Fractional Hedonic Games

Introduction
An important issue in multi-agent systems is the exploitation of synergies via coalition formation. We initiate the formal study of fractional hedonic games. In fractional hedonic games, the utility of a player in a coalition structure is the average value he ascribes to the members of his coalition. Among other settings, this covers situations in which there are several types of agents and each agent desires to be in a coalition in which the fraction of agents of his own type is minimal. Fractional hedonic games not only constitute a natural class of succinctly representable coalition formation games, but also provide an interesting framework for network clustering. We propose a number of conditions under which the core of fractional hedonic games is non-empty and provide algorithms for computing a core stable outcome.

Fractional Hedonic Games
A hedonic game is a tuple \((N, \succsim_1, \ldots, \succsim_n)\), where \(N = \{1, \ldots, n\}\) is a set of players, \(\succsim_i\) a transitive and complete preference relation over the coalitions \(i\) is a member of. A coalition structure \(\pi\) is a partition of the players into coalitions. By \(\pi(i)\) we denote the coalition \(i\) is in under coalition structure \(\pi\).

Question Which coalitions can reasonably be expected to form?
A hedonic game is a fractional hedonic game if for each player \(i\) there is a valuation function \(v_i: N \to \mathbb{R}\) of the players, such that
\[
S \succsim T \text{ if and only if } \frac{\sum_{j\in S} v_i(j)}{|S|} \geq \frac{\sum_{j\in T} v_i(j)}{|T|}.
\]
Intuition Each player \(i\) has value \(v_i(j)\) for player \(j\) being in the same coalition. The utility of a coalition \(S\) for player \(i\) is the average value of its coalition members.

Preferences are symmetric if players like/dislike each other with the same intensity. Preferences are simple if \(v_i(j)\) is either 0 or 1 for all players \(i\) and \(j\).

Observation Simple and symmetric fractional hedonic games can naturally be represented by undirected and unweighted graphs, with vertices corresponding to players and an edge \((i, j)\) indicating that \(v_i(j) = 1\).

Negative Results

Theorem For fractional hedonic games, the core can be empty.

Corollary For every Bakers and Millers game, there is a unique finest strict core stable partition (up to renaming players of the same type), which, moreover, can be computed in linear time.

Graphs with Large Girth

Theorem For simple symmetric fractional hedonic games represented by graphs with girth at least five, the core is non-empty.

References

Bakers and Millers

Players are divided into types \(\theta_1, \ldots, \theta_k\), e.g., bakers and millers. Player \(i\) of type \(\theta(i)\) prefers to be in a coalition with as many players of other types as and if few players of the same type as possible:
\[
S \succsim T \text{ if and only if } \frac{|S \cap \theta(i)|}{|S|} \leq \frac{|T \cap \theta(i)|}{|T|}.
\]

These Bakers and Millers games are naturally modelled as complete \(k\)-partite graphs.