Computational Aspects of Covering in Dominance Graphs*

Felix Brandt and **Felix Fischer**
Computer Science Department
University of Munich
80538 Munich, Germany
{brandtf,fischerf}@tcs.ifi.lmu.de

Abstract

Various problems in AI and multiagent systems can be tackled by finding the "most desirable" elements of a set given some binary relation. Examples can be found in areas as diverse as voting theory, game theory, and argumentation theory. Some particularly attractive solution sets are defined in terms of a *covering* relation—a transitive subrelation of the original relation. We consider three different types of covering (upward, downward, and bidirectional) and the corresponding solution concepts known as the *uncovered set* and the *minimal covering set*. We present the first polynomial-time algorithm for finding the minimal bidirectional covering set (an acknowledged open problem) and prove that deciding whether an alternative is in a minimal upward or downward covering set is NP-hard. Furthermore, we obtain various set-theoretical inclusions, which reveal a strong connection between von Neumann-Morgenstern stable sets and upward covering on the one hand, and the Banks set and downward covering on the other hand. In particular, we show that every stable set is also a minimal upward covering set.

Introduction

Various problems in AI and multiagent systems can be tackled by identifying the "most desirable" elements of a set of alternatives according to some binary dominance relation. Examples are diverse and include finding valid arguments in argumentation theory (*e.g.*, Dung, 1995), selecting socially preferred candidates in social choice settings (*e.g.*, Fishburn, 1977; Laslier, 1997; Brandt, Fischer, & Harrenstein, 2007), determining the winner of a competition (*e.g.*, Dutta & Laslier, 1999), choosing the optimal strategy in a symmetric two-player zero-sum game (Duggan & Le Breton, 1996), and investigating which coalitions will form in cooperative game theory (Brandt & Harrenstein, 2007). In social choice theory, where dominance-based solutions are most prevalent (see, *e.g.*, Fishburn, 1977; Laslier, 1997; Brandt, Fischer, & Harrenstein, 2007), the dominance relation can simply be defined as the pairwise majority relation, *i.e.*, an alternative a is said to dominate another alternative b if the number of individuals preferring a to b exceeds the number of individuals preferring b to a. McGarvey (1953) has shown that *any* asymmetric dominance relation can be realized by a particular preference profile, even if individual preferences are required to be linear. As is well known from Condorcet's paradox, however, the dominance relation may very well contain cycles. This implies that the dominance relation need not have a maximum, or even a maximal, element, even if the underlying individual preferences do. Thus, the concept of maximality is rendered untenable in most cases. As a consequence, various so-called *solution concepts* that take over the role of maximality in non-transitive relations have been suggested (*e.g.*, von Neumann & Morgenstern, 1944; Fishburn, 1977; Miller, 1980; Banks, 1985; Dutta, 1988). Some particularly attractive solution sets are defined in terms of a *covering* relation—a subrelation of the dominance relation (Gillies, 1959; Fishburn, 1977; Miller, 1980; Dutta, 1988). There are three natural conceptions of covering:

- *upward covering*, where an alternative a is said to cover another alternative b if a dominates b and the alternatives dominating a form a subset of those dominating b,
- *downward covering*, where a covers b if a dominates b and the alternatives dominated by b form a subset of those dominated by a, and
- *bidirectional covering*, where a covers b if a covers b upward and downward.

In tournaments, *i.e.*, complete dominance relations, all notions of covering coincide.[1] Tournaments have received particular attention in social choice theory because the pairwise majority relation is guaranteed to be complete given an odd number of voters with linear preferences.

Since each of the covering relations is transitive, maximal (*i.e.*, uncovered) elements are guaranteed to exist if the set of alternatives is finite. Consequently, the set of uncovered alternatives for a given covering relation constitutes a natural solution concept. In tournaments, the resulting *uncovered set* consists precisely of those alternatives that dominate any

*This material is based upon work supported by the Deutsche Forschungsgemeinschaft under grant BR 2312/3-1.

[1] Additional covering relations due to Fishburn (1977) and Miller (1980), which do not require that a dominates b for a to cover b, were introduced in the context of tournaments where they coincide with all other covering relations. Since they possess some undesirable properties for *incomplete* dominance relations (see, *e.g.*, Dutta & Laslier, 1999), we will not consider them in this paper.

other alternative along a domination path of length one or two and is the finest solution concept that satisfies the so-called *expansion* property (Moulin, 1986). Dutta & Laslier (1999) generalize Moulin's result and provide an appealing axiomatic characterization of the bidirectional uncovered set for incomplete dominance relations.

Uncovered sets tend to be rather large and are not idempotent. Thus, a natural refinement of the uncovered set can be obtained by repeatedly computing the uncovered set until no more alternatives can be removed. This solution is called the *iterated uncovered set* (see Laslier, 1997). Unfortunately, the iterated uncovered set does not satisfy several criteria that are considered essential for any solution concept (such as monotonicity). As a solution to these problems, Dutta (1988) proposed the *minimal covering set*, which is the smallest set of alternatives (with respect to set inclusion) that satisfies specific notions of internal and external stability (with respect to the underlying covering relation). Minimal covering sets are always contained in their corresponding iterated uncovered set. The minimal *bidirectional* covering set of any dominance relation is unique and considered especially attractive because it satisfies an outstanding number of desirable criteria (Laslier, 1997; Dutta & Laslier, 1999; Peris & Subiza, 1999). In addition to the minimal bidirectional covering set, minimal *upward* and *downward* covering sets are considered for the first time in this paper.

Naturally, computational tractability is a crucial property of any solution concept, simply because intractability renders a concept useless for large instances that do not possess additional structure. While for some solution sets either efficient algorithms or hardness results have been put forward (see, *e.g.*, Woeginger, 2003; Conitzer, 2006; Conitzer, Davenport, & Kalagnanam, 2006; Brandt, Fischer, & Harrenstein, 2007), very little is known about the computational complexity of solution concepts based on covering relations. In fact, Laslier states that the "computational needs for the different methods to be applied also vary a lot. [...] Unfortunately, no algorithm has yet been published for finding the Minimal Covering set [...] of large tournaments. For tournaments of order 10 or more, it is almost impossible to find (in the general case) these sets at hand" (Laslier, 1997, p. 8). In this paper, we provide polynomial-time algorithms for finding the minimal bidirectional covering set (the set Laslier was referring to), the essential set (an attractive subset of the minimal bidirectional covering set), and iterated uncovered sets. Moreover, we show that deciding whether an alternative is in a minimal upward or downward covering set is NP-hard.

Preliminaries

Let A be a finite set of *alternatives* and let $\succ \subseteq A \times A$ be an *asymmetric* and *irreflexive* relation on A, the *dominance relation*. The fact that an alternative a dominates another alternative b, denoted $a \succ b$, means that a is "strictly better than" b or "beats" b in a pairwise comparison. We do *not* in general assume *completeness* or *transitivity* of $\succ$ but allow for ties among alternatives and cyclical dominance. A dominance relation that does satisfy completeness is called a *tournament*. In the literature, the more general case of an incomplete dominance relation as studied in this paper is sometimes referred to as a *weak tournament*. We will sometimes find it convenient to view $\succ$ as a directed *dominance graph* (V, E) with vertex set $V = A$ and $(a, b) \in E$ if and only if $a \succ b$, or as a (skew-symmetric) *adjacency matrix* $M_{A,\succ} = (m_{ij})_{i,j\in A}$ where $m_{ij} = 1$ if $i \succ j$, $m_{ij} = -1$ if $j \succ i$, and $m_{ij} = 0$ otherwise.

We say that an alternative $a \in A$ is *undominated* relative to $\succ$ whenever there is no alternative $b \in A$ with $b \succ a$. A special type of undominated alternative is the *Condorcet winner*, an alternative that dominates every other alternative. The concept of a *maximal element* we reserve in this paper to denote an undominated element of a transitive relation. Given its asymmetry, transitivity of the dominance relation implies its acyclicity. The implication in the other direction holds for tournaments but not for the general case. Failure of transitivity or completeness makes that a Condorcet winner need not exist; failure of acyclicity, moreover, that the dominance relation need not even contain maximal elements. As such, the obvious notion of maximality is no longer available to single out the "best" alternatives. Other concepts have been devised to take over its role. In the context of this paper, a *choice set* is a function f from the set of ordered pairs $(A, \succ)$ into the set of nonempty subsets of A. While choice sets are always computed for a pair $(A, \succ)$, we will often omit $\succ$ where the meaning is obvious from the context.

Covering Relations and Choice Sets

In this paper we focus on choice sets based on transitive subrelations of the dominance relation called covering relations.

Definition 1 (covering) *Let A be a set of alternatives, $\succ$ a dominance relation on A. Then, for any $x, y \in A$,*

- *x* upward covers *y, denoted xC_uy, if $x \succ y$ and for all $z \in A$, $z \succ x$ implies $z \succ y$,*
- *x* downward covers *y, denoted xC_dy, if $x \succ y$ and for all $z \in A$, $y \succ z$ implies $x \succ z$, and*
- *x* bidirectionally covers *y, denoted xC_by, if xC_uy and xC_dy.*

It is easily verified that each of these covering relations is asymmetric and transitive, and thus a *strict partial order* on A. The set of maximal elements of such an ordering is referred to as the *uncovered set*.

Definition 2 (uncovered set) *Let A be a set of alternatives, C a covering relation on A. Then, the* uncovered set *of A with respect to C is defined as*

$$UC_C(A) = \{\, x \in A \mid yCx \textit{ for } \text{no } y \in A \,\}.$$

In particular, we will write $UC_u = UC_{C_u}$ for the upward uncovered set, $UC_d = UC_{C_d}$ for the downward uncovered set, and $UC_b = UC_{C_b}$ for the bidirectional uncovered set.

For an example of uncovered sets according to the different covering relations, consider the dominance graph of Figure 1. Here, a upward covers b because f, the only alternative that dominates a, also dominates b. a itself is not

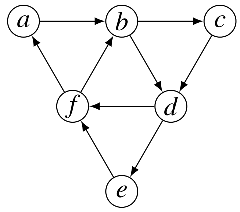

Figure 1: Example of upward, downward, and bidirectional covering. The set A of alternatives is partitioned into the upward uncovered set $UC_u(A) = \{a, c, e\}$ and the downward uncovered set $UC_d(A) = \{b, d, f\}$.

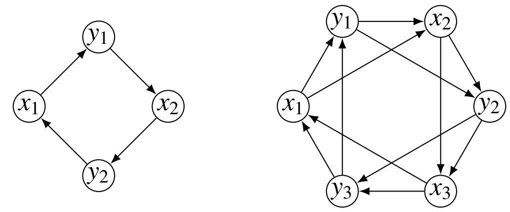

Figure 2: Minimal upward and downward covering sets need not be unique. There are two minimal upward covering sets $\{x_1, x_2\}$ and $\{y_1, y_2\}$ in the dominance graph on the left, and two minimal downward covering sets $\{x_1, x_2, x_3\}$ and $\{y_1, y_2, y_3\}$ in the dominance graph on the right.

upward covered by f because d and e dominate f but not a. On the other hand, f downward covers a because it dominates b, the only alternative dominated by a. Neither a nor f downward covers b, because the latter is the only alternative that dominates c. By symmetry of the graph, we have $UC_u(A) = \{a, c, e\}$, $UC_d(A) = \{b, d, f\}$, and $UC_b(A) = A$.

The uncovered set is not idempotent and may be applied iteratively to obtain finer solutions. We write $UC_C^k(A) = UC_C(UC_c^{k-1}(A))$ for the kth iteration of UC_C on A and define the iterated uncovered set as the fixed point $UC_C^\infty(A) = UC_C^m(A)$ if $UC_C^m(A) = UC_C^{m+1}(A)$ for some m.

Dutta (1988) proposes a further refinement of the iterated uncovered set in tournaments, which is based on the notion of a *covering set*.

Definition 3 (covering set) *Let A be a set of alternatives, $\succ$ a dominance relation on A, and C a covering relation based on $\succ$. Then, $B \subseteq A$ is a* covering set *for A under C if*

(i) *$UC_C(B) = B$, and*

(ii) *for all $x \in A \setminus B$, $x \notin UC_C(B \cup \{x\})$.*

Properties (*i*) and (*ii*) are referred to as *internal* and *external stability* of a covering set, respectively.

For tournaments, where the different notions of covering and uncovered sets coincide, Dutta (1988) proves the existence of a unique *minimal* covering set with respect to set inclusion. Peris & Subiza (1999) and Dutta & Laslier (1999) extend this result to incomplete dominance graphs by showing that there is always a unique minimal *bidirectional* covering set. We will denote this set by MC. The minimal bidirectional covering set is regarded as particularly attractive because it satisfies an outstanding number of desirable criteria (Laslier, 1997; Dutta & Laslier, 1999; Peris & Subiza, 1999). Furthermore, Duggan & Le Breton (1996) have pointed out that the minimal covering set of a tournament coincides with the *weak saddle* of the corresponding adjacency game—a solution concept that was proposed independently (and much earlier) by Shapley (1964). Similar *set-valued* solution concepts such as CURB sets and Set-Nash equilibria have recently received increased attention in computational contexts (Benisch, Davis, & Sandholm, 2006; Lavi & Nisan, 2005).

Figure 2 illustrates that uniqueness of a minimal covering set is not guaranteed for upward or downward covering. $\{x_1, x_2\}$ is a minimal upward covering set for the dominance graph on the left because $x_1 C_u y_1$ in $\{x_1, x_2, y_1\}$ and $x_2 C_u y_2$ in $\{x_1, x_2, y_2\}$, while no single alternative can cover the remaining three alternatives. By a symmetric argument, the same holds for $\{y_1, y_2\}$. For the dominance graph on the right, x_i is downward uncovered in $\{x_1, x_2, x_3\}$ and y_i is downward covered in $\{x_1, x_2, x_3, y_i\}$ for all $i \in \{1, 2, 3\}$. Since any proper subset of $\{x_1, x_2, x_3\}$ fails to cover y_i for some $i \in \{1, 2, 3\}$, we have actually found a minimal downward covering set. We leave it to the reader to verify that apart from the symmetric set $\{y_1, y_2, y_3\}$ there are no additional minimal downward covering sets.

A serious defect of downward covering sets is that they may fail to exist for very simple instances. For example, it is easily verified that no subset of $A = \{a, b, c\}$ with $a \succ b \succ c$ satisfies both internal and external stability. As we will see later on, deciding the existence of a downward covering set is NP-hard. Minimal upward covering sets, on the other hand, are guaranteed to exist. The following theorem can be proven by showing that any iteration of the upward uncovered set is externally stable. We omit the proof.

Theorem 1 *There always exists a minimal upward covering set.*

A refinement of the minimal bidirectional covering set can be obtained by considering the *adjacency game* (called tournament game by Dutta & Laslier, 1999) in which two parties propose alternatives $x, y \in A$. The first party wins if $x \succ y$, the second party wins if $y \succ x$, and the game ends in a tie if neither of the two alternatives dominates the other. In other words, the adjacency game $\Gamma(A, \succ)$ is a symmetric two-player zero-sum game where actions correspond to elements of A and the payoff of the first player is given by the adjacency matrix $M_{A,\succ}$ of the dominance graph for A and $\succ$. A *strategy* in such a game consists of a probability distribution over the different actions. A pair of strategies is called a *Nash equilibrium* if none of the two players can increase his (expected) payoff by changing his strategy, given that the strategy of the other player remains the same (see, *e.g.*, Osborne & Rubinstein, 1994). Dutta & Laslier (1999) define the *essential set* as the set of alternatives for which the corresponding action is played with positive probability in some Nash equilibrium of the adjacency game. It suffices to restrict attention to symmetric equilibria because the set of equilibria in a zero-sum game is convex.

Definition 4 (essential set) *Let A be a set of alternatives, $\succ$ a dominance relation on A. Then, the* essential set *of A is defined as*

$$ES(A) = \{\, a \in A \mid s_a > 0 \text{ for some } (s, s) \in N(\Gamma(A, \succ)) \,\},$$

where $N(\Gamma)$ denotes the set of Nash equilibria of game Γ and s_a the probability of action a under strategy s.

The essential set generalizes the *bipartisan set*, which is defined in terms of the unique Nash equilibrium of the adjacency game in the case of a complete dominance relation (Laslier, 1997). The essential set and the solution concepts based on bidirectional covering can be ordered linearly with respect to set inclusion: $ES \subseteq MC \subseteq UC_b^\infty \subseteq UC_b$.

Set-theoretical Relationships

By analyzing set-theoretical inclusions and disjunctions between choice sets, one can gain additional insight into the reasons why, and to which extent, particular choice sets are different. A complete characterization of the relationships between various choice sets in tournaments is given by Laslier (1997), including all the choice sets studied in this paper. Bordes (1983) investigates relationships between the different variants of the uncovered set in general dominance graphs. We extend these results for the three variants of the minimal covering set.

It is rather straightforward to show that every minimal covering set has to be contained in the iterated uncovered set for the same dominance relation, and that MC is both an upward and a downward covering set. However, we were able to construct dominance graphs with ten alternatives where an additional minimal upward or downward covering set exists that does *not* intersect with MC. Figure 1 illustrates that upward and downward uncovered sets, and hence the corresponding minimal covering sets, can have an empty intersection. This example also reveals an interesting relationship between covering sets and two well-known choice sets, which we introduce next.

A set S of alternatives is called *stable* if no element inside the set can be removed because it is dominated by some other element in the set, while no element outside the set can be included in the set because some element inside the set dominates it (von Neumann & Morgenstern, 1944).

Definition 5 (stable set) *Let A be a set of alternatives, $\succ$ a dominance relation on A. Then $S \subseteq A$ is a* (von Neumann-Morgenstern) stable set *if*

(i) $a \succ b$ for no $a, b \in S$ and

(ii) for all $a \notin S$ there is some $b \in S$ with $b \succ a$.

Stable sets are neither guaranteed to exist nor to be unique. Elementary counterexamples are cycles consisting of three or four alternatives, respectively.

The *Banks set* consists of those elements that are the maximal element of $\succ$ for some subset of the alternatives on which $\succ$ is complete and transitive and which is itself maximal with respect to set inclusion (Banks, 1985).

Definition 6 (Banks set) *Let A be a set of alternatives, $\succ$ a dominance relation on A. Then, an alternative a is in the* Banks set *of A, denoted $a \in B(A)$, if there exists $X \subseteq A$ such that $\succ$ is complete and transitive on X with maximal element a and there is no $b \in A$ such that $b \succ x$ for all $x \in X$.*

Returning to the dominance graph of Figure 1, it is easily verified that there exists a unique stable set $S = \{a, c, e\}$, and that $B(A) = \{b, d, f\}$. The relationship between upward covering and stable sets on the one hand and downward covering and the Banks set on the other is no mere coincidence. We state the following two theorems without proof.

Theorem 2 *Every stable set is a minimal upward covering set and thus contained in the upward uncovered set.*

It is worth noting that there can be additional minimal upward covering sets, which may have an empty intersection with all stable sets.

Theorem 3 *The Banks set intersects with every downward covering set and is contained in the downward uncovered set.*[2]

Computing Choice Sets

Naturally, computational tractability is a crucial property of any choice set, simply because intractability renders a concept useless for large instances that do not possess additional structure. In the following, we assume the reader to be familiar with the well-known chain of complexity classes $AC^0 \subset P \subseteq NP$, and the notion of polynomial-time reducibility (see, *e.g.*, Johnson, 1990). AC^0 is the class of problems solvable by uniform constant-depth Boolean circuits with unbounded fan-in and a polynomial number of gates. P and NP are the classes of problems that can be solved in polynomial time by deterministic and nondeterministic Turing machines, respectively.

We start by showing that all variants of the uncovered set are very easy to compute and amenable to parallel computation. We omit the straightforward proof.

Theorem 4 *Deciding whether an alternative is contained in the upward, downward, or bidirectional uncovered set is in AC^0.*[3]

We continue with the finest solution concept studied in this paper, the essential set. By Definition 4, the essential set can be computed by finding those actions of the adjacency game that are played with positive probability in any Nash equilibrium. Algorithm 1 determines this set by first computing the (expected) payoff of the first player in *some* Nash equilibrium of the game, which constitutes the solution of a linear programming problem. The minimax theorem (see, *e.g.*, von Neumann, 1928) implies that this payoff equals the

[2]The second part of this theorem is due to Banks & Bordes (1988).

[3]If the input is a preference profile, as it is usually the case in the context of social choice, the problem becomes TC^0-complete (under constant depth reducibility).

Algorithm 1 Essential set

procedure $ES(A, >)$
  $B \leftarrow \emptyset$; $(m_{ij})_{i,j \in A} \leftarrow M_{A,>}$
  minimize v
  subject to $\sum_{j \in A} s_j \cdot m_{ij} \leq v \quad \forall i \in A$
    $\sum_{j \in A} s_j = 1$
    $s_j \geq 0 \quad \forall j \in A$
  for all $k \in A$ **do**
    maximize s_k
    subject to $\sum_{j \in A} s_j \cdot m_{ij} \leq v \quad \forall i \in A$
      $\sum_{j \in A} s_j = 1$
      $s_j \geq 0 \quad \forall j \in A$
    if $s_k > 0$ **then** $B \leftarrow B \cup \{k\}$ **end if**
  end for
  return B

Algorithm 2 Minimal bidirectional covering set

procedure $MC(A, >)$
  $B \leftarrow ES(A, >)$
  loop
    $A' \leftarrow \{ a \in A \setminus B \mid a$ uncovered in $B \cup \{a\} \}$
    if $A' = \emptyset$ **then return** B **end if**
    $B \leftarrow B \cup ES(A', >)$
  end loop

payoff in *every* Nash equilibrium in a two-player zero-sum game. We can thus search for a strategy of the second player in which a particular action is played with the highest possible probability while ensuring that the payoff of the first player for any of his actions does not exceed the payoff computed in the first stage of the algorithm. It is straightforward to show that this algorithm is correct and runs in polynomial time.

Theorem 5 *The essential set can be computed in time polynomial in the number of alternatives.*

By inclusion of *ES* in *MC*, Algorithm 1 also provides a way to efficiently compute *some* element of *MC*. While this cannot always be exploited to efficiently compute the whole set, it is of great benefit in our context. Algorithm 2 determines *MC* by starting with the essential set and iteratively adding specific elements outside the set that are still uncovered. The crux of the matter is to only add elements that may not be covered in a later iteration and it is not at all obvious which elements these should be. It turns out that these are precisely the elements in the minimal bidirectional covering set of the *subrelation* induced by the uncovered alternatives. Recalling that we can efficiently compute a subset of any minimal covering set, the algorithm is complete.

Theorem 6 *The minimal* bidirectional *covering set can be computed in time polynomial in the number of alternatives.*

Proof: We prove that Algorithm 2 computes the minimal bidirectional covering set and runs in time polynomial in the number of alternatives. In each iteration of the algorithm, at least one element is added to B, so the algorithm is guaranteed to terminate after a linear number of iterations. Each iteration consists of a single call to *ES* for a subset of the alternatives, which by Theorem 5 requires only polynomial time.

As for correctness, we show by induction on the size of B that $B \subseteq MC(A)$ holds at any time. When the algorithm terminates, B is a covering set for A, so we must actually have $B = MC(A)$. The base case follows directly from the fact that $ES(A) \subseteq MC(A)$ (Dutta & Laslier, 1999). Now assume that $B \subseteq MC(A)$ at the beginning of a particular iteration. We will argue that every element of $MC(A')$ has to be part of every superset of B that is a covering set for A, thus $B \cup ES(A') \subseteq B \cup MC(A') \subseteq MC(A)$. Assume for contradiction that for some $x \in MC(A')$ there exists a set $B' \supseteq B$, $x \notin B'$, such that x is covered in $B' \cup \{x\}$. Obviously, x cannot be covered in $B' \cup \{x\}$ by any element of B since x is uncovered in B, *i.e.*, for every $y \in B$ we either have $y \not> x$ or there is some $z \in B$ such that $x > z$ and $y \not> z$ or $z > y$ and $z \not> x$. The same holds for elements of $A \setminus (B \cup A')$. If x was covered in B' by an element $y \in A \setminus (B \cup A')$, this would imply that $y > z$ if $x > z$ and $z \not> y$ if $z \not> x$ for all $z \in B$ and thus $y \in A'$. Finally, if x was covered in B' by $y \in A'$, we would have that $y > x$ and $y > z$ if $x > z$ and $z \not> y$ if $z \not> x$ for all $z \in A'$. In this case, $A' \setminus \{x\}$ would be a covering set for A' and, since *MC* lies in the intersection of all bidirectional covering sets (Dutta & Laslier, 1999), $x \notin MC(A')$. This is a contradiction. □

A potential problem of bidirectional covering is that it is not very discriminatory. One might thus try to obtain smaller choice sets by considering covering in one direction only. It turns out that this renders the computational problems hard. The following theorem can be shown using reductions from satisfiability. We again omit the proof.

Theorem 7 *It is NP-hard to decide (i) whether an alternative is contained in some minimal* upward *covering set, (ii) whether an alternative is contained in some minimal* downward *covering set, and (iii) whether there exists a downward covering set.*

Conclusions

We have investigated solution concepts for dominance graphs that are based on the notion of covering and analyzed their computational complexity. It turned out that polynomial-time algorithms exist for computing (iterated) uncovered sets, the minimal bidirectional covering set, and the essential set. In contrast, we proved that deciding whether an alternative is in some minimal upward or downward covering set is NP-hard. This is particularly intriguing, because we further showed that these sets are related to von Neumann-Morgenstern stable sets and to the Banks set, respectively, which are also known to be computationally intractable (Brandt, Fischer, & Harrenstein, 2007; Woeginger, 2003). Table 1 summarizes our results.

Our algorithm for computing the minimal bidirectional covering set *MC* underlines the significance of *MC* as a practical solution concept. *MC* was originally introduced as a

	existence	complexity
UC_b, UC_u, UC_d	unique	in AC^0
$UC_b^\infty, UC_u^\infty, UC_d^\infty$	unique	in P
MC	unique	in P
minimal upward covering	exists	NP-hard
minimal downward covering	—	NP-hard
ES	unique	in P

Table 1: Existence, uniqueness, and complexity of the choice sets studied in this paper

refinement of the uncovered set that is superior to the Kemeny set because the latter fails to satisfy a very mild consistency criterion (Dutta, 1988). Now it has turned out that, besides this advantage, MC can be computed in polynomial time whereas computing the Kemeny set is NP-hard and all known algorithms have exponential worst-case complexity (Hemaspaandra, Spakowski, & Vogel, 2005; Conitzer, Davenport, & Kalagnanam, 2006). Moreover, due to the equivalence pointed out by Duggan & Le Breton (1996), our algorithm for computing MC can also be applied for finding the unique weak saddle in a subclass of symmetric two-player zero-sum games.

Acknowledgements

We thank Hermann Gruber for taking part in a fruitful discussion on the hardness of upward and downward covering and Paul Harrenstein for reading an early draft of this paper. We further thank the anonymous referees for useful comments.

References

Banks, J. S., and Bordes, G. A. 1988. Voting games, indifference, and consistent sequential choice rules. *Social Choice and Welfare* 5:31–44.

Banks, J. S. 1985. Sophisticated voting outcomes and agenda control. *Social Choice and Welfare* 3:295–306.

Benisch, M.; Davis, G. B.; and Sandholm, T. 2006. Algorithms for rationalizability and CURB sets. In *Proc. of 21st AAAI Conference*, 598–604. AAAI Press.

Bordes, G. 1983. On the possibility of reasonable consistent majoritarian choice: Some positive results. *Journal of Economic Theory* 31:122–132.

Brandt, F., and Harrenstein, P. 2007. Dominance in social choice and coalitional game theory. Presented at the 5th International Conference on Logic, Game Theory and Social Choice (LGS).

Brandt, F.; Fischer, F.; and Harrenstein, P. 2007. The computational complexity of choice sets. In Samet, D., ed., *Proc. of 11th TARK Conference*.

Conitzer, V.; Davenport, A.; and Kalagnanam, J. 2006. Improved bounds for computing Kemeny rankings. In *Proc. of 21st AAAI Conference*, 620–627. AAAI Press.

Conitzer, V. 2006. Computing Slater rankings using similarities among candidates. In *Proc. of 21st AAAI Conference*, 613–619. AAAI Press.

Duggan, J., and Le Breton, M. 1996. Dutta's minimal covering set and Shapley's saddles. *Journal of Economic Theory* 70:257–265.

Dung, P. M. 1995. On the acceptability of arguments and its fundamental role in nonmonotonic reasoning, logic programming and n-person games. *Artificial Intelligence* 77:321–357.

Dutta, B., and Laslier, J.-F. 1999. Comparison functions and choice correspondences. *Social Choice and Welfare* 16(4):513–532.

Dutta, B. 1988. Covering sets and a new Condorcet choice correspondence. *Journal of Economic Theory* 44:63–80.

Fishburn, P. C. 1977. Condorcet social choice functions. *SIAM Journal on Applied Mathematics* 33(3):469–489.

Gillies, D. B. 1959. Solutions to general non-zero-sum games. In Tucker, A. W., and Luce, R. D., eds., *Contributions to the Theory of Games IV*, volume 40 of *Annals of Mathematics Studies*. Princeton University Press. 47–85.

Hemaspaandra, E.; Spakowski, H.; and Vogel, J. 2005. The complexity of Kemeny elections. *Theoretical Computer Science* 349:382–391.

Johnson, D. S. 1990. A catalog of complexity classes. In van Leeuwen, J., ed., *Handbook of Theoretical Computer Science*, volume A. Elsevier. chapter 2, 67–161.

Laslier, J.-F. 1997. *Tournament Solutions and Majority Voting*. Springer.

Lavi, R., and Nisan, N. 2005. Online ascending auctions for gradually expiring items. In *Proc. of 16th SODA*, 1146–1155. SIAM.

McGarvey, D. C. 1953. A theorem on the construction of voting paradoxes. *Econometrica* 21(4):608–610.

Miller, N. R. 1980. A new solution set for tournaments and majority voting: Further graph-theoretical approaches to the theory of voting. *American Journal of Political Science* 24(1):68–96.

Moulin, H. 1986. Choosing from a tournament. *Social Choice and Welfare* 3:271–291.

Osborne, M. J., and Rubinstein, A. 1994. *A Course in Game Theory*. MIT Press.

Peris, J. E., and Subiza, B. 1999. Condorcet choice correspondences for weak tournaments. *Social Choice and Welfare* 16(2):217–231.

Shapley, L. 1964. Some topics in two-person games. In Dresher, M.; Shapley, L. S.; and Tucker, A. W., eds., *Advances in Game Theory*, volume 52 of *Annals of Mathematics Studies*. Princeton University Press. 1–29.

von Neumann, J., and Morgenstern, O. 1944. *The Theory of Games and Economic Behavior*. Princeton University Press.

von Neumann, J. 1928. Zur Theorie der Gesellschaftspiele. *Mathematische Annalen* 100:295–320.

Woeginger, G. J. 2003. Banks winners in tournaments are difficult to recognize. *Social Choice and Welfare* 20:523–528.