freeness. Recently, Brandl et al. (2016a) gave a computer-aided proof that shows the incompatibility of SD-efficiency and weak strategyproofness in the more general domain of social choice. It is open whether the same statement also holds for random assignment (when agents have weak preferences). For the case of strict preferences, it would be interesting to see whether Theorem 4 can be strengthened by replacing popularity with PC-efficiency.

References

D. K. Abraham, R. W. Irving, T. Kavitha, and K. Mehlhorn. Popular matchings. *SIAM Journal on Computing*, 37(4):1030–1034, 2007.

D. Austen-Smith and J. S. Banks. *Positive Political Theory I: Collective Preference*. University of Michigan Press, 2000.

H. Aziz, F. Brandt, and P. Stursberg. On popular random assignments. In *Proceedings of the 6th International Symposium on Algorithmic Game Theory (SAGT)*, volume 8146 of *Lecture Notes in Computer Science (LNCS)*, pages 183–194. Springer-Verlag, 2013.

H. Aziz, F. Brandl, and F. Brandt. On the incompatibility of efficiency and strategyproofness in randomized social choice. In *Proceedings of the 28th AAAI Conference on Artificial Intelligence (AAAI)*, pages 545–551. AAAI Press, 2014.

H. Aziz, F. Brandl, and F. Brandt. Universal Pareto dominance and welfare for plausible utility functions. *Journal of Mathematical Economics*, 60:123–133, 2015.

P. Biró, R. W. Irving, and D. Manlove. Popular matchings in the marriage and roommates problems. In *Proceedings of the 7th Italian Conference on Algorithms and Complexity (CIAC)*, pages 97–108, 2010.

A. Bogomolnaia and H. Moulin. A new solution to the random assignment problem. *Journal of Economic Theory*, 100(2):295–328, 2001.

A. Bogomolnaia and H. Moulin. A simple random assignment problem with a unique solution. *Economic Theory*, 19(3):623–635, 2002.

F. Brandl, F. Brandt, and J. Hofbauer. Welfare maximization entices participation. 2015. Working paper.

F. Brandl, F. Brandt, and C. Geist. Proving the incompatibility of efficiency and strategyproofness via SMT solving. In *Proceedings of the 25th International Joint Conference on Artificial Intelligence (IJCAI)*, 2016a. Forthcoming.

F. Brandl, F. Brandt, and H. G. Seedig. Consistent probabilistic social choice. *Econometrica*, 2016b. Forthcoming.

D. E. Critchlow, M. A. Fligner, and J. S. Verducci. Probability models on rankings. *Journal of Mathematical Psychology*, 35:294–318, 1991.

Á. Cseh, C.-C. Huang, and T. Kavitha. Popular matchings with two-sided preferences and one-sided ties. In *Proceedings of the 42nd International Colloquium on Automata, Languages, and Programming (ICALP)*, volume 9134 of *Lecture Notes in Computer Science (LNCS)*, pages 367–379, 2015.

B. Debord. Caractérisation des matrices des préférences nettes et méthodes d'agrégation associées. *Mathématiques et sciences humaines*, 97:5–17, 1987.

P. Gärdenfors. Match making: Assignments based on bilateral preferences. *Behavioral Science*, 20(3):166–173, 1975.

W. K. B. Hofstee. Allocation by lot: a conceptual and empirical analysis. *Social Science Information*, 29(4):745–763, 1990.

C.-C. Huang and T. Kavitha. Popular matchings in the stable marriage problem. In *Proceedings of the 38th International Colloquium on Automata, Languages, and Programming (ICALP)*, volume 6755 of *Lecture Notes in Computer Science (LNCS)*, pages 666–677, 2011.

A.-K. Katta and J. Sethuraman. A solution to the random assignment problem on the full preference domain. *Journal of Economic Theory*, 131(1):231–250, 2006.

T. Kavitha. A size-popularity tradeoff in the stable marriage problem. *SIAM Journal on Computing*, 43(1):52–71, 2014.

T. Kavitha and M. Nasre. Optimal popular matchings. *Discrete Applied Mathematics*, 157: 3181–3186, 2009.

T. Kavitha, J. Mestre, and M. Nasre. Popular mixed matchings. *Theoretical Computer Science*, 412(24):2679–2690, 2011.

B. Klaus, D. Manlove, and F. Rossi. Matching under preferences. In F. Brandt, V. Conitzer, U. Endriss, J. Lang, and A. D. Procaccia, editors, *Handbook of Computational Social Choice*, chapter 14. Cambridge University Press, 2016.

G. Laffond, J.-F. Laslier, and M. Le Breton. A theorem on symmetric two-player zero-sum games. *Journal of Economic Theory*, 72(2):426–431, 1997.

M. Le Breton. On the uniqueness of equilibrium in symmetric two-player zero-sum games with integer payoffs. *Économie publique*, 17(2):187–195, 2005.

M. Mahdian. Random popular matchings. In *Proceedings of the 7th ACM Conference on Electronic Commerce (ACM-EC)*, pages 238–242, 2006.

D. Manlove. *Algorithmics of Matching Under Preferences.* World Scientific Publishing Company, 2013.

J. I. Marden. *Analyzing and Modeling Rank Data.* Number 64 in Monographs on Statistics and Applied Probability. Chapman & Hall, 1995.

D. C. McGarvey. A theorem on the construction of voting paradoxes. *Econometrica*, 21(4): 608–610, 1953.

P. C. Ordeshook. The spatial analysis of elections and committees: four decades of research. Technical report, California Institute of Technology, 1993.

A. E. Roth. *Who Gets What — and Why: The New Economics of Matchmaking and Market Design.* Eamon Dolan/Houghton Mifflin Harcourt, 2015.

Felix Brandt
Institut für Informatik
Technische Universität München
85748 München, Germany
Email: `brandtf@in.tum.de`

Johannes Hofbauer
Institut für Informatik
Technische Universität München
85748 München, Germany
Email: `hofbauej@in.tum.de`

Martin Suderland
Institut für Informatik
Technische Universität München
85748 München, Germany
Email: `martin.suderland@freenet.de`

APPENDIX

A Omitted Proofs

A.1 Proof of Theorem 1

Theorem 1 states that two assignment problems induce the same majority graph if and only if their decompositions are rotation equivalent. We first prove one direction, namely that two assignment problems whose decompositions are rotation equivalent induce identical majority graphs.

Lemma 1. *Let $(\mathcal{A}, \mathcal{H}, \succsim)$ and $(\mathcal{A}, \mathcal{H}, \succsim')$ be two assignment problems whose decompositions are rotation equivalent. We then have for the induced majority graphs G and G' that $G = G'$.*

Proof. Let $(\mathcal{A}, \mathcal{H}, \succsim)$ and $(\mathcal{A}, \mathcal{H}, \succsim')$ be two assignment problems whose decompositions are rotation equivalent meaning that $\succsim^k = \succsim'^{(k+d) \bmod_1 m}$ for all $k \in [m]$. Moreover, denote by $G = (V, E, w)$ and $G' = (V', E', w')$ the induced majority graphs. We have that $V = V'$ and $E = E'$ by definition of the majority graph, so it only remains to show that we indeed have $w = w'$. Recall that $w(M, M') = \phi_{\succsim}(M, M')$ and by the definition of $\phi_{\succsim}(M, M')$,

$$\phi_{\succsim}(M, M') = \sum_{a \in \mathcal{A}} \phi_{\succsim_a}(M(a), M'(a)).$$

For the sake of readability, we introduce a permutation π mapping the house h an agent receives in M to the house h' he receives in M'. Formally, for two arbitrary matchings $M, M' \in \mathcal{M}(n)$ we define the permutation $\pi \colon \mathcal{H} \to \mathcal{H}$, $\pi(h) = M'(M(h))$. As every agent is assigned exactly one house, we can equivalently sum over all houses.

$$\sum_{a \in \mathcal{A}} \phi_{\succsim_a}(M(a), M'(a)) = \sum_{h \in \mathcal{H}} \phi_{\succsim_{M(h)}}(h, \pi(h))$$

We now partition $\mathcal{H}$ into four subsets. For $h \in \mathcal{H}$ assume $h \in H_k$ and $\pi(h) \in H_{k'}$ when considering the decomposition of $(\mathcal{A}, \mathcal{H}, \succsim)$ and $h \in H'_l$ and $\pi(h) \in H'_{l'}$ when considering the decomposition of $(\mathcal{A}, \mathcal{H}, \succsim')$. We distinguish whether

(*i*) $k > l$ and $k' > l'$,

(*ii*) $k > l$ and $k' \leq l'$,

(*iii*) $k \leq l$ and $k' > l'$, or

(*iv*) $k \leq l$ and $k' \leq l'$.

Obviously $k = l$ if and only if $k' = l'$. Thus, for instance $k > l$ and $l' = k'$ is technically impossible but theoretically captured by (*ii*) for the sake of completeness.

The four cases correspond to different situations that may occur when the subsets of $\mathcal{H}$ obtained in the decomposition of $(\mathcal{A}, \mathcal{H}, \succsim)$ rotate to match those obtained when decomposing $(\mathcal{A}, \mathcal{H}, \succsim')$. Note that by the definition of rotation equivalence, this rotation is always 'to the right'. We say that the rotation of $h \in H_k$ *exceeds* m if $k + d > m$, i.e., the $\bmod_1$ function is actually applied.

In this respect, (*i*) prescribes that the rotation of both h and $\pi(h)$ exceeds m. In case (*ii*) the rotation of h exceeds m while the rotation of $\pi(h)$ does not. For (*iii*) the converse holds. Lastly, (*iv*) defines the situation where both rotations do not exceed m. To this end, let the function δ assign to every $h \in \mathcal{H}$ the corresponding k such that $h \in H_k$, $\delta\colon \mathcal{H} \to [m]$, $\delta(h) = k \in [m]$ such that $h \in H_k$. Partitioning $\mathcal{H}$ in that way, we obtain

$$\sum_{h\in\mathcal{H}} \phi_{\succsim_{M(h)}}(h,\pi(h)) = \sum_{\substack{h\in\mathcal{H}\\ \delta(h)+d\le m\\ \delta(\pi(h))+d\le m}} \phi_{\succsim_{M(h)}}(h,\pi(h)) + \sum_{\substack{h\in\mathcal{H}\\ \delta(h)+d\le m\\ \delta(\pi(h))+d> m}} \phi_{\succsim_{M(h)}}(h,\pi(h)) + \sum_{\substack{h\in\mathcal{H}\\ \delta(h)+d> m\\ \delta(\pi(h))+d\le m}} \phi_{\succsim_{M(h)}}(h,\pi(h)) + \sum_{\substack{h\in\mathcal{H}\\ \delta(h)+d> m\\ \delta(\pi(h))+d> m}} \phi_{\succsim_{M(h)}}(h,\pi(h)).$$

We now make use of some equalities in order to link $\succsim$ and $\succsim'$. First note that if both the rotations of h and $\pi(h)$ do not exceed m, then we have that either $h \succsim_{M(h)} \pi(h)$ and $h \succsim'_{M(h)} \pi(h)$ or $\pi(h) \succsim_{M(h)} h$ and $\pi(h) \succsim'_{M(h)} h$. The same holds for the case where the rotations of h and $\pi(h)$ exceed m. Therefore,

$$\sum_{\substack{h\in\mathcal{H}\\ \delta(h)+d\le m\\ \delta(\pi(h))+d\le m}} \phi_{\succsim_{M(h)}}(h,\pi(h)) = \sum_{\substack{h\in\mathcal{H}\\ \delta(h)+d\le m\\ \delta(\pi(h))+d\le m}} \phi_{\succsim'_{M(h)}}(h,\pi(h)) \text{ and}$$
$$\sum_{\substack{h\in\mathcal{H}\\ \delta(h)+d> m\\ \delta(\pi(h))+d> m}} \phi_{\succsim_{M(h)}}(h,\pi(h)) = \sum_{\substack{h\in\mathcal{H}\\ \delta(h)+d> m\\ \delta(\pi(h))+d> m}} \phi_{\succsim'_{M(h)}}(h,\pi(h)).$$

Furthermore, we have that the number of cases where the rotation of h exceeds m but the rotation of $\pi(h)$ does not has to equal the number of cases where the rotation of $\pi(h)$ exceeds m and the rotation of h does not. Note that this equivalence holds no matter if we consider $\succsim$ or $\succsim'$. Consequently,

$$\sum_{\substack{h\in\mathcal{H}\\ \delta(h)+d\le m\\ \delta(\pi(h))+d> m}} \phi_{\succsim_{M(h)}}(h,\pi(h)) = -\sum_{\substack{h\in\mathcal{H}\\ \delta(h)+d> m\\ \delta(\pi(h))+d\le m}} \phi_{\succsim_{M(h)}}(h,\pi(h))$$
$$\sum_{\substack{h\in\mathcal{H}\\ \delta(h)+d> m\\ \delta(\pi(h))+d\le m}} \phi_{\succsim'_{M(h)}}(h,\pi(h)) = -\sum_{\substack{h\in\mathcal{H}\\ \delta(h)+d\le m\\ \delta(\pi(h))+d> m}} \phi_{\succsim'_{M(h)}}(h,\pi(h)).$$

Employing these equalities, we obtain that

$$\begin{aligned}
&\sum_{\substack{h\in\mathcal{H}\\ \delta(h)+d\le m\\ \delta(\pi(h))+d\le m}} \phi_{\succsim_{M(h)}}(h,\pi(h)) + \sum_{\substack{h\in\mathcal{H}\\ \delta(h)+d\le m\\ \delta(\pi(h))+d> m}} \phi_{\succsim_{M(h)}}(h,\pi(h))\\
&+ \sum_{\substack{h\in\mathcal{H}\\ \delta(h)+d> m\\ \delta(\pi(h))+d\le m}} \phi_{\succsim_{M(h)}}(h,\pi(h)) + \sum_{\substack{h\in\mathcal{H}\\ \delta(h)+d> m\\ \delta(\pi(h))+d> m}} \phi_{\succsim_{M(h)}}(h,\pi(h))\\
&= \sum_{\substack{h\in\mathcal{H}\\ \delta(h)+d\le m\\ \delta(\pi(h))+d\le m}} \phi_{\succsim'_{M(h)}}(h,\pi(h)) + \sum_{\substack{h\in\mathcal{H}\\ \delta(h)+d> m\\ \delta(\pi(h))+d\le m}} \phi_{\succsim'_{M(h)}}(h,\pi(h))\\
&+ \sum_{\substack{h\in\mathcal{H}\\ \delta(h)+d\le m\\ \delta(\pi(h))+d> m}} \phi_{\succsim'_{M(h)}}(h,\pi(h)) + \sum_{\substack{h\in\mathcal{H}\\ \delta(h)+d> m\\ \delta(\pi(h))+d> m}} \phi_{\succsim'_{M(h)}}(h,\pi(h))\\
&= \phi_{\succsim'}(M,M').
\end{aligned}$$

Note that the last equality holds by transformations similar to those at the beginning of the proof. As $\phi'_{\succsim}(M, M') = w'(M, M')$ we hence deduce that $w(M, M') = w'(M, M')$ for all $M, M' \in \mathcal{M}(\widetilde{n})$ which proves Lemma 1. □

We now show the converse direction, i.e., given a majority graph, we can uniquely determine a group of decompositions of assignment problems that are rotation equivalent. This statement and the corresponding proof will be split into three parts: First, we show that given a majority graph G, we can compute a maximal partition of $\mathcal{H}$ such that all agents have identical preferences over two houses h, h' that are contained in different subsets of this partition, $h \in H, h' \in H', H \neq H'$. We afterwards show that we can determine all agents' individual preferences over all houses contained in the same subset H. Finally, we prove that G implies a unique cyclic ordering of the partition's subsets.

Lemma 2. *Let $G = (V, E, w)$ be the majority graph of an assignment problem. G uniquely determines $\mathcal{A}$, $\mathcal{H}$, and the maximal partition of $\mathcal{H} = \dot{\bigcup}_{k \in [m]} H_k$ of an assignment problem $(\mathcal{A}, \mathcal{H}, \succsim)$ inducing G, where $\{H_1, \ldots, H_m\} = \bar{H}$ are of the form that for all $H \neq H' \in \bar{H}$ either $H \succ_a H'$ for all $a \in \mathcal{A}$ or $H' \succ_a H$ for all $a \in \mathcal{A}$.*

Proof. Let $G = (V, E, w)$ be the induced majority graph of an assignment problem. First note that $\mathcal{A}$ and $\mathcal{H}$ are implicitly defined by the vertex set V. In order to find a partition of $\mathcal{H}$ into a maximal number of subsets $\{H_1, \ldots, H_m\} = \bar{H}$ such that for every pair of subsets $H \neq H'$ we have that either $H \succ_a H'$ for all $a \in \mathcal{A}$ or $H' \succ_a H$ for all $a \in \mathcal{A}$ we employ the following algorithm:

Algorithm 1 Find a partition of $\mathcal{H}$.

for all $h \neq h' \in \mathcal{H}$ **do**
  for all $a \neq a' \in \mathcal{A}$ **do**
    Select two matchings $M, M' \in \mathcal{M}(n) = V$ such that $a = M(h) = M'(h')$, $a' = M(h') = M'(h)$, and $M(h'') = M'(h'')$ for all $h'' \in \mathcal{H} \setminus \{h, h'\}$.
    Set $w(h, h', a, a') = w(M, M')$.
  end for
  if $w(h, h', a, a') \neq 0$ for some $a \neq a' \in \mathcal{A}$ **then**
    h and h' are to be part of the same $H \in \bar{H}$.
  end if
end for

As to the correctness of Algorithm 1, note that $w(h, h', a, a') \neq 0$ for some $a \neq a' \in \mathcal{A}$ implies that agents a and a' disagree on h being preferred to h' or vice versa. Consequently, h and h' cannot be part of disjoint subsets of $\bar{H}$. The converse, however, is not true. If $w(h, h', a, a') = 0$ for all $a \neq a' \in \mathcal{A}$, all agents agree on which of h or h' is preferred. Still, it might be the case that a prefers some h'' to both h and h' while another agent a' prefers h and h' to h''. In this setting h and h' have to be contained in the same subset of $\bar{H}$, no matter that all agents agree on one of them being better.

This case is captured in the way that Algorithm 1 indirectly gives that h and h' have to be part of equal subsets of $\bar{H}$ by yielding that h has to be contained in the same subset as h'' which in turn has to be contained in the same subset as h'. If no such indirect connection via an h'' exists, h and h' are placed in disjoint subsets. It is worth noting that the described indirect connection via h'' is not necessarily limited to a single house but might also employ a sequence of houses $h''_1, \ldots, h''_k$.

In both cases, the algorithm constructs $\bar{H}$ as claimed.[2] □

[2]Note that antisymmetry of the agents' preferences is crucial for the present proof. Antisymmetry ensures that equality to zero implies strict preferences.

Lemma 3. *Let $G = (V, E, w)$ be the majority graph of an assignment problem where $\mathcal{A}$ denotes the set of agents and $\mathcal{H}$ the set of houses. Furthermore, let $\bar{H} = \{H_1, \ldots, H_m\}$ be a partition of $\mathcal{H}$ into the maximal number of subsets such that for all $H \neq H' \in \bar{H}$ we have that either $H \succ_a H'$ for all $a \in \mathcal{A}$ or $H' \succ_a H$ for all $a \in \mathcal{A}$. Then G uniquely determines $(\mathcal{A}, H, \succsim^H)$ for all $H \in \bar{H}$.*

Proof. Let $G, \mathcal{A}, \mathcal{H}, \bar{H}$ be as assumed in Lemma 3. In order to determine $\succsim_a^H$ for an arbitrary $H \in \bar{H}, |H| \geq 2$, and $a \in \mathcal{A}$ employ Algorithm 2.

Algorithm 2 Determine individual preferences over the houses contained in each of the partition's subsets.

Step 1.
for all $h \neq h' \in H$ **do**
  for all $a \neq a' \in \mathcal{A}$ **do**
    Select two matchings $M, M' \in \mathcal{M}(n) = V$ such that $a = M(h) = M'(h')$, $a' = M(h') = M'(h)$, and $M(h'') = M'(h'')$ for all $h'' \in \mathcal{H} \setminus \{h, h'\}$.
    if $w(M, M') = 2$ **then**
      Let $h \succsim_a h'$.
    else if $w(M, M') = -2$ **then**
      Let $h' \succsim_a h$.
    end if
  end for
end for
Step 2.
for all $h \neq h' \in H$ left incomparable by some agent in *Step 1* **do**
  for all $a \neq a' \in \mathcal{A}$ **do**
    Select $h''_1, \ldots, h''_k \in H$ such that for a and a' individual preferences over $\{h, h''_1\}, \{h''_1, h''_2\}, \ldots, \{h''_{k-1}, h''_k\}$, and $\{h''_k, h'\}$ have already been devised in *Step 1*.
    Select two matchings $M, M' \in \mathcal{M}(n) = V$ and $a''_1, \ldots, a''_k \in \mathcal{A}$ such that $a = M(h) = M'(h')$, $a' = M(h') = M'(h''_1)$, $a''_1 = M(h''_1) = M(h''_2)$, $\ldots$, $a''_k = M(h''_k) = M'(h)$, and $M(h''') = M'(h''')$ for all $h''' \in \mathcal{H} \setminus \{h, h', h''_1, \ldots, h''_k\}$.
    if $w(M, M') - \phi_{\succsim_{a'}}(h', h''_1) - \sum_{i \in [k]} \phi_{\succsim_{a''_i}}(M(a''_i), M'(a''_i)) = 1$ **then**
      Let $h \succsim_a h'$.
    else if $w(M, M') - \phi_{\succsim_{a'}}(h', h''_1) - \sum_{i \in [k]} \phi_{\succsim_{a''_i}}(M(a''_i), M'(a''_i)) = -1$ **then**
      Let $h' \succsim_a h$.
    end if
  end for
end for

First note that for $|H| = 1$, determining individual preferences over all houses contained in H is trivial. We continue with $|H| \geq 2$ and the algorithm.

In *Step 1*, $w(M, M') \in \{-2, 2\}$ occurs if agents a, a' disagree on which of h, h' is more preferred. In this case, $w(M, M') \in \{-2, 2\}$ holds by the antisymmetry of individual preferences. More precisely, $w(M, M') = 2$ means that both a and a' prefer M to M', i.e., a prefers h to h' and a' prefers h' to h. For $w(M, M') = -2$, the converse holds.

However, as discussed in the proof of Lemma 2, it is possible that all agents agree on the pairwise comparison between h and h' and they are still contained in the same H. We easily see that this is only possible if $|H| \geq 3$ and thus implicitly $|\mathcal{A}| \geq 3$. In addition, we note that for each pair of houses $h, h' \in H$, *Step 1* of Algorithm 2 either determines preferences for all agents $a \in \mathcal{A}$, or for no agent at all.

Concerning *Step 2*, for every pair h, h' for which no individual preferences were fixed in *Step 1*, there has to exist a sequence of houses $h''_1, \dots, h''_k \in H$ such that h can be compared to h''_1, h''_i can be compared to h''_{i+1}, $i \in [k-1]$ and finally h''_k can be compared to h'. Assume no such sequence exists, then h and h' cannot be contained in the same subset H by Algorithm 1. By the construction of M, M' we have that $w(M, M') = \phi_{\succsim_a}(h, h') + \phi_{\succsim_{a'}}(h', h''_1) + \sum_{i \in [k]} \phi_{\succsim_{a''_i}}(M(a''_i), M'(a''_i))$. Here, all terms but $\phi_{\succsim_a}(h, h')$ are set in *Step 1* and known by the choice of $h''_1, \dots, h''_k$, and $w(M, M')$ is given by G. Consequently, we can directly compute the remaining summand that corresponds to a's preferences over h, h'.

It is easy to see that Algorithm 2 uniquely determines complete, transitive and anti-symmetric preferences $\succsim_a^H$ over all houses in H for all agents $a \in \mathcal{A}$. Applying the same algorithm for all $H \in \bar{H}$ gives m disjoint $(\mathcal{A}, H, \succsim^H)$ as claimed in Lemma 3. □

Lemma 4. *Let $G = (V, E, w)$ be the majority graph of an assignment problem $(\mathcal{A}, H, \succsim)$. Furthermore, let $\bar{H}$ be a partition of $\mathcal{H}$ into the maximal number of subsets such that for all $H, H' \in \bar{H}$ we have that either $H \succ_a H'$ for all $a \in \mathcal{A}$ or $H' \succ_a H$ for all $a \in \mathcal{A}$.*

Then, for all $H, H' \in \bar{H}$, G uniquely determines transitive orderings $H_1 \succ H_2 \succ \dots \succ H_m$, $H = H_1$, and $H'_1 \succ H'_2 \succ \dots \succ H'_m$, $H' = H'_1$, with $H_k \succ_a H_{k'}$ and $H'_k \succ_a H'_{k'}$ for all $k < k', a \in \mathcal{A}$, such that there exists $d \in [m]$ for which we have that $H_k = H'_{(k+d) \bmod_1 m}$ for all $k \in [m]$.

Proof. Let $G, \mathcal{A}, \mathcal{H}, \bar{H}$ be defined as above.

In the case that $|\bar{H}| = 1$, Lemma 4 is trivially satisfied. For $|\bar{H}| = 2$, there only exist two different orderings of the subsets of $\bar{H}$, each is uniquely determined if one subset is fixed as the more preferred one. In addition, there obviously exists $d \in \{1, 2\}$ for which the given identities hold. We thus focus on $|\bar{H}| \geq 3$.

First, we choose $H \in \bar{H}$ at random and show that for any two $H' \neq H'' \in \bar{H} \setminus \{H\}$, G determines whether $H \succ H' \succ H''$ or $H \succ H'' \succ H'$. Note that this ordering is transitive by the transitivity of individual preferences. To determine which of the two holds we randomly select $h \in H, h' \in H', h'' \in H''$ and two matchings $M, M' \in \mathcal{M}(n) = V$ such that $a = M(h) = M'(h')$, $a' = M(h') = M'(h'')$, $a'' = M(h'') = M'(h)$, for some $a, a', a'' \in \mathcal{A}$, and $M(h''') = M'(h''')$ for all $h''' \in \mathcal{H} \setminus \{h, h', h''\}$.

It trivially holds that $w(M, M') \in \{-1, 1\}$. If $w(M, M') = 1$, two agents prefer their assigned house in M over their house in M' while one agent has reversed preferences. We conclude that $H \succ H' \succ H''$. If on the other hand $w(M, M') = -1$, we have that $H \succ H'' \succ H'$.

Repeatedly applying the above procedure for different choices of $H' \neq H'' \in \bar{H} \setminus \{H\}$ yields a complete ordering $H = H_1 \succ H_2 \succ \dots \succ H_m$.

Next, assume that for two different choices $H \neq H' \in \bar{H}$ we have identified different orderings $H = H_1 \succ H_2 \succ \dots \succ H_m$ and $H' = H'_1 \succ H'_2 \succ \dots \succ H'_m$ with $H_k \succ_a H_{k'}$ and $H'_k \succ_a H'_{k'}$ for all $k < k', a \in \mathcal{A}$. Without loss of generality let $H_1 = H'_{k'}$ and set $d = k' - 1$. We show that if $H_k = H'_{(k+d) \bmod_1 m}$, then $H_{(k+1) \bmod_1 m} = H'_{(k+1+d) \bmod_1 m}$, $k \in [m]$.

Hence, suppose $H_k = H'_{(k+d) \bmod_1 m}$ for some $k \in [m]$ and assume for contradiction

$$\underbrace{H_{(k+1) \bmod_1 m}}_{=H''} \neq \underbrace{H'_{(k+1+d) \bmod_1 m}}_{=H'''}.$$

We deduce with respect to the ordering $H = H_1 \succ H_2 \succ \dots \succ H_m$ that either $H_k \succ H'' \succ H'''$, $H''' \succ H_k \succ H''$, or $H'' \succ H''' \succ H_k$. At the same time, regarding $H' = H'_1 \succ H'_2 \succ \dots \succ H'_m$, we have that either $H_k \succ H''' \succ H''$, $H'' \succ H_k \succ H'''$, or $H''' \succ H'' \succ H_k$.

Now, consider $h \in H_k, h'' \in H'', h''' \in H'''$ and two matchings $M, M' \in \mathcal{M}(n) = V$ such that $a = M(h) = M'(h'')$, $a' = M(h'') = M'(h''')$, $a'' = M(h''') = M'(h)$, for some

$a, a', a'' \in \mathcal{A}$, and $M(h') = M'(h')$ for all $h' \in \mathcal{H} \setminus \{h, h'', h'''\}$. It is straightforward to compute that all of $H_k \succ H'' \succ H'''$, $H''' \succ H_k \succ H''$, and $H'' \succ H''' \succ H_k$ imply $w(M, M') = 1$. On the other hand, $H_k \succ H''' \succ H''$, $H'' \succ H_k \succ H'''$, and $H''' \succ H'' \succ H_k$ all imply $w(M, M') = -1$

We conclude that $H_{(k+1) \bmod_1 m} \neq H'_{(k+1+d) \bmod_1 m}$ means that $1 = w(M, M') = -1$, a contradiction. Thus, if $H_k = H'_{(k+d) \bmod_1 m}$, then $H_{(k+1) \bmod_1 m} = H'_{(k+1+d) \bmod_1 m}$, which in total proves Lemma 4. □

Lemmas 1 to 4 together imply Theorem 1:

Theorem 1. *Let $(\mathcal{A}, \mathcal{H}, \succsim)$ and $(\mathcal{A}, \mathcal{H}, \succsim')$ be two assignment problems that induce majority graphs G and G', respectively. Then, $G = G'$ if and only if the decompositions of $(\mathcal{A}, \mathcal{H}, \succsim)$ and $(\mathcal{A}, \mathcal{H}, \succsim')$ are rotation equivalent.*

Proof. The direction from right to left follows directly from Lemma 1.
For the converse direction, Lemma 2 argues that G uniquely determines $\mathcal{A}$, $\mathcal{H}$, and a maximal partition $\mathcal{H} = \dot{\bigcup}_{k \in [m]} H_k$ with the property that for all $H \neq H' \in \{H_1, \ldots, H_m\}$ either $H \succ_a H'$ for all $a \in \mathcal{A}$ or $H' \succ_a H$ for all $a \in \mathcal{A}$. By Lemma 3, G uniquely defines preference profiles $\succsim^k$ for all $k \in [m]$. Finally, Lemma 4 shows that G determines m different orderings of $\{H_1, \ldots, H_m\}$, each of them giving rise to the decomposition of an assignment problem, all of which are rotation equivalent. □

We emphasize that Algorithms 1 and 2 are not optimized with respect to runtime, but readability.

A.2 Proof of Theorem 2

Theorem 2. *Let $(\mathcal{A}, \mathcal{H}, \succsim)$ be an assignment problem where all agents have identical preferences. Then, the random assignment $p \in \mathcal{R}(n)$ is popular if and only if $L(L(p)) = p$.*

The proof of Theorem 2 will be split into various lemmas to improve readability. First, Lemmas 5 and 6 show that for every popular random assignment p, $L(L(p)) = p$. The converse direction, i.e., every random assignment p that satisfies $L(L(p)) = p$ is popular, follows from Lemmas 7 and 8.

Lemma 5. *Let $n \geq 3$ and $A(n) \in \mathbb{Z}^{n \times n}$ be the matrix defined by*

$$A(n)_{i,j} = \begin{cases} -2n+4 & \text{if } i = j, \\ -n+4 & \text{if } |i-j| = 1 \text{ or } (i,j) \in \{(1,n),(n,1)\}, \\ 4 & \text{otherwise.} \end{cases}$$

Then, for all $x \in \mathbb{R}^n$ we have that $x^T A(n)\, x \leq 0$, i.e., $A(n)$ is negative semi-definite.

Proof. Let $n \geq 3$ and $A(n)$ as defined in Lemma 5. For better illustration also consider the following representation of $A(n)$:

$$A(n) = \begin{pmatrix} -2n+4 & -n+4 & 4 & \ldots & 4 & -n+4 \\ -n+4 & -2n+4 & -n+4 & 4 & \ldots & 4 \\ 4 & -n+4 & -2n+4 & -n+4 & 4 & \vdots \\ \vdots & 4 & \ddots & \ddots & \ddots & 4 \\ 4 & \ldots & 4 & -n+4 & -2n+4 & -n+4 \\ -n+4 & 4 & \ldots & 4 & -n+4 & -2n+4 \end{pmatrix}.$$

We have that

$$
\begin{aligned}
x^{\mathrm{T}} A(n)\, x = &\sum_{i,j\in[n]} A(n)_{i,j} x_i x_j \\
= &\, 4 \sum_{i,j\in[n]} x_i x_j - 2n \sum_{i=j\in[n]} x_i x_j - n(x_1 x_n + x_n x_1) - n \sum_{i,j\in[n],|i-j|=1} x_i x_j .
\end{aligned}
$$

For the sake of presentation, we split the right side and consider the two resulting terms separately. First, consider $4\sum_{i,j\in[n]} x_i x_j$ and compute

$$
\begin{aligned}
4 \sum_{i,j\in[n]} x_i x_j = &\, 4 \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{j=1}^{n} x_j\right) \\
= &\, 8 \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{j=1}^{n} x_j\right) - 4 \left(\sum_{j=1}^{n} x_j\right)^2 \\
= &\, 4 \left(\sum_{j=1}^{n} x_j\right)\left(x_1 + x_n + \sum_{i=1}^{n-1} (x_i + x_{i+1})\right) - 4 \left(\sum_{j=1}^{n} x_j\right)^2 \\
= &\, {}^4\!/\!_n \left(\sum_{j=1}^{n} x_j\right)(n x_1 + n x_n) + {}^4\!/\!_n \left(\sum_{j=1}^{n} x_j\right)\left(\sum_{i=1}^{n-1} (n x_i + n x_{i+1})\right) \\
&- {}^4\!/\!_n \left(\sum_{j=1}^{n} x_j\right)^2 - \underbrace{{}^{4n-4}\!/\!_n \left(\sum_{j=1}^{n} x_j\right)^2}_{= {}^4\!/\!_n \sum_{i=1}^{n-1} \left(\sum_{j=1}^{n} x_j\right)^2} .
\end{aligned}
$$

Now, for the remaining terms, we compute

$$
\begin{aligned}
&- 2n \sum_{i=j\in[n]} x_i x_j - n(x_1 x_n + x_n x_1) - n \sum_{i,j\in[n],|i-j|=1} x_i x_j \\
= &- 2n \sum_{i=1}^{n} x_i^2 - 2n x_1 x_n - 2n \sum_{i=1}^{n-1} x_i x_{i+1} \\
= &- n(x_1^2 + 2x_1 x_n + x_n^2) - n \sum_{i=1}^{n-1} (x_i^2 + 2 x_i x_{i+1} + x_{i+1}^2) \\
= &- {}^1\!/\!_n (n x_1 + n x_n)^2 - {}^1\!/\!_n \sum_{i=1}^{n-1} (n x_i + n x_{i+1})^2 .
\end{aligned}
$$

In order to complete the proof, we combine the final terms of both calculations, rearrange them, and obtain

$$\begin{aligned} x^{\mathrm{T}} A(n)\, x =& -{}^1\!/\!_n(nx_1+nx_n)^2 + {}^4\!/\!_n\left(\sum_{j=1}^{n} x_j\right)(nx_1+nx_n) - {}^4\!/\!_n\left(\sum_{j=1}^{n} x_j\right)^2 \\ & - {}^1\!/\!_n\sum_{i=1}^{n-1}(nx_i+nx_{i+1})^2 + {}^4\!/\!_n\left(\sum_{j=1}^{n} x_j\right)\left(\sum_{i=1}^{n-1}(nx_i+nx_{i+1})\right) - {}^4\!/\!_n\sum_{i=1}^{n-1}\left(\sum_{j=1}^{n} x_j\right)^2 \\ =& -{}^1\!/\!_n\left(nx_1+nx_n-2\sum_{j=1}^{n} x_j\right)^2 - {}^1\!/\!_n\sum_{i=1}^{n-1}\left(nx_i+nx_{i+1}-2\sum_{j=1}^{n} x_j\right)^2 \qquad (1) \\ \le& \, 0. \end{aligned}$$

□

Lemma 6. *Let $(\mathcal{A}, \mathcal{H}, \succsim)$ be an assignment problem where all agents have identical preferences. Then, for every popular random assignment $p \in \mathcal{R}(n)$ we have $L(L(p)) = p$.*

Proof. Let $(\mathcal{A}, \mathcal{H}, \succsim)$ be an assignment problem as assumed in Lemma 6. Note that as long as $n \le 2$, every random assignment is invariant under double application of the left shift, so Lemma 6 trivially holds.

We thus focus on $n \ge 3$. The proof is divided into two parts: First, we show that the left shift of a random assignment p always is at least as popular as p itself, i.e., $\phi(L(p), p) \ge 0$. The second step shows that $\phi(L(p), p) = 0$ holds if and only if $L(L(p)) = p$ which in total proves Lemma 6.

We begin by computing

$$\begin{aligned} \phi(L(p), p) &= \sum_{i,j,j' \in [n]} (L(p))_{i,j} p_{i,j'} \phi_{\succsim_{a_i}}(h_j, h'_j) \\ &= \sum_{\substack{i,j,j' \in [n] \\ j \le n-1}} p_{i,j+1} p_{i,j'} \phi_{\succsim_{a_i}}(h_j, h_{j'}) + \sum_{i,j' \in [n]} p_{i,1} p_{i,j'} \phi_{\succsim_{a_i}}(h_n, h_{j'}) \\ &= \sum_{\substack{i,j,j' \in [n] \\ j \le n-1 \\ j < j'}} p_{i,j+1} p_{i,j'} - \sum_{\substack{i,j,j' \in [n] \\ j \le n-1 \\ j' < j}} p_{i,j+1} p_{i,j'} - \sum_{\substack{i,j' \in [n] \\ j' < n}} p_{i,1} p_{i,j'}. \end{aligned}$$

For the last equality we make use of the fact that $\phi_{\succsim_{a_i}}(h_j, h_{j'}) = 1$ for all $j < j'$, $\phi_{\succsim_{a_i}}(h_j, h_{j'}) = -1$ for all $j' < j$, and $\phi_{\succsim_{a_i}}(h_j, h_{j'}) = 0$ for $j = j'$. In addition, we have that $\phi_{\succsim_{a_i}}(h_n, h_{j'}) = -1$ for all $j' < n$ and $\phi_{\succsim_{a_i}}(h_n, h_{j'}) = 0$ for $j' = n$. We now shift j and split the first sum.

$$\begin{aligned} & \sum_{\substack{i,j,j' \in [n] \\ j \le n-1 \\ j < j'}} p_{i,j+1} p_{i,j'} - \sum_{\substack{i,j,j' \in [n] \\ j \le n-1 \\ j' < j}} p_{i,j+1} p_{i,j'} - \sum_{\substack{i,j' \in [n] \\ j' < n}} p_{i,1} p_{i,j'} \\ =& \sum_{\substack{i,j,j' \in [n] \\ 2 \le j \le j' \le n-1}} p_{i,j} p_{i,j'} + \sum_{\substack{i,j \in [n] \\ 2 \le j}} p_{i,j} p_{i,n} - \sum_{\substack{i,j,j' \in [n] \\ 3 \le j \\ j' \le j-2}} p_{i,j} p_{i,j'} - \sum_{\substack{i,j' \in [n] \\ j' \le n-1}} p_{i,1} p_{i,j'}. \qquad (2) \end{aligned}$$

We continue with the second and the fourth sum and ignore the rest for the moment.

$$
\begin{aligned}
\sum_{\substack{i,j\in[n]\\2\leq j}} p_{i,j}p_{i,n} - \sum_{\substack{i,j'\in[n]\\j'\leq n-1}} p_{i,1}p_{i,j'} &= \sum_{i\in[n]} \left(\sum_{\substack{j\in[n]\\2\leq j}} p_{i,j}\right)\left(1-\sum_{\substack{j'\in[n]\\j'\leq n-1}} p_{i,j'}\right) - \sum_{\substack{i,j'\in[n]\\j'\leq n-1}} p_{i,1}p_{i,j'}\\
&= \sum_{i\in[n]} (1-p_{i,1})\left(1-\sum_{\substack{j'\in[n]\\j'\leq n-1}} p_{i,j'}\right) - \sum_{\substack{i,j'\in[n]\\j'\leq n-1}} p_{i,1}p_{i,j'}\\
&= \sum_{i\in[n]} 1 - \sum_{i\in[n]} p_{i,1} - \sum_{\substack{i,j'\in[n]\\j'\leq n-1}} p_{i,j'} + \sum_{\substack{i,j'\in[n]\\j'\leq n-1}} p_{i,1}p_{i,j'} - \sum_{\substack{i,j'\in[n]\\j'\leq n-1}} p_{i,1}p_{i,j'}\\
&= n-1-(n-1)\\
&= 0.
\end{aligned}
$$

Now, we go back to (2) and consider the first sum.

$$
\begin{aligned}
\sum_{\substack{i,j,j'\in[n]\\2\leq j\leq j'\leq n-1}} p_{i,j}p_{i,j'} &= \sum_{\substack{i,j,j'\in[n]\\j\leq j'\leq n-1}} p_{i,j}p_{i,j'} - \sum_{\substack{i,j,j'\in[n]\\j=1\\j'\leq n-1}} p_{i,j}p_{i,j'}\\
&= \sum_{\substack{i,j,j'\in[n]\\j'\leq j\leq n-1}} p_{i,j}p_{i,j'} - \sum_{\substack{i,j,j'\in[n]\\j=1\\j'\leq n-1}} p_{i,j}p_{i,j'}. \qquad (3)
\end{aligned}
$$

Note that the second equality holds because of symmetry. We go on with the third sum of (2).

$$
\begin{aligned}
-\sum_{\substack{i,j,j'\in[n]\\3\le j\\j'\le j-2}} p_{i,j}p_{i,j'} &= -\sum_{\substack{i,j,j'\in[n]\\3\le j\le n-1\\j'\le j-2}} p_{i,j}p_{i,j'} - \sum_{\substack{i,j'\in[n]\\j'\le n-2}} p_{i,n}p_{i,j'}\\
&= -\sum_{\substack{i,j,j'\in[n]\\3\le j\le n-1\\j'\le j-2}} p_{i,j}p_{i,j'} - \sum_{\substack{i,j'\in[n]\\j'\le n-2}} \left(1-\sum_{\substack{j\in[n]\\j\le n-1}} p_{i,j}\right) p_{i,j'}\\
&= -\sum_{\substack{i,j,j'\in[n]\\3\le j\le n-1\\j'\le j-2}} p_{i,j}p_{i,j'} - \underbrace{\sum_{\substack{i,j'\in[n]\\j'\le n-2}} p_{i,j'}}_{=n-2} + \sum_{\substack{i,j,j'\in[n]\\j\le n-1\\j'\le n-2}} p_{i,j}p_{i,j'}\\
&= 2-n-\sum_{\substack{i,j,j'\in[n]\\3\le j'\le n-1\\j\le j'-2}} p_{i,j}p_{i,j'} + \sum_{\substack{i,j,j'\in[n]\\j\le j'\le n-2}} p_{i,j}p_{i,j'} + \sum_{\substack{i,j,j'\in[n]\\j'\le n-2\\j'+1\le j\le n-1}} p_{i,j}p_{i,j'}\\
&= 2-n-\left(\sum_{\substack{i,j,j'\in[n]\\j\le n-2\\j\le j'-2\le n-3}} p_{i,j}p_{i,j'} + \sum_{\substack{i,j,j'\in[n]\\j\le n-2\\j'=n}} p_{i,j}p_{i,j'}\right) + \sum_{\substack{i,j,j'\in[n]\\j\le j'\le n-2}} p_{i,j}p_{i,j'}\\
&\quad + \left(\sum_{\substack{i,j,j'\in[n]\\j'\le n-2\\j'+1\le j\le n-1}} p_{i,j}p_{i,j'} + \sum_{\substack{i,j,j'\in[n]\\j\le n-2\\j'=n}} p_{i,j}p_{i,j'}\right)\\
&= 2-n-\sum_{\substack{i,j,j'\in[n]\\j\le n-2\\j\le j'-2}} p_{i,j}p_{i,j'} + \sum_{\substack{i,j,j'\in[n]\\j\le j'\le n-1}} p_{i,j}p_{i,j'} - \sum_{\substack{i,j,j'\in[n]\\j\le n-1\\j'=n-1}} p_{i,j}p_{i,j'}\\
&\quad + \sum_{\substack{i,j,j'\in[n]\\j'\le n-2\\j'+1\le j}} p_{i,j}p_{i,j'} \qquad (4)\\
&= 2-n+\sum_{\substack{i,j,j'\in[n]\\j\le n-2\\j'=j+1}} p_{i,j}p_{i,j'} + \sum_{\substack{i,j,j'\in[n]\\j\le n-1\\j'=j}} p_{i,j}p_{i,j'} + \sum_{\substack{i,j,j'\in[n]\\j<j'\le n-1}} p_{i,j}p_{i,j'}\\
&\quad - \sum_{\substack{i,j,j'\in[n]\\j\le n-1\\j'=n-1}} p_{i,j}p_{i,j'} \qquad (5)
\end{aligned}
$$

In the last step we combine the first and the fourth sum of (4) to form the first sum of (5) and simultaneously split the second sum of (4) into two parts, namely the second and third sum of (5).

At this point, we recall (2) and combine (3) and (5) in order to obtain

$$\begin{aligned}
\phi(L(p),p) = {} & 2 - n + \sum_{\substack{i,j,j' \in [n] \\ j \le n-1 \\ j'=j}} p_{i,j} p_{i,j'} + \left(\sum_{\substack{i,j,j' \in [n] \\ j < j' \le n-1}} p_{i,j} p_{i,j'} + \sum_{\substack{i,j,j' \in [n] \\ j' \le j \le n-1}} p_{i,j} p_{i,j'} \right) \\
& + \sum_{\substack{i,j,j' \in [n] \\ j \le n-2 \\ j'=j+1}} p_{i,j} p_{i,j'} - \sum_{\substack{i,j,j' \in [n] \\ j \le n-1 \\ j'=n-1}} p_{i,j} p_{i,j'} - \sum_{\substack{i,j,j' \in [n] \\ j=1 \\ j' \le n-1}} p_{i,j} p_{i,j'} \\
= {} & 2 - n + \sum_{\substack{i,j \in [n] \\ j \le n-1}} p_{i,j}^2 + \sum_{\substack{i \in [n] \\ j,j' \in [n-1]}} p_{i,j} p_{i,j'} + \sum_{\substack{i,j \in [n] \\ j \le n-2}} p_{i,j} p_{i,j+1} - \sum_{\substack{i,j \in [n] \\ j \le n-1}} p_{i,j} p_{i,n-1} \\
& - \sum_{\substack{i,j \in [n] \\ j \le n-1}} p_{i,j} p_{i,1}. \qquad (6)
\end{aligned}$$

For the second sum of (6) we calculate

$$\begin{aligned}
\sum_{\substack{i \in [n] \\ j,j' \in [n-1]}} p_{i,j} p_{i,j'} &= \sum_{i \in [n]} \left(\sum_{j \in [n-1]} p_{i,j} \right) \left(\sum_{j' \in [n-1]} p_{i,j'} \right) \\
&= \sum_{i \in [n]} (1 - p_{i,n})(1 - p_{i,n}) \\
&= \sum_{i \in [n]} \left(1 - 2p_{i,n} + p_{i,n}^2\right) \\
&= n - 2 + \sum_{i \in [n]} p_{i,n}^2. \qquad (7)
\end{aligned}$$

For the fourth sum of (6) we calculate

$$- \sum_{\substack{i,j \in [n] \\ j \le n-1}} p_{i,j} p_{i,n-1} = -1 + \sum_{i \in [n]} p_{i,n-1} p_{i,n} \qquad (8)$$

and similarly for the fifth sum

$$- \sum_{\substack{i,j \in [n] \\ j \le n-1}} p_{i,j} p_{i,1} = -1 + \sum_{i \in [n]} p_{i,1} p_{i,n}. \qquad (9)$$

We furthermore have that

$$\begin{aligned}
0 &= 2 - {}^{4n}\!/_{2n} \\
&= 2 - {}^{1}\!/_{2n} \sum_{i \in [n]} 4 \left(\sum_{j \in [n]} p_{i,j} \right) \left(\sum_{j' \in [n]} p_{i,j'} \right). \qquad (10)
\end{aligned}$$

The last step is to recombine (6) and our calculations (7), (8), (9), and (10).

$$\begin{aligned}
\phi(L(p),p) =& 2-n+n-2-1-1+2-{}^1\!/\!{}_{2n}\sum_{i\in[n]}4\left(\sum_{j\in[n]}p_{i,j}\right)\left(\sum_{j'\in[n]}p_{i,j'}\right)\\
&+\left(\sum_{\substack{i,j\in[n]\\ j\le n-1}}p_{i,j}^2+\sum_{i\in[n]}p_{i,n}^2\right)+\left(\sum_{\substack{i,j\in[n]\\ j\le n-2}}p_{i,j}p_{i,j+1}+\sum_{i\in[n]}p_{i,n-1}p_{i,n}\right)+\sum_{i\in[n]}p_{i,1}p_{i,n}\\
=& -{}^1\!/\!{}_{2n}\left(\sum_{i,j,j'\in[n]}4p_{i,j}p_{i,j'}-\sum_{i,j\in[n]}2np_{i,j}^2-\sum_{\substack{i,j\in[n]\\ j\le n-1}}2np_{i,j}p_{i,j+1}-\sum_{i\in[n]}2np_{i,1}p_{i,n}\right)\\
=& -{}^1\!/\!{}_{2n}\sum_{i\in[n]}p_{[i]}\,A(n)\,p_{[i]}^{\mathrm{T}}.
\end{aligned}$$

Here, $A(n)$ is the matrix defined in Lemma 5. As $A(n)$ is negative semidefinite by Lemma 5, we deduce that $\phi(L(p),p)\geq 0$, i.e., the left shift $L(p)$ of a random assignment p is always at least as popular as p itself.

In order to show that $\phi(L(p),p)=0$ if and only if $L(L(p))=p$ we recall Equation 1 in the proof of Lemma 5. We see that $\sum_{i\in[n]}p_{[i]}\,A(n)\,p_{[i]}^{\mathrm{T}}=0$ holds if and only if

$$\begin{aligned}
np_{i,1}+p_{i,n}-2\sum_{j\in[n]}p_{i,j} &= 0 \text{ for all } i\in[n] \text{ and}\\
np_{i,j}+p_{i,j+1}-2\sum_{j'\in[n]}p_{i,j'} &= 0 \text{ for all } i\in[n], j\in[n-1].
\end{aligned}$$

These equations imply that for all $i,j\in[n]$ it has to hold that

$$\begin{aligned}
np_{i,j}+np_{i,(j+1)\bmod_1 n}-2\sum_{j'\in[n]}p_{i,j'} &= np_{i,(j+1)\bmod_1 n}+np_{i,(j+2)\bmod_1 n}-2\sum_{j'\in[n]}p_{i,j'}\\
\Leftrightarrow\qquad p_{i,j} &= p_{i,(j+2)\bmod_1 n}.
\end{aligned}$$

This is equivalent to $L(L(p))=p$.

Put differently, if $L(L(p))\neq p$ then $\phi(L(p),p)>0$. So, for every random assignment p with $L(L(p))\neq p$ there exists another random assignment which is strictly more popular. We deduce that no random assignment p with $L(L(p))\neq p$ can be popular meaning that every popular random assignment p also has to satisfy $L(L(p))=p$. This finishes the proof of Lemma 6. □

We now show the converse direction, i.e., every random assignment p which satisfies $L(L(p))=p$ is popular.

First consider odd $n\in\mathbb{N}$. Note that repeated application of $L(L(p))=p$ directly implies that $p_{i,1}=p_{i,3}=\cdots=p_{i,n}=p_{i,2}=\cdots=p_{i,n-1}$ for all $i\in[n]$. Consequently, $p_{i,j}={}^1\!/\!{}_n$ for all $i,j\in[n]$. Recall that popular random assignments are guaranteed to exist and satisfy $L(L(p))=p$ by Lemmas 5 and 6. We deduce that p with $p_{i,j}={}^1\!/\!{}_n$ for all $i,j\in[n]$ is the unique popular random assignment if all agents have identical preferences and the number of agents is odd.

When taking even n into account, we first define the set of *extremal* random assignments $E(n)$:

$$\begin{aligned} E(n) &= \{e^I \in \mathcal{R}(n) \colon I \subseteq [n], |I| = {}^{n}/_{2}\} \\ e^I_{i,j} &= \begin{cases} {}^{2}/_{n} & \text{if either } i \in I \text{ and } j \text{ odd, or } i \notin I \text{ and } j \text{ even} \\ 0 & \text{else} \end{cases} \end{aligned}$$

$E(n)$ thus consists of random assignments e^I with $e^I_{i,j} \in \{0, {}^{2}/_{n}\}$ for all $i, j \in [n]$ where for every agent probabilities alternate throughout his preference list.

The remaining proofs are structured as follows: In Lemma 7 we first show that all assignments $e^I \in E(n)$ are popular. Next, we show in Lemma 8 that all $p \in \mathcal{R}(n)$ that satisfy $L(L(p)) = p$ can be represented as convex combination of random assignments in $E(n)$. Using the convexity of the set of popular random assignments, we get that every $p \in \mathcal{R}(n)$ that satisfies $L(L(p)) = p$ is popular as well for even n, which completes the proof.

Lemma 7. *Let $(\mathcal{A}, \mathcal{H}, \succsim)$ be an assignment problem where all agents have identical preferences. Furthermore assume that the number of agents is even. Then, all extremal random assignments $e^I \in E(n)$ are popular, i.e., we have that $\phi(e^I, q) \geq 0$ for all $e^I \in E(n), p \in \mathcal{R}(n)$.*

Proof. Let $(\mathcal{A}, \mathcal{H}, \succsim)$ be an assignment problem of size n where all agents have identical preferences and let n be even. Furthermore, choose $e^I \in E(n)$ and $p \in \mathcal{R}(n)$ arbitrarily. We have that

$$\begin{aligned} \phi(e^I, p) &= \sum_{i,j,j' \in [n]} e^I_{i,j} p_{i,j'} \phi_i(h_j, h_{j'}) \\ &= \sum_{\substack{i,j,j' \in [n] \\ i \in I}} e^I_{i,j} p_{i,j'} \phi_i(h_j, h_{j'}) + \sum_{\substack{i,j,j' \in [n] \\ i \notin I}} e^I_{i,j} p_{i,j'} \phi_i(h_j, h_{j'}) \end{aligned}$$

By assumption, we have that $\phi_i(h_j, h_{j'}) = 1$ if $j < j'$, $\phi_i(h_j, h_{j'}) = -1$ if $j > j'$ and $\phi_i(h_j, h_{j'}) = 0$ if $j = j'$. For the case that $i \in I$ recall that $e^I_{i,j} = {}^{2}/_{n}$ for odd j and $e^I_{i,j} = 0$ for even j. Consequently, there are ${}^{n}/_{2}$ entries in the ith row of e^I larger than zero, out of which $\lfloor {}^{j'}/_{2} \rfloor$ correspond to houses preferred to $h_{j'}$ while $\lfloor {}^{n-j'}/_{2} \rfloor = {}^{n}/_{2} - \lceil {}^{j'}/_{2} \rceil$ correspond to houses less preferred than $h_{j'}$. In total, this gives

$$\begin{aligned} \sum_{\substack{i,j,j' \in [n] \\ i \in I}} e^I_{i,j} p_{i,j'} \phi_i(h_j, h_{j'}) &= \sum_{\substack{i,j' \in [n] \\ i \in I}} {}^{2}/_{n} \left(\lfloor {}^{j'}/_{2} \rfloor - ({}^{n}/_{2} - \lceil {}^{j'}/_{2} \rceil)\right) p_{i,j'} \\ &= {}^{2}/_{n} \sum_{\substack{i,j' \in [n] \\ i \in I}} (j' - {}^{n}/_{2}) p_{i,j'}. \end{aligned}$$

If on the other hand $i \notin I$, we have that $e^I_{i,j} = 0$ for odd j and $e^I_{i,j} = {}^{2}/_{n}$ for even j. Following similar arguments as before, we here have $\lfloor {}^{j'-1}/_{2} \rfloor$ non-negative entries in the ith row of e^I that correspond to houses preferred to $h_{j'}$ and $\lfloor {}^{n-j'+1}/_{2} \rfloor = {}^{n}/_{2} - \lceil {}^{j'-1}/_{2} \rceil$ entries corresponding to houses less preferred than $h_{j'}$. Thus, we have that

$$\begin{aligned} \sum_{\substack{i,j,j' \in [n] \\ i \notin I}} e^I_{i,j} p_{i,j'} \phi_i(h_j, h_{j'}) &= \sum_{\substack{i,j' \in [n] \\ i \notin I}} {}^{2}/_{n} \left(\lfloor {}^{j'-1}/_{2} \rfloor - ({}^{n}/_{2} - \lceil {}^{j'-1}/_{2} \rceil)\right) p_{i,j'} \\ &= {}^{2}/_{n} \sum_{\substack{i,j' \in [n] \\ i \notin I}} (j' - 1 - {}^{n}/_{2}) p_{i,j'}. \end{aligned}$$

In total, we obtain

$$\begin{aligned}\phi(e^I, p) &= {}^2\!/n \left(\sum_{\substack{i,j' \in [n] \\ i \in I}} (j' - {}^n\!/2) p_{i,j'} + \sum_{\substack{i,j' \in [n] \\ i \notin I}} (j' - 1 - {}^n\!/2) p_{i,j'} \right) \\ &= {}^2\!/n \left(\sum_{i,j' \in [n]} (j' - {}^n\!/2) p_{i,j'} - \sum_{\substack{i,j' \in [n] \\ i \notin I}} p_{i,j'} \right) \\ &= {}^2\!/n \left(\sum_{j' \in [n]} (j' - {}^n\!/2) - {}^n\!/2 \right) \\ &= {}^2\!/n \left({}^1\!/2\, n(n+1) - {}^n\!/2\, n - {}^n\!/2 \right) \\ &= 0. \end{aligned}$$

Hence, every extremal random assignment is popular. □

Lemma 8. *Let $(\mathcal{A}, \mathcal{H}, \succsim)$ be an assignment problem where all agents have identical preferences. Furthermore assume that the number of agents is even. Then, all random assignments $p \in \mathcal{R}(n)$ that satisfy $L(L(p)) = p$ can be represented as convex combinations of extremal random assignments.*

Proof. Let $(\mathcal{A}, \mathcal{H}, \succsim)$ be an assignment problem of size n where all agents have identical preferences and let n be even. Furthermore, let $p \in \mathcal{R}(n)$ be a random assignment that satisfies $L(L(p)) = p$. We show that there exist $m \leq 2n$, $\lambda \in \mathbb{R}^m$, and a sequence $(I_k)_{k \in [m]}$ with $I_k \subseteq [n], |I_k| = {}^n\!/2$ for all $k \in [m]$ such that $\lambda_k \geq 0$ for all $k \in [m]$, $\sum_{k \in [m]} \lambda_k = 1$ and $\sum_{k \in [m]} \lambda_k e^{I_k} = p$.

To this end, let p be a random assignment such that $L(L(p)) = p$ and determine m, λ, and $(I_k)_{k \in [m]}$ with the help of Algorithm 3.

Algorithm 3 Find m extremal random assignments together with positive weights such that the convex combination equals p.

Set $p^0 = p, k = 0$.
while $p^k \neq 0$ **do**
  Set $k = k + 1$
  Choose $I_k \subseteq [n]$ such that $|I_k| = {}^n\!/2$ and $p^{k-1}_{i,1} \geq p^{k-1}_{i',1}$ for all $i \in I_k, i' \in [n] \setminus I_k$.
  Set $\lambda_k = {}^n\!/2 \min_{i \in I_k, i' \in [n] \setminus I_k} \{p^{k-1}_{i,1}, p^{k-1}_{i',2}\}$.
  Set $p^k = p^{k-1} - \lambda_k e^{I_k}$.
end while

In order to see that Algorithm 3 works as claimed first note that by $L(L(p)) = p$ we have that $p_{i,1} = p_{i,3} = \cdots = p_{i,n-1}$ and $p_{i,2} = p_{i,4} = \cdots = p_{i,n}$ for all $i \in [n]$. As this also holds for all $e^I \in E(n)$ it suffices to consider the first two columns of all p^k. We directly obtain $p_{i,1} + p_{i,2} = {}^2\!/n$ for all $i \in [n]$ and correspondingly $p^k_{i,1} + p^k_{i,2} = p^k_{i',1} + p^k_{i',2}$ for all k and $i, i' \in [n]$. Furthermore, $\sum_{i \in [n]} p_{i,1} = \sum_{i \in [n]} p_{i,2} = 1$ by the definition of a random assignment.

We begin by arguing that for all p^k we have that $p^k \geq 0$ implying that also $\lambda_k \geq 0$. Next, we show that Algorithm 3 indeed terminates after $m \leq 2n$ steps with $p^m = 0$ and $\sum_{k \in [m]} \lambda_k = 1$.

For $k = 0$, $p^k \geq 0$ trivially holds by p being a random assignment. For $k > 0$, note that $\lambda_k = {}^{n}\!/_{2} \min_{i \in I_k, i' \in [n] \setminus I_k} \{p_{i,1}^{k-1}, p_{i',2}^{k-1}\}$. Thus, for $i \in I_k$,

$$\begin{aligned} p_{i,1}^k &= p_{i,1}^{k-1} - \lambda_k e^{I_k} \\ &= p_{i,1}^{k-1} - {}^{2}\!/_{n} \cdot {}^{n}\!/_{2} \min_{i' \in I_k, i'' \in [n] \setminus I_k} \{p_{i',1}^{k-1}, p_{i'',2}^{k-1}\} \\ &= p_{i,1}^{k-1} - \min_{i' \in I_k, i'' \in [n] \setminus I_k} \{p_{i',1}^{k-1}, p_{i'',2}^{k-1}\} \\ &\geq 0 \end{aligned}$$

and $p_{i,1}^k = p_{i,1}^{k-1} \geq 0$ for $i \in [n] \setminus I_k$. Similarly, for $i \in [n] \setminus I_k$,

$$\begin{aligned} p_{i,2}^k &= p_{i,2}^{k-1} - \lambda_k e^{I_k} \\ &= p_{i,2}^{k-1} - {}^{2}\!/_{n} \cdot {}^{n}\!/_{2} \min_{i' \in I_k, i'' \in [n] \setminus I_k} \{p_{i',1}^{k-1}, p_{i'',2}^{k-1}\} \\ &= p_{i,2}^{k-1} - \min_{i' \in I_k, i'' \in [n] \setminus I_k} \{p_{i',1}^{k-1}, p_{i'',2}^{k-1}\} \\ &\geq 0 \end{aligned}$$

and $p_{i,2}^k = p_{i,2}^{k-1} \geq 0$ for all $i \in I_k$. Consequently, for all k we have that $p^k \geq 0$ and also $\lambda_k \geq 0$.

To see that indeed $\lambda_k > 0$ for all k assume for contradiction that there exists k' such that $\lambda_{k'} = 0$. We deduce that for k', $\min_{i \in I_{k'}, i' \in [n] \setminus I_{k'}} \{p_{i,1}^{k'-1}, p_{i',2}^{k'-1}\} = 0$. Set $i_{\min} \in [n], j_{\min} \in \{1, 2\}$ such that the minimum is achieved for $p_{i_{\min}, j_{\min}}^{k'-1}$, i.e., $p_{i_{\min}, j_{\min}}^{k'-1} \in \arg \min_{i \in I_{k'}, i' \in [n] \setminus I_{k'}} \{p_{i,1}^{k'-1}, p_{i',2}^{k'-1}\}$ and in addition let $j'_{\min} = 3 - j_{\min}$. By the general definition of I_k we have that $p_{i,1}^{k'-1} \geq p_{i',1}^{k'-1}$ for all $i \in I_{k'}, i' \in [n]$. As we have that $p_{i,1}^{k'-1} + p_{i,2}^{k'-1} = p_{i',1}^{k'-1} + p_{i',2}^{k'-1}$ for all $i, i' \in [n]$, we trivially also have the dual condition $p_{i',2}^{k'-1} \geq p_{i,2}^{k'-1}$ for all $i' \in [n] \setminus I_{k'}, i \in [n]$. The identical argument additionally gives that if $p_{i_{\min}, j_{\min}}^{k'-1} = 0$ then by $p^{k'-1} \neq 0$ we obtain that $p_{i_{\min}, j'_{\min}}^{k'-1} > 0$. Let $p_{1,1}^{k'-1} + p_{1,2}^{k'-1} = \alpha$. We now have $\sum_{i \in [n]} p_{i, j_{\min}}^{k-1} < {}^{n}\!/_{2}\, \alpha$ while simultaneously $\sum_{i \in [n]} p_{i, j'_{\min}}^{k'-1} > {}^{n}\!/_{2}\, \alpha$. Recall that on the other hand $\sum_{i \in [n]} p_{i,1} = \sum_{i \in [n]} p_{i,2}$ and by the definition of $E(n)$ consequently also $\sum_{i \in [n]} p_{i,1}^{k'-1} = \sum_{i \in [n]} p_{i,2}^{k'-1}$, a contradiction. Hence, no such k' can exist and we have that $\lambda_k > 0$ for all k.

Now, observe that in the kth iteration of Algorithm 3 all $p_{i_{\min}, j_{\min}}^{k-1} \in \arg \min_{i \in I_k, i' \in [n] \setminus I_k} \{p_{i,1}^{k'-1}, p_{i',2}^{k'-1}\}$ are set to zero. As this happens with at least one entry $p_{i,j}^{k-1}$ of the first two columns per iteration and at most once per $p_{i,j}^{k-1}$, the algorithm terminates after $m \leq 2n$ iterations.[3]

$\sum_{k \in [m]} \lambda_k = 1$ follows trivially from $p_{i,1} + p_{i,2} = {}^{2}\!/_{n}$ for all $i \in [n]$, $e_{i,1}^I + e_{i,2}^I = {}^{2}\!/_{n}$ for all $i \in [n], I \subseteq [n], |I| = {}^{n}\!/_{2}$, and $p_{i,1}^m + p_{i,2}^m = (p_{i,1} + p_{i,2}) - \sum_{k \in [m]} \lambda_k \left(e_{i,1}^{I_k} + e_{i,2}^{I_k}\right) = 0$. □

A.3 Proof of Theorem 3

Theorem 3. *There exist assignment problems for which no popular random assignment satisfies weak envy-freeness when $n \geq 5$.*

[3]It can even be argued that in the mth iteration of Algorithm 3 all $p_{i,1}^m$ and $p_{i',2}^m$, $i \in I_m, i' \in [n] \setminus I_m$ are set to zero simultaneously. Hence, $m \leq n + 1$.

Proof. Consider the assignment problem $(\mathcal{A}, \mathcal{H}, \succsim)$ with five agents $\mathcal{A} = \{a_1, \ldots, a_5\}$, five houses $\mathcal{H} = \{h_1, \ldots, h_5\}$, and

$$\succsim = \begin{array}{ll} a_1\colon & h_1, h_2, h_3, h_4, h_5 \\ a_2\colon & h_1, h_2, h_3, h_4, h_5 \\ a_3\colon & h_1, h_2, h_3, h_4, h_5 \\ a_4\colon & h_4, h_1, h_2, h_3, h_5 \\ a_5\colon & h_1, h_4, h_2, h_5, h_3 \end{array} .$$

With the aid of a computer or by solving several inequalities, it can be shown that for all popular random assignments $p \in \mathcal{R}(n)$, $p_{i,j} = 1/3$ for all $i, j \in [3]$.

Consequently, only a_4 and a_5 are competing for houses h_4 and h_5. Even though they share the strict preference $h_4 \succ h_5$, a_4 ranks h_4 higher and h_5 lower in comparison to a_5. We compute that popularity of p implies $2/3 \leq p_{5,5} \leq 1$. Thus, every popular random assignment p is of the form

$$p = \begin{pmatrix} 1/3 & 1/3 & 1/3 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 1-\lambda \\ 0 & 0 & 0 & 1-\lambda & \lambda \end{pmatrix}$$

with $2/3 \leq \lambda \leq 1$. For all such assignments p, a_5 SD-prefers a_4's allocation to his own.

Note that similar profiles can also be constructed for all $n \geq 5$. □

A.4 Proof of Theorem 4

Theorem 4. *No popular random assignment rule satisfies weak strategyproofness when $n \geq 7$.*

Proof. Consider the assignment problem $(\mathcal{A}, \mathcal{H}, \succsim)$ with seven agents $\mathcal{A} = \{a_1, \ldots, a_7\}$, seven houses $\mathcal{H} = \{h_1, \ldots, h_7\}$, and

$$\succsim = \begin{array}{ll} a_1\colon & h_1, h_2, h_3, h_6, h_4, h_5, h_7 \\ a_2\colon & h_1, h_2, h_3, h_6, h_4, h_5, h_7 \\ a_3\colon & h_1, h_2, h_3, h_6, h_4, h_5, h_7 \\ a_4\colon & h_4, h_5, h_1, h_2, h_3, h_6, h_7 \\ a_5\colon & h_4, h_5, h_1, h_2, h_3, h_6, h_7 \\ a_6\colon & h_1, h_6, h_4, h_3, h_5, h_2, h_7 \\ a_7\colon & h_1, h_4, h_6, h_7, h_2, h_5, h_3 \end{array} .$$

With the aid of a computer or by solving several inequalities, one can compute the vertices of the convex polytope containing all popular random assignments p. For all those, we deduce that $1/2 \leq p_{7,7} = 1 - p_{7,6} \leq 1$. Put differently, a_7 receives h_7 with probability at least $1/2$ and h_1 to h_5 with probability 0.

Now, let a_7 alter his preferences in a way such that h_6 shall be his most preferred house while h_7 becomes the least preferred one leaving everything else unchanged, i.e.,

$$a_7'\colon \quad h_6, h_1, h_4, h_2, h_5, h_3, h_7.$$

For the new assignment problem $(\mathcal{A}, \mathcal{H}, \succsim')$ with $\succsim'_a = \succsim_a$ for all $a \in \mathcal{A} \setminus \{a_7\}$ we once more compute all popular random assignments p'. Now, we obtain that $0 \leq p'_{7,7} = 1 - p'_{7,6} \leq 2/5$. Hence, in all random assignments $p' \in \mathcal{R}(7)$ that are popular with respect to $(\mathcal{A}, \mathcal{H}, \succsim')$, a_7

receives h_6 with strictly more probability than in p while getting h_7 less frequently. We see that a_7 prefers his new allocation to the one he would have received before, $p'_{[7]} \succ^{\text{SD}}_{a_7} p_{[7]}$.

Introducing additional agents and houses such that each agent a_k has house h_k as first preference, $k \geq 8$, allows us to construct preference profiles for $n \geq 8$, each admitting the same manipulation beneficial for a_7. Thus, no random assignment rule can satisfy popularity and weak strategyproofness at the same time when $n \geq 7$. □