A Principal-Agent Model of Bidding Firms in Multi-Unit Auctions

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Abstract. Principal-agent relationships between the supervisory board and the management of bidding firms in auctions are widespread in high-stakes auctions. Often only the agent has information about the value of the objects being sold. The board wants to maximize the profit, but the management wants to win the package with the highest value. In environments where it is efficient for firms to coordinate, we show that the principals would coordinate, while the agents would not. We analyze markets with decreasing levels of information that the principal has about the valuations. Sometimes it can be impossible to set budget constraints which align the agents’ strategies in equilibrium. The analysis helps explain price wars in high-stakes auctions.

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1 Introduction

Price wars have frequently been discussed in high-stakes auctions such as spectrum sales by governments. Often such price wars are hard to explain with standard assumptions of payoff-maximizing bidders.\(^2\)

Principal-agent relationships have been mentioned as one reason (Schmidt 2005). In this paper, we introduce a principal-agent model of a bidding firm, which helps explain this phenomenon. We model a wide-spread hidden information problem in high-stakes auctions, where the agent knows the valuation of different objects or packages, but the principal does not. What is more, the payments in these markets are such that the principal needs to pay. As a consequence, the agent wants to maximize value, while the principal wants to maximize payoff. We analyze equilibrium strategies and discuss the agency dilemma that arises and possibilities for the principal to align the agent’s strategy. Depending on the information that the principal has about the valuations and the auction format, this can be hard or even impossible to do, which can be a significant source of inefficiency in high-stakes auctions.

Let us first motivate our model by looking at spectrum auctions, which are an important economic activity (generating hundreds of billions worldwide) and which have been an important catalyst for theoretical research in auctions. It is well-known that the relationship between the supervisory board (principal) and the management (agent) of a telecom in spectrum play an important role in spectrum auctions (Chakravorti et al. 1995; Shapiro et al. 2013). Similar relationships arise between the management of a multinational telecom and the management of a national subsidiary bidding in the auction. The payments made by telecommunication firms in spectrum auctions are often billions of dollars, and thus the management cannot cover the cost of the auction. This means, the agent in these markets has limited liability and the principal has to pay in the auction. The budgets that need to be reserved for such an auction by the board are also such that they cannot just be transferred in total to the agent in order to induce a profit-maximizing motive. The residual budget after the auction might be in the billions, and there can be much more efficient investments elsewhere.
In spectrum auctions, firms have preferences over different packages of spectrum licenses. Each of these packages can be assigned a business case with a net present value. The management knows the market best, it knows the technology, the competition, and the end consumer market, and so they can compute business cases which allow for a good estimate of the net present value of each package. The board of directors does not have this information, and the management has no incentive to reveal it truthfully. Principals often need to rely on analyst estimates, which typically have an enormous variance.\(^3\) The principal will also not learn the true valuations of the licenses after the auction, as the future profits of the firm depend on many other decisions. The situation is nicely summarized in a report by a consulting firm in this field (Friend 2015):

“... The amount of money spent by mobile operators at auction is often staggering. The money needed to pay for spectrum cannot usually be funded from the agreed capital expenditure budget of the business. As a result, spectrum payments are usually treated as a separate amount that does not impact the Key Performance Indicators of the business upon which the management team's bonuses are often based. However, the management team of a mobile business usually prefers to have more spectrum rather than less. The Chief Marketing Officer prefers more spectrum than his competitors as it allows him to advertise a bigger, faster and better network which helps him achieve his sales target. The Chief Technical Officer prefers more spectrum as it means he needs to build fewer sites to provide the same capacity and helps him achieve his capex to sales targets. The CEO is happy because the business is hitting its targets. So the management team will typically prefer to win more rather than less spectrum at auction.”

Empire-building motives are a widespread reason for such value-maximizing behavior of agents in the principal-agent literature (Jensen 1986). Note that spectrum auctions are only one example of value maximizing agents. Engelbrecht-Wiggans (1987) writes “in bidding for mineral leases, a firm may wish to maximize expected profits while its bidder feels it should maximize the firm's proven reserves.” In addition, he discusses auctions for defense systems and construction. Payoff maximization is hard to defend for an agent in such relationships, and agents typically try to “win within budget.” In contrast, value maximization is a good approximation of such agent motives. If agents do not maximize payoff, it is
important to understand their bias and means for the principal to align the agent with his equilibrium bidding strategy, even in case of hidden information.

1.1 Contributions and Outline

We introduce a principal-agent model of a firm participating in a multi-unit auction, where the agent has hidden information about the valuations of the goods. Interesting strategic problems arise in multi-unit auctions. The principal then screens the agent optimally with budget constraints or payments to overcome the adverse selection problem. Principal-agent relationships of bidding teams of spectrum auctions have only recently become a topic in auction theory, but prior models focus on single object auctions and the agents’ motives are different (Burkett 2015; Burkett 2016).\(^4\)

First, we describe the environment formally as the \emph{principal-agent \(2 \times 2\) package auction model}, where 2 bidders compete for 2 units of a homogeneous good in Section 2.\(^5\) While being tractable, the \(2 \times 2\) model captures central strategic challenge that can also arise in larger markets with value-maximizing agents. We largely focus on package auctions in our analysis due to their relevance in spectrum auctions,\(^6\) however, we also discuss traditional multi-unit auctions and where they differ. Then we discuss equilibrium bidding strategies. If there was a strategy-proof and welfare-maximizing deterministic auction mechanism for both, the principal and the agent, no agency dilemma would arise. Unfortunately, such a mechanism does not exist.

In Section 3, we analyze the different auction formats as Bayesian games in order to analyze the agency problems that arise. To demonstrate the agency bias, we derive equilibrium bidding strategies of value-maximizing agents, and show that the agent does not bid on a single-unit package in the unique ex-post Nash equilibrium. In our analysis, we assume the agents do not bid beyond the firm’s package valuations. Although agents do not internalize the package prices paid they cannot bid more than the valuation. This “no overbidding” assumption helps to understand the strategic bias of the agent and it is frequently made in the literature (Bhawalkar and Roughgarden 2011; Caragiannis et al. 2011; Leme and
Tardos 2010; Lucier and Borodin 2010). There will always be some budget constraint such that the agent cannot bid infinity, and the analysis only serves to understand the bias of the agent and how his equilibrium bidding strategy differs from that of the principal.

Next, we analyze the equilibrium bidding strategies of quasilinear principals in case they had exact information about the valuations of the firm and their prior distributions. Bayesian Nash equilibrium analysis of multi-object auctions has turned out challenging (Goeree and Lien 2014; Krishna and Rosenthal 1996). We assume prior information about the efficiency environment (i.e., an allocation with two winners is efficient) and show that coordination on the efficient outcome constitutes a Bayesian Nash equilibrium for a risk-neutral principal in a first-price sealed-bid package auction in this environment. This environment is strategically interesting, because bidding firms need to coordinate in the efficient equilibrium, but value-maximizing bidders would not. Efficient equilibria for payoff-maximizing bidders exist for both, the ascending and the first-price sealed-bid auction and this constitutes the agency dilemma.

In Section 4, we analyze the possibilities for the principal to implement his equilibrium strategy with a budget constraining the agent. Whether the principal can implement his strategy with such a constraint or not, this depends crucially on the level of information he has about the valuations. We show that even without hidden information, there cannot always be a budget constraint, which sets incentives for the agents to bid their allowance and at the same time is in equilibrium for the principal in the first-price sealed-bid auction. In the ascending package auction this is easier. However, the principal would still need to know at least the efficiency environment in the market.

In Section 5, we analyze a combination of wages and budget constraints as part of the optimal contract to implement the principal’s strategy in the asymmetric information setting. We focus on a situation, where there is uncertainty about the package valuation as well as their corresponding ranges, but not about the efficiency environment. The situation is practically relevant and non-trivial. In the first-price sealed-bid auction, the principal needs to pay wages, which can be very high. In contrast, in the ascending
package auction, the principal can again set a zero budget constraint for two units to implement the efficient equilibrium, and he does not incur agency costs. So, with information about the efficiency in the market, it is much easier for the principal to align the agent in an ascending auction compared to a first-price sealed-bid auction in our model.

2 The Model

In our model we consider 2 ex-ante symmetric firms $i, j \in I = \{1, 2\}$ competing in a multi-object package auction for 2 units $l \in L = \{1, 2\}$ of a homogeneous good. We denote this setting as $2 \times 2$ principal-agent package auction model. Let us first outline the $2 \times 2$ first-price sealed-bid package auction and the $2 \times 2$ ascending package auction, before discussing the payoff environment.

2.1 The Auctions

It is straightforward to see that the VCG mechanism is not incentive-compatible for agents, who do not internalize payments in their utility function and, moreover, that there cannot be a strategy-proof and deterministic mechanism for value-maximizing agents (Fadaei and Bichler 2016).

We focus on simple $2 \times 2$ first-price sealed-bid package auction and $2 \times 2$ ascending package auction with 2 objects and 2 bidders. These are also being used in spectrum sales and other high-stakes auctions. We mainly analyze multi-unit package auction formats which allow bidders to submit multiple all-or-nothing package bids. In multi-unit package auctions, each bidder $i \in I$ submits an all-or-nothing bid for every package. Here, each package is identified by the number of units, $l \in L$, it contains. In these package auctions, we assume an XOR bid language, because it is the most general bid language allowing the expression of complements and substitutes (Nisan 2006), and it is regularly used in spectrum auctions. An XOR bid language allows a bidder to win at most one of his bids. In both auctions, a risk-neutral auctioneer selects the revenue-maximizing combination of package bids.
In the $2 \times 2$ first-price sealed-bid package auction, both bidders simultaneously submit their bids to the auctioneer without knowing the opponent’s bids. We use the iBundle auction format (Parkes and Ungar 2000) as a theoretical model for the $2 \times 2$ ascending package auction, because this auction format is always efficient with truthful and payoff-maximizing bidders, and truthfulness is an equilibrium in our specific model. We get the same result for the wide-spread combinatorial clock auction (Porter et al. 2003) in our model, but the linear prices in this auction can lead to significant inefficiencies with general valuations, even if bidders are payoff-maximizing and truthful (Bichler et al. 2013).

In the ascending iBundle auction, bidders observe discriminatory and non-linear prices for each package after each round. In each round, every bidder can place bids on one or more packages. After the round, the auctioneer determines the winning allocation. The auction ends if there are no new bids. For all bundle bids from losing bidders, the ask price is the last bid price plus a minimum bid increment. If every bidder bids on his payoff-maximizing packages in each round, then this auction will end in a competitive equilibrium (Bikhchandani and Ostroy 2002). Although, we focus on sealed-bid and ascending package auctions in the paper, we also compare our results to a traditional (non-package) multi-unit auction which serves as a model for the wide-spread simultaneous multiple round auction (SMRA).

### 2.2 The Payoff Environment

Each firm $i \in I$ has a value for one unit of $v_i(1) \in V(1) = [\overline{v}(1), \underline{v}(1)]$ and a package valuation for two units of $v_i(2) \in V(2) = [\overline{v}(2), \underline{v}(2)]$. Let us define the vector of package values as $v_i = (v_i(1), v_i(2)) \in V = V(1) \times V(2)$. We normalize the reservation utility $v_i(0) = 0$ and assume $v_i(1) < v_i(2)$ for all $i \in I$. Note that the assumption implies $\overline{v}(2) > \overline{v}(1)$ and $\underline{v}(2) > \underline{v}(1)$, and for supports of $\overline{v}(1) < \overline{v}(2)$, for example, is always satisfied.

The risk-neutral principal wants to maximize expected profit, where his profit of winning a package of $l$ units is given by $\pi_i(l) = v_i(l) - \beta_i(l)$. In this expression $\beta_i(l) = \beta(v_i(l))$ denotes the final price paid in the auction and $\beta: V(l) \rightarrow \mathbb{R}$ $\forall l \in L$ is a weakly increasing function. Let us define the vector of
package prices as $\beta_i = (\beta_i(1), \beta_i(2))$. The agent’s gross utility includes his value-maximizing motives and is denoted as $u_i(l) = w(v_i(l))$ when winning a package of $l$ units. The function $w: V(l) \rightarrow \mathbb{R}$ is strictly increasing in package value $v_i(l)$ and thus, an agent always prefers winning two units to one unit.

The principal determines a budget constraint for the package of $l$ units of $\alpha_i(l) = \alpha(v_i(l))$ with which to provide his agent to bid in the auction. The constraint $\alpha: V(l) \rightarrow \mathbb{R} \forall l \in L$ is weakly increasing in package value. We refer to the vector of all package budget constraints as $\alpha_i = (\alpha_i(1), \alpha_i(2))$. As long as the price for a bundle of $l$ units is weakly lower than his respective budget constraint of $\alpha_i(l)$, the agent obtains a utility of $w_i(v_i(l))$. Any price $\beta_i(l) > \alpha_i(l)$ is an unacceptable action for him, as he will be fired, for example, if payments exceed the budget constraint. The principal-agent package auction model is illustrated in Figure 1 below.

![Figure 1 Illustration of the principal-agent package auction model](image)

The overall market within and between firms shown in Figure 1 is modeled as a sequential game of incomplete information. The auction itself corresponds to a game of incomplete information between firms. Given the standard symmetry assumption, each firm $i$’s vector of valuation draws $v_i$ is a priori distributed according to a monotonically increasing joint cumulative distribution function $F(v_i)_{v_i(1)<v_i(2)}: V \rightarrow [0,1]$ with corresponding marginal distribution function for value $v_i(l) \in V(l) \forall l \in L$ of the form $F_l(v_i(l)): V(l) \rightarrow [0,1]$. We assume the distribution functions $F(\cdot)_{v_i(1)<v_i(2)}$ and $F_l(\cdot)$, the price function $\beta(\cdot)$ as well as the budget function $\alpha(\cdot)$ to be common knowledge within a firm $i$ and between both firms $i$ and $j$. 
The relationship between principal and agent within a firm corresponds to an adverse selection problem and consists of two subsequent stages. First, nature determines every firm $i$’s vector of package valuation draws, $v_i$, which are private information to agent $i$. In stage 1, each principal decides on a vector of budget constraints, $\alpha_i$, with which to provide his agent. In stage 2, agents compete against each other and decide on a vector of prices, $\beta_i$, in the auction. The risk-neutral auctioneer selects the revenue maximizing set of packages. We apply backward induction starting at the second stage and then continue with the first stage to examine Perfect Bayesian equilibria of this game.

In order to understand the adverse selection problem in more detail we first derive Bayesian Nash equilibria of agents if they were to bid alone in the auction without being restricted by their principals. In this setting we assume agents not to overbid the firms’ package valuations. Note that the “no overbidding” assumption simply corresponds to a special case of budget constraints with $\alpha_i(l) = v_i(l) \in V(l) \forall l \in L$.

To analyze the principals’ equilibrium strategies in the first-best solution in which the principal has complete information about the firm’s valuations we restrict our analysis to dual-winner efficiency. With dual-winner efficiency, it is always efficient to have the dual-winner outcome: $v_i(1) + v_j(1) - \max\{v_i(2), v_j(2)\} > 0$ for all $(v_i, v_j) \in V$. The condition $2 \cdot v(1) > \bar{v}(2)$ ensures dual-winner efficiency for all $(v_i, v_j) \in V$ and enables us to define the principal’s Bayesian Nash equilibrium strategies to coordinate on the efficient solution.

3 The Agency Dilemma

In this section, we analyze equilibrium bidding strategies of value-maximizing agents, before we discuss the equilibrium strategies of the principal in the first-best solution, and demonstrate the agency dilemma.
3.1 Equilibrium Bidding Strategies of the Agent

Let us first provide two useful lemmata for the $2 \times 2$ first-price sealed-bid package auction. We will then discuss how these results extend to the $2 \times 2$ ascending package auction.

**Lemma 1:** It is a weakly dominant action for any agent $i$ to submit a bid in the amount of his valuation on the two-unit package $2 \times 2$ first-price sealed-bid package auction: $\beta_i(2) = v_i(2)$.

All proofs are in Appendix I. After having restricted the set of rationalizable bids for two units we continue the analysis for the single-unit bids.

**Lemma 2:** An agent $i$’s set of weakly dominant actions is to submit a single-unit package bid of either zero or his valuation in the $2 \times 2$ first-price sealed-bid package auction: $\beta_i(1) \in \{0, v_i(1)\}$.

**Lemma 1** and **Lemma 2** also hold for the $2 \times 2$ ascending package auction in a slightly different way. To use identical notation for both auction formats, let us denote the highest package price for $l$ units that bidder $i$ is willing to accept as $\beta_i(l)$. Then, it is a weakly dominant action for any agent $i$ to remain active for the package of two units until its price reaches his valuation in the $2 \times 2$ ascending package auction: $\beta_i(2) = v_i(2)$. Furthermore, any agent $i$’s optimal action for the single-unit package in the $2 \times 2$ ascending package auction is either to not bid on the package at all, or to remain active until he is winning or its price reaches his corresponding valuation: $\beta_i(1) = v_i(1)$.

3.1.1 First-Price Sealed-Bid Package Auction

With these lemmata, we can derive the agents’ equilibrium strategies in both $2 \times 2$ package auction formats. Our first observation of the first-price sealed-bid package auction is that agents would not coordinate in equilibrium.

**Theorem 1:** It is the unique symmetric ex-post equilibrium for any agent $i$ to submit a vector of bids $\beta_i = [0, v_i(2)]$ in the $2 \times 2$ first-price sealed-bid package auction for any vector of package values $v_i \in V$. 

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Intuitively, any agent $i$’s opponent $j$ would only be willing to coordinate on one unit if his valuation for two packages was low. In this case, however, it would be a best response for bidder $i$ not to coordinate, but try to win the package of both units independent of both agents’ actual values. Interestingly, this is an ex-post equilibrium, which is robust against risk aversion. Even arbitrary risk-averse bidders cannot coordinate on winning one unit with certainty.

### 3.1.2 Ascending Package Auction

In the analysis of the $2 \times 2$ ascending package auction, let us first introduce an adapted definition of straightforward bidding for agents in the $2 \times 2$ ascending package auction model.

**Definition 1 (Straightforward bidding of value-maximizing agents):** An agent $i$ begins to bid on the most valuable package of two units and remains active until he is overbid at his corresponding valuation of $v_i(2)$. As long as he is winning, he does not bid for the smaller single-unit package. If he is overbid, he starts to bid for the less valuable package and again remains active until he is overbid at his respective valuation of $v_i(1)$.

Remember that an agent prefers a larger package to a smaller one independent of its price, as long as the price is weakly lower than his valuation.

**Theorem 2:** In the $2 \times 2$ ascending package auction model, straightforward bidding of the agent constitutes an ex-post equilibrium. In this equilibrium the agent with the highest valuation for two units does not get active on one unit.

Similar to Theorem 1 of the $2 \times 2$ first-price sealed-bid package auction, Theorem 2 describes an ex-post equilibrium that is robust against risk aversion. In both auction formats, agents never coordinate on winning one unit each. This result is independent of the efficiency environment. The analysis shows that in the $2 \times 2$ package auction, agents will only bid on the large package in equilibrium.
3.2 Equilibrium Bidding Strategies of the Principal

Let us now analyze how a quasilinear principal would bid in equilibrium. This will provide a baseline to compare with the strategies of the agent. Theorem 3 and Theorem 6 show that an efficient dual-winner outcome can be supported as equilibrium for all possible valuations \((v_i, v_j) \in V\) in case of dual-winner efficiency in the \(2 \times 2\) first-price sealed-bid and \(2 \times 2\) ascending package auction, respectively. Theorem 4 and Theorem 7 derive conditions under which the single-winner outcome constitutes an equilibrium for all possible valuations \((v_i, v_j) \in V\) in dual-winner efficiency. Finally, Theorem 5 and Corollary 1 establish conditions when the efficient dual-winner equilibrium is payoff-dominant.

3.2.1 First-Price Sealed-Bid Package Auction

We first define the dual-winner equilibrium in Theorem 3 below.

**Theorem 3:** Any bidder \(i\)’s vector of bids \(\beta_i\) is a symmetric dual-winner equilibrium in the \(2 \times 2\) first-price sealed-bid package auction, if the following conditions hold:

1. \(\overline{v}(2) < 2 \cdot v(1)\)
2. \(\beta_i(1)\) is constant over \(v_i(1) \in V(1)\) and denoted by \(\beta_i(1) = \beta(1)\) for all \(i \in I\)
3. \(\beta(1) \in [\overline{v}(2) - v(1), v(1)]\)
4. \(\beta(\overline{v}(2)) = 2 \cdot \beta(1)\)
5. \(G(v_i(2), \beta(1)) \leq \beta(v_i(2))\) for all \(v_i(2) \in V(2)\) and all \(i \in I\)

The proof draws on techniques by Anton and Yao (1992), but is independent of a publicly known efficiency parameter. First, condition (1.) ensures dual-winner efficiency. Second, according to condition (2.) both bidders pool at a constant single-unit bid of \(\beta(1)\) out of its range from condition (3.). Third, condition (4.) ensures that the auctioneer always selects the dual-winner outcome in equilibrium. Finally, note that condition (5.) restrains any bidder \(i\)’s equilibrium bidding function for two units. It is not
allowed to fall below the lower bound of $G(v_i(2), \beta(1))$ in order to support the pooling bid for one unit. The lower bound is defined as follows:

$$G(v_i(2), \beta(1)) \equiv \beta(1) + \frac{\beta(1) - v(1) \cdot (1 - F_2(v_i(2)))}{F_2(v_i(2))}$$

This restriction ensures that winning the double-unit package is less profitable in expectation than obtaining a single unit in equilibrium. For our analysis we focus on the lowest pooling bid for one unit of $\beta(1) = \overline{v}(2) - v(1)$. This maximizes the utility of both bidders and therefore serves as a natural focal point for implicit coordination in the dual-winner equilibrium. This is not the only equilibrium for payoff-maximizing principals in our model, and there is also a single winner equilibrium.

**Theorem 4:** Any bidder $i$’s vector of bids $\beta_i$ is a symmetric single-winner equilibrium in the $2 \times 2$ first-price sealed-bid package auction, if the following conditions hold:

1. $\beta_i(2) = v_i(2) - F_2(v_i(2))^{-1} \cdot \int_{v_i(2)}^{\overline{v}(2)} F_2(v_j(2)) \cdot dv_j(2)$
2. $\beta_i(1) \in [0, \overline{v}(2) - v(1)]$

The equilibrium bid on the double-unit package in the single-winner equilibrium of the $2 \times 2$ first-price sealed-bid package auction from condition (1.) corresponds to the equilibrium strategy of the well-known standard first-price sealed-bid auction, in which two units are sold as the sole package to two bidders. Condition (2.) ensures any bidder can enforce the single-winner equilibrium by making the dual-winner outcome unprofitable for the opponent. Using payoff-dominance, we can show that for a range of valuations, payoff-maximizing bidders prefer to coordinate in the dual-winner equilibrium rather than select the single-winner equilibrium.

**Theorem 5:** Any principal $i$ prefers the dual-winner equilibrium to the single-winner equilibrium in the $2 \times 2$ first-price sealed-bid package auction if the expected value of two objects exceeds twice the equilibrium bid, $2 \cdot (\overline{v}(2) - v(1)) < E(v_j(2))$. 

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Intuitively, the expected double-unit valuation exceeds twice the pooling bid if the probability for large double-unit value draws is high. If high value draws for the package of two units are likely, however, any bidder prefers the *dual-winner equilibrium* to the *single-winner equilibrium*, because he is likely to lose in the latter equilibrium. Let us now turn to the analysis of the principals’ equilibrium strategy in the 2 × 2 *ascending package auction*.

### 3.2.2 Ascending Package Auction

We start with the characterization of the *dual-winner equilibrium*.

**Theorem 6:** In a symmetric ex-post dual-winner equilibrium of the 2 × 2 ascending package auction, no bidder *i* bids on the double-unit package with dual-winner efficiency. Any bidder *i* only bids on one unit until the respective price reaches his valuation of \( v_i(1) \).

Similar to the 2 × 2 *first-price sealed-bid package auction* auction, there is also a single-winner equilibrium.

**Theorem 7:** In a symmetric ex-post single-winner equilibrium of the 2 × 2 ascending package auction, any bidder *i* remains active on the single-unit package as long as its price is strictly below \( v(2) - v(1) \) and continues to bid on the package of two units until the respective price reaches his valuation of \( v_i(2) \).

The *single-winner equilibrium* of the 2 × 2 ascending package auction corresponds to the equilibrium strategy of the well-known single-object ascending (English) auction, in which two units are sold as the sole package to two bidders. Again, any bidder can enforce the *single-winner equilibrium* by making the *dual-winner outcome* unprofitable for the opponent. However, in the ascending auction the *dual-winner equilibrium* always strictly dominates the *single-winner equilibrium* in payoff.

**Corollary 1:** Any bidder *i* always strictly prefers the *dual-winner equilibrium* to the *single-winner equilibrium* in the 2 × 2 ascending package auction in dual-winner efficiency.
In general, the agents’ equilibrium behavior of not bidding on one unit leads to a conflict of interest with the principals’ dual-winner equilibrium. However, the principals still face an equilibrium selection problem in the $2 \times 2$ first-price sealed-bid package auction. In this respect, the $2 \times 2$ ascending package auction possesses two advantages: First, the dual-winner equilibrium strictly dominates the single-winner equilibrium in profit and therefore serves as a natural focal point for the bidders to coordinate. Second, bidders can observe their opponents’ equilibrium choices and adjust accordingly. This means they can see if the opponent wants to coordinate on a dual-winner equilibrium. If this is not the case, they can switch and aim for a single-winner equilibrium.

To illustrate the last point, suppose bidder $i$ plays the dual-winner equilibrium, defined in Theorem 6, and let opponent $j$ chose the single-winner equilibrium from Theorem 7. As all package prices are publicly observable, $i$ is able to recognize that his opponent is playing a different equilibrium and can adjust his own equilibrium strategy to the single-winner equilibrium. Note that this robustness of the $2 \times 2$ ascending package auction against the equilibrium selection problem might serve as an additional reason for each bidder to start trying to coordinate on the dual-winner equilibrium. If this coordination fails, an adjustment towards the single-winner equilibrium is always possible. Overall, the natural focal point equilibrium of the ascending package auction also leads to a conflict of interest between principals and agents. In the next section, we discuss how to overcome this conflict via budget constraints.

4 Budget Constraints

Budget constraints are a widely used as a means to discipline the bidding agent in high-stakes auctions (Engelbrecht-Wiggans 1987; Shapiro et al. 2013). Actually, the principal-agent problem can be seen as a form of optimal delegation, where an uninformed principal delegates decision rights to an informed but biased agent. Holmström (1977) showed that the optimal mechanism for the principal when utility is not transferable consists of choosing a subset of actions, from among which the agent is allowed to pick the most desired one.
Let us now analyze if a principal could set budget constraints, i.e., constrain the actions of the agent in the first-price sealed-bid package auction, so that the agent implements his strategy. In this subsection, we assume an environment where the supports of the prior distributions and the value draws are known to the principals, and we show that even in this information setting budget constraints can be insufficient to implement the principal’s equilibrium bidding strategy in the first-price auction with an agent. With less information about the valuations available to the principal, it is not even clear how the principal would determine a budget constraint or his equilibrium bidding strategy optimally.

4.1 First-Price Sealed-Bid Package Auction

So far, we assumed agent $i$’s budget constraints, $\alpha_i(I)$, to be an increasing function of the package values. However, agents would not bid on a single-unit package in equilibrium with these types of budget constraints. We will therefore analyze budget constraints $\hat{\alpha}_i(2) < \alpha(1)$ for the package of two units. We define a proxy value $\hat{v}_i(I) \in \mathbb{R}$ and assign a respective budget constraint of $\hat{\alpha}_i(I) = \alpha(\hat{v}_i(I))$. This proxy value $\hat{v}_i(I)$ could also be zero, for example. This would force the agent to bid on a single unit with a strictly positive budget constraint.

**Lemma 3:** A principal $i$ can direct his agent on winning one unit together with his opponent with certainty in the second stage of the principal-agent $2 \times 2$ first-price sealed-bid package auction model by assigning him package-dependent budget constraints of the form $\alpha_i = [\alpha_i(1), \hat{\alpha}_i(2)]$, with $\hat{\alpha}_i(2) = \alpha(\hat{v}_i(2))$ that satisfy inequalities $\alpha(v_i(1)) + \alpha\left(v(1)\right) \geq \alpha(\hat{v}_i(2))$ and $w(v_i(1)) \geq w(v_i(2)) \cdot F_1(\hat{v}_i(2))$.

Following backward induction, we need to analyze whether the budget constraint scheme from **Lemma 3** actually permits the implementation of the principals’ dual-winner equilibrium. In other words, we need to understand when these budget constraints violate the principals’ equilibrium bid functions.
from Theorem 3. In Theorem 8, we derive a condition under which the principals cannot direct their agents to truthfully reveal their profit-maximizing equilibrium strategies.

**Theorem 8:** There is no vector of budget constraints $\alpha_i = [\alpha_i(1), \hat{\alpha}_i(2)]$, which constitutes a Perfect Bayesian Equilibrium for the principal-agent $2 \times 2$ first-price sealed-bid package auction model under dual-winner efficiency, in which the principal implements his dual-winner equilibrium strategies if inequality $2 \cdot (\overline{v}(2) - v(1)) > \overline{v}(1)$ is true.

It follows that a Perfect Bayesian equilibrium, in which principal and agent are incentive aligned, cannot exist under reasonable ranges of valuations. This means that budget constraints are not always sufficient to align agent strategies in a first-price sealed-bid auction, even if the principal knows the valuations. Intuitively, any firm faces the following trade-off: On the one hand, the principal has to bid high enough on two units in equilibrium to prohibit the opponent from making a profit by deviating from the dual-winner equilibrium. On the other hand, the agent can only be directed on bidding for one unit if his budget constraint on the double-unit package is low enough. Both requirements cannot always be met simultaneously.

### 4.2 Ascending Package Auction

In contrast to the $2 \times 2$ first-price sealed-bid package auction, the principals’ dual-winner equilibrium can easily be implemented as a Perfect Bayesian equilibrium of the principal-agent $2 \times 2$ ascending package auction model, as long as dual-winner efficiency is known.

**Theorem 9:** The vector of budget constraints $\alpha_i = [\alpha_i(1), \hat{\alpha}_i(2)]$, with $\alpha_i(1) = v_i(1)$ and $\hat{\alpha}_i(2) = 0$, constitutes a Perfect Bayesian equilibrium for the principal-agent $2 \times 2$ ascending package auction model, in which any principal $i$ implements the dual-winner equilibrium.
The difficulty to set budget constraints in equilibrium in the first-price package auction is an argument for the ascending auctions in our model, which is different to traditional arguments for ascending auctions such as the linkage principle (Milgrom and Weber 1982).

Apart from ascending package auctions, the simultaneous multiple-round auction (SMRA) is used worldwide to sell spectrum licenses. It is interesting to understand whether the results for ascending package auctions carry over to SMRA. A detailed analysis in the context of our model with budget constraints is provided in Appendix II. Similar to the ascending package auction, principals could provide a zero budget constraint for two units to implement the dual-winner equilibrium with their agents, and the auction would stop in round two. Suppose dual-winner efficiency in the market is not certain, and the agents get some set of budget constraints $\alpha_i(1) \leq \alpha_i(2)$. Then, they would bid up to their budget constraint for two units instead of reducing demand, because they could actually win both units. This leads to much higher prices than if bidders were quasilinear and reduced demand early. This helps explain demand inflation in SMRA as it has been observed in the field, as we will discuss in the conclusion.

5 Contracts with Wages

This section determines the optimal contract between principal and agent in an asymmetric information setting. The principal knows that there is dual-winner efficiency, but he is not informed about the package value draws and their exact ranges. He faces bounds for these supports.

We assume any principal $i$ neither to know the firm’s vector of package valuations, $v_i$, nor the range bounds, $v(l)$ and $\overline{v}(l)$, of any package valuation, $v_i(l) \in V(l) = [v(l), \overline{v}(l)]$. In particular, we model these bounds as a random draw $d \in D = [d, \overline{d}]$ with $d \equiv \overline{v}(2) - v(1)$ that is unknown to the principal. However, he does know its supports of $d \equiv \overline{v}(2) - v(1)$ and $\overline{d} = \overline{v}(2) - v(1)$. We assume $\overline{v}(2) > v(1)$ and $\overline{d} > v(1)$, and that $d$ is commonly known, within and between firms, to satisfy dual-winner
efficiency such that $2 \cdot \underline{v}(1) > \bar{v}(2)$ still always holds. Note, however, that we do not assume $2 \cdot \underline{v}(1) > \bar{v}(2)$ such that dual-winner efficiency is not necessarily given for any possible range bound combinations from point of view of the principal. In addition to the vector of package constraints, $\alpha_i$, the principal employs a vector of payments for the agent in the asymmetric setting, $m_i = (m_i(1), m_i(2))$, with payment for the package of $l$ units of $m_i(l)$.

In the $2 \times 2$ ascending package auction, the principal can assign his agent a budget constraint of zero for the package of two units and a constraint of $\bar{v}(1)$ for the package of one unit. In equilibrium both agents can only bid on the single lot, and thus the auction stops immediately with dual-winner efficiency. The high budget for the package of one unit serves as a credible threat for the opposing principal not to deviate from the proposed equilibrium strategy. Note that in the ascending package auction, principals can implement the dual-winner equilibrium without any payments to the agents and do not incur agency costs.

In the $2 \times 2$ first-price sealed-bid auction the optimal contract to implement the dual winner equilibrium is more difficult to design. A principal $i$ has to design a menu of budget constraints and payments, $(\alpha_i, m_i)$, to direct his agent to submit the dual winner equilibrium strategies of $\beta(1) = \bar{v}(2) - \underline{v}(1)$ and $\beta(2) = 2 \cdot (\bar{v}(2) - \underline{v}(1))$ in the auction. As $2 \cdot \underline{v}(1) > \bar{v}(2)$ does not necessarily hold a principal $i$ cannot set budget constraints based on the bounds of $d$ to implement his dual-winner equilibrium strategy.

**Theorem 10:** A principal $i$ can implement his dual-winner equilibrium with budget constraints of $\alpha_i(1) = d$ and $\alpha_i(2) = 2 \cdot \alpha_i(1)$, and payments of $m_i(1) = w(\bar{v}(2)) - w(\underline{v}(1))$ and $m_i(2) = 0$ in the $2 \times 2$ first-price sealed-bid package auction.

The principal does not need to set incentives for bidding on the package, but he needs to pay an amount, which compensates the agent for not winning two units. Unfortunately, these agency costs caused by wages in the first-price sealed-bid auction can be very high.
6 Conclusions

In our hidden information model, we show that there is a conflict of interest between principal and agent in efficiency settings where it is payoff dominant for the principals to coordinate. The types of manipulation discussed in this paper are specific to multi-object auctions, and differ from the problems in single-object auctions (Burkett 2015).

Budget constraints are widely used by principals in high-stakes auctions, which might be due to the fact that there is often considerable uncertainty about the efficiency environment and the prior type distributions for the principal in the field. With uncertainty about the efficiency environment, the equilibrium bidding strategies in our model are unknown. In such an environment, principals often try to at least limit the risk of the agent overbidding substantially via budget constraints. However, budget constraints are insufficient to make the agent bid payoff-maximizing in general. We show that even if the principal knew the type distribution and its supports, it can be impossible for him to set a budget constraint in a first-price sealed-bid auction, which also constitutes a Perfect Bayesian equilibrium. In our principal-agent package auction model, the principal would need to know that there is dual-winner efficiency to set budget constraints appropriately in an ascending auction. In a FPSB auction, this might not even be possible in equilibrium.

Our results help explain some observations of high-stakes auctions in the field. There are many examples of auctions where bidders bid aggressively on large packages and the auction ended with very high prices. This is hard to explain with profit maximization. For example, Ausubel et al. (2014) discuss strategic demand reduction and provide examples of high-stakes spectrum auctions. They use a sealed-bid uniform price auction to model the simultaneous multiple-round auction (SMRA), which has been used worldwide for spectrum sales (Milgrom 2000). The model demonstrates that profit-maximizing bidders have an incentive to bid less than their marginal willingness to pay.
Demand reduction did not happen, in the famous German spectrum auction in 2000, where seven bidders could win two or three out of twelve licenses. The remaining bidders could have reduced demand to two blocks after the seventh bidder dropped out, so that the auction would have ended at a price of EUR ~2.5 bn. per block. Eventually two bidders continued to fight for the third block until the price reached EUR ~4.2 bn. per block. Jehiel and Moldovanu (2001) describe the bidding behavior as “bizarre,” and question whether the outcome is in equilibrium. Klemperer (2001) described the aborted attempt to acquire large licenses as irrational behavior. The attempt to drive out the sixth bidder and the resultant exposure problem was used as an explanation, but even with such externalities “the total of the bids exceeded expectations by reaching the staggering amount of € 50.8 bn.” Schmidt (2004) already mentions principal-agent relationships as a reason for high prices in the context of the German spectrum auction in 2000.

An attempt to drive out competitors from the market cannot be used to explain the recent German spectrum auction in 2015 with three bidders, where only one out of 23 licenses was over-demanded after round 34 at prices corresponding to a total revenue of €1,993 bn. However, prices continued to rise until round 181, when a total revenue of more than €5 bn. was reached. In total, the three operators spent around €3 billion more than if one of them would have reduced demand by only one license already in round 34. This points to “demand inflation” rather than demand reduction, which was also discussed by telecom experts: “It is likely that the marginal price paid for the 5th 1800MHz block is well above the marginal value to of that 5th block.” Note that in both auctions in 2000 and 2015 the analyst estimates differed substantially and it is likely that the principals setting the budget constraints had little information about the efficiencies in the market.

One can also observe high bids on large packages and resulting high prices in combinatorial auctions, which have been used in recent years to sell spectrum all over the world (Kroemer et al. 2014). For example, the regulator in the Austrian auction in 2013 revealed that bids were mostly submitted on very large packages. The auction resulted in very high prices compared to those paid in other countries.
Different reasons have been discussed for bidding behavior in the combinatorial clock auction (Janssen 2014), but principal-agent relationships can serve as one explanation as to why bidders do not try to coordinate with small packages in such auctions.

In summary, the information asymmetries in bidding firms and the limited knowledge about the prior type distributions can make it very difficult for principals to incentivize the payoff maximization of agents. This has implications for auctioneers and their choice of auction format as well. There is an ongoing policy discussion about sealed-bid vs. ascending spectrum auctions. For example, in a spectrum auction in Norway, the allocation of a first-price sealed-bid combinatorial auction was such that some items remained unsold, whereas one incumbent did not win sufficient spectrum and had to leave the market later. There can be many reasons for such an outcome. However, our analysis suggests that principal-agent relationships can be an important reason why bidders do not coordinate in a first-price sealed-bid package auction. In markets, where principals do not have enough information about the valuations and the efficiency in the market, there is a danger that agents will not bid on smaller packages, even in situations where this is payoff-maximal for the firm.

References


Appendix I: Proofs

Lemma 1: It is a weakly dominant action for any agent \(i\) to submit a bid in the amount of his valuation on the two-unit package in the \(2 \times 2\) first-price sealed-bid package auction: \(\beta_i(2) = v_i(2)\).

Proof: For any of opponent \(j\)’s bids, \(\beta_j\), suppose agent \(i\) bids strictly less than his valuation on the package of 2 units. Increasing his bid to \(\beta_i(2) = v_i(2)\) does not reduce his payoff but strictly raises the chances of winning the respective bundle. This comes at the opportunity cost of proportionately lowering the chances to win the single-unit package. The action leads to a shift in probability from agent \(i\) winning one unit to agent \(i\) winning two units. It is independent of the probability of losing against the opposing agent \(j\). However, as agent \(i\) strictly prefers two units to one unit this shift constitutes a utility-increasing deviation. Bids of \(\beta_i(2) > v_i(2)\) are excluded by assumption. QED.

Lemma 2: An agent \(i\)’s set of weakly dominant actions is to submit a single-unit package bid of either zero or his valuation in the \(2 \times 2\) first-price sealed-bid package auction: \(\beta_i(1) \in \{0, v_i(1)\}\).

Proof: Agent \(i\)’s single-unit bids of the form \(\beta_i(1) \in (0, v_i(1)]\) do not affect his payoff when winning the unit. For any of opponents \(j\)’s bids, \(\beta_j\), a bid of \(\beta_i(1) = v_i(1)\), however, maximizes the chances of winning one unit. It thereby minimizes the chances of winning the two-unit package. A bid of \(\beta_i(1) = 0\) eliminates the possibility to win one unit and maximizes the chances of winning the two-unit package. If the agent wants to coordinate on the dual-winner award in equilibrium, a bid of \(\beta_i(1) = v_i(1)\) is a weakly dominant action. Otherwise \(\beta_i(1) = 0\) weakly dominates any other bid on the single unit. Any bid from...
within the range of \((0, v_i(1))\) neither maximizes nor minimizes the chances of obtaining the one-unit package and therefore cannot be optimal. A bid exceeding the valuation is excluded by assumption. QED.

**Theorem 1**: It is the unique symmetric ex-post equilibrium for any agent \(i\) to submit a vector of bids \(\beta_i = [0, v_i(2)]\) in the \(2 \times 2\) first-price sealed-bid package auction for any vector of package values \(v_i \in V\).

**Proof**: We prove Theorem 1 by contradiction. Assume there is a set of valuation combinations for which bidders submit bids on one unit and the package for two units. Denote this set by \(S \subset V\). For all value draw combinations not in \(S\) bidders bid on the large package only. Let us focus on a bidder \(i\) with any value draws of \(v_i \in V\) and suppose his opponent \(j\) possesses value draws of \(v_j \in S\). According to Lemma 1 and Lemma 2 opponent \(j\) submits bids of \(v_j(1)\) and \(v_j(2)\) on one and two units, respectively. Bidder \(i\)’s expected payoff of bidding on the large package only, \(\pi_{\text{package}}(v_i(1), v_i(2))\), corresponds to

\[
\pi_{\text{package}}(v_i(1), v_i(2)) = v_i(2) \cdot P(v_i(2) \geq v_j(2)).
\]

According to Lemma 1 bidder \(i\) always bids his entire double-unit valuation on the package of two units. His expected payoff of bidding on the singleton and package, \(\pi_{\text{singleton and package}}(v_i(1), v_i(2))\), is

\[
\pi_{\text{singleton and package}}(v_i(1), v_i(2)) = v_i(2) \cdot P(v_i(2) \geq v_j(2) \cap v_i(2) \geq v_i(1) + v_j(1))
\]

\[+ v_i(1) \cdot P(v_i(1) + v_j(1) \geq v_i(2) \cap v_i(1) + v_j(1) \geq v_j(2)).\]

Following Lemma 2, if bidder \(i\) submits a bid on one unit it corresponds to his single-unit valuation. Define the difference between bidder \(i\)’s expected payoffs, \(\Delta \pi(v_i(1), v_i(2))\), as

\[
\Delta \pi(v_i(1), v_i(2)) = \pi_{\text{singleton and package}}(v_i(1), v_i(2)) - \pi_{\text{package}}(v_i(1), v_i(2)).
\]

We now demonstrate that \(\Delta \pi(v_i(1), v_i(2)) \leq 0 \ \forall \ v_i \in v_i\) which corresponds to showing that set \(S\) cannot exist.
First, observe that bidder $i$’s value draw for two-units of $v_i(2) \in [\hat{v}_i(2), \bar{v}(2)]$ cannot belong to set $S$ independent of the value for one unit. For the highest possible value draw for two units, $v_i(2) = \bar{v}(2)$, the difference in bidder $i$’s expected payoffs is weakly negative for all possible single-unit valuations, i.e.,

$$\Delta \pi(v_i(1), \bar{v}(2)) \leq 0 \forall v_i(1) \in V(1),$$

as

$$\bar{v}(2) \cdot P\left(\bar{v}(2) \geq v_i(1) + v_j(1)\right) + v_i(1) \cdot P\left(v_i(1) + v_j(1) \geq \bar{v}(2)\right) \leq \bar{v}(2). \quad (I)$$

Let us distinguish two different cases. If $\bar{v}(2) < 2 \cdot \bar{v}(1)$ then $P\left(\bar{v}(2) \geq v_i(1) + v_j(1)\right) < 1$ and $P\left(v_i(1) + v_j(1) \geq \bar{v}(2)\right) > 0$. As $v_i(1)$ is strictly smaller than $v_i(2) = \bar{v}(2)$ the LHS is strictly smaller than the RHS and (I) holds strictly for bidder $i$ with highest double-unit valuation and any single-unit value draw. If $\bar{v}(2) \geq 2 \cdot \bar{v}(1)$ then $P\left(\bar{v}(2) \geq v_i(1) + v_j(1)\right) = 1$ and $P\left(v_i(1) + v_j(1) \geq \bar{v}(2)\right) = 0$ and (I) holds with equality. Bidder $i$ is indifferent between bidding on the package only and bidding on one and two units for all possible single-unit value draws because he wins the large package anyway. WLOG in this case we can assume bidder $i$ with double-unit value draws $v_i(2) \geq \bar{v}_i(2)$ to bid on the large package only independent of the single-unit value draw, with $\bar{v}_i(2)$ being defined as the lowest double-unit value such that $P\left(\bar{v}_i(2) \geq v_i(1) + v_j(1)\right) = 1$. Note that for both cases (I) also holds strictly for slightly lower two-unit values, $v_i(2) \in [\hat{v}_i(2), \bar{v}(2))$, independent of the value draw for one unit.

Second, define the set of valuation combinations with the highest double-unit value draw in $S$ as

$$H \subseteq S.$$ Let us from now on focus on bidder $i$ with value draws of $v_i \in H$. By definition, if his double-unit value draw $i$ is marginally increased he does not belong to set $S$ anymore. This observation is guaranteed by the existence of high enough double-unit value draws, $v_i(2) \in [\hat{v}_i(2), \bar{v}(2)]$, for which bidder $i$ bids on the large package only independent of his value for one unit. Bidder $i$’s expected payoff from bidding on the large package only remains unaltered whereas his expected payoff from bidding on both packages corresponds to
\[ \pi_{\text{singleton and package}}(v_i(1), v_i(2)) = v_i(2) \cdot P \left( v_i(2) \geq v_j(2) \land v_i(2) \geq v_i(1) + v_j(1) \right) \\
+ v_i(1) \cdot P \left( v_i(2) \geq v_j(2) \land v_i(1) + v_j(1) \geq v_i(2) \right). \]

The bidder’s probability of winning the single unit must take into account the opponent not having a higher double-unit value draw than himself, \( v_j(2) \leq v_i(2) \). Otherwise, by symmetry, opponent \( j \) would not bid on the single unit independent of his corresponding valuation and bidder \( i \) could not win one unit anyway. Note at this stage, there might be value combinations with lower value draws for two units for which opponent \( j \) bids on the large package only. However, if we can show that bidder \( i \) prefers to bid on the large package only if we assume all bidders with lower double-unit value draws to bid on both packages he will not change his preferences if some bidders with lower double-unit values bid on the large package only.

By definition for bidder \( i \) out of set \( S \) the difference in his expected payoff must be positive for all his value combinations, i.e. \( \Delta \pi(v_i(1), v_i(2)) > 0 \ \forall \ v_i \in H \subseteq S \), which corresponds to

\[ v_i(2) \cdot P \left( v_i(2) \geq v_j(2) \land v_i(2) \geq v_i(1) + v_j(1) \right) + v_i(1) \cdot P \left( v_i(2) \geq v_j(2) \land v_i(1) + v_j(1) \geq v_i(2) \right) \\
> v_i(2) \cdot P(v_i(2) \geq v_j(2)). \quad (I') \]

Use conditional probability to rewrite the LHS of (I’) and cancel out \( P(v_i(2) \geq v_j(2)) \) to obtain

\[ v_i(2) \cdot P \left( v_i(2) \geq v_i(1) + v_j(1) | v_i(2) \geq v_j(2) \right) + v_i(1) \cdot P \left( v_i(2) \leq v_i(1) + v_j(1) | v_i(2) \geq v_j(2) \right) > v_i(2) \]

As \( v_i(1) \) is strictly smaller than \( v_i(2) \) the LHS is strictly smaller than the RHS in (I’) and in fact \( \Delta \pi(v_i(1), v_i(2)) < 0 \ \forall \ v_i \in H \subseteq S \). Hence, bidder \( i \) has an incentive to deviate and bid for the large package only. Thus, set \( H \) cannot belong to \( S \). Finally, as it is always possible to define a subset \( H \) in \( S \), in which bidder \( j \) has the highest valuation for two units in the set \( S \), there cannot be a set \( S \subseteq V \) as defined above, and the argument unravels for all types. QED.
Theorem 2: In the $2 \times 2$ ascending package auction model, straightforward bidding of the agent constitutes an ex-post equilibrium. In this equilibrium the agent with the highest valuation for two units does not get active on one unit.

Proof: Lemma 1 implies that both agents start bidding on the two-unit package immediately. Each of them remains active until his valuation for two units is reached. According to Lemma 2, each bidder has to decide whether to get active on one unit or not. If a bidder bids on one unit, he stays active until his valuation for one unit is reached. Without loss of generality, assume agent $i$ is the last remaining active bidder on the package of two units. Suppose he decides to bid for the single unit. Let opponent $j$ be active on this unit as well. Then the sum of both agents’ single-unit prices might eventually exceed bidder $i$’s valuation for two units. This cannot happen if agent $i$ does not bid on one unit. In this case, it is a weakly dominant action for bidder $i$ not to start bidding on the single unit. Remember, agent $i$ strictly prefers two units to one unit. Given this behavior, opponent $j$ is in fact indifferent between becoming active and not bidding on the single-unit package, as he cannot win anyway.

Even knowing the opponent’s type, no agent can benefit by deviating from his equilibrium strategy. Thus, straightforward bidding constitutes an ex-post equilibrium in the second stage of the principal-agent $2 \times 2$ ascending package auction model. QED.

Theorem 3: Any bidder $i$’s vector of bids $\beta_i$ is a symmetric dual-winner equilibrium in the $2 \times 2$ first-price sealed-bid package auction, if the following conditions hold:

$(1.) \overline{v}(2) < 2 \cdot \underline{v}(1)$

$(2.) \beta_i(1)$ is constant over $v_i(1) \in V(1)$ and denoted by $\beta_i(1) = \beta(1)$ for all $i \in I$

$(3.) \beta(1) \in [\overline{v}(2) - \underline{v}(1), \underline{v}(1)]$

$(4.) \beta(\overline{v}(2)) = 2 \cdot \beta(1)$
(5.) \( G(v_i(2), \beta(1)) \leq \beta(v_i(2)) \) for all \( v_i(2) \in V(2) \) and all \( i \in I \)

**Proof:** Condition (1.) ensures dual-winner efficiency for all bidders’ possible package value combinations. Regarding condition (2.), suppose any bidder \( i \)’s equilibrium bidding function for one unit \( \beta_i(1) \) varies with \( v_i(1) \) in the dual-winner equilibrium, so that \( \beta(v_i(1)) < \beta(\hat{v}_i(1)) \) with \( v_i(1) \neq \hat{v}_i(1) \). Then bidder \( i \) with value \( \hat{v}_i(1) \) for one unit has an incentive to bid \( \beta_i(1) = \beta(v_i(1)) \) to raise his profit: \( \hat{v}_i(1) - \beta(v_i(1)) > \hat{v}_i(1) - \beta(\hat{v}_i(1)) \). Thus, any bidder \( i \) with single-unit values of \( v_i(1) \) or \( \hat{v}_i(1) \) bids \( \beta(v_i(1)) \) independent of his valuation. This reasoning is true for any bidder with any value and results in the equilibrium bidding function for one unit of \( \beta(v_i(1)) = \beta(\hat{v}_i(1)) = \beta(1) \) for all \( v_i(1), \hat{v}_i(1) \in V(1) \), the pooling bid.

The upper bound in condition (3.) ensures that any bidder with the lowest single-unit value \( \underline{v}(1) \) will not make a negative profit in the dual-winner equilibrium: \( \underline{v}(1) \geq \beta(1) \). Note further that any bidder \( i \) with vector of valuations \( v_i = [\underline{v}(1), \overline{v}(2)] \) has the highest incentive to deviate from the dual-winner outcome. The lower bound in condition (3.) makes sure this bidder does not deviate at any pooling price \( \beta(1) \geq \overline{v}(2) - \underline{v}(1) \). The respective bidder could marginally overbid twice the pooling bid with his bid on the package of two units to obtain the profit of the single-winner outcome with certainty: \( \pi_i(2) = \overline{v}(2) - 2 \cdot \beta(1) - \epsilon \) for \( \epsilon \to 0 \). For this deviation not to be profitable, the pooling bid has to be of the form \( \beta(1) \geq \overline{v}(2) - \underline{v}(1) \). Note that bidder \( i \)’s profit in the dual-winner equilibrium is given by \( \pi_i(1) = \underline{v}(1) - \beta(1) \). If this bidder with vector of valuations \( v_i = [\underline{v}(1), \overline{v}(2)] \) has no incentive to deviate from the dual-winner equilibrium, then no other bidder \( j \) with valuations \( v_j(1) \geq \underline{v}(1) \) and \( v_j(2) \leq \overline{v}(2) \) deviates either.

For the dual-winner outcome to be chosen by the revenue-maximizing auctioneer for all possible double-unit package bids, it has to be true that \( 2 \cdot \beta(1) \geq \sup_{v_i(2)} \{ \beta(v_i(2)) \} \). The supremum is defined as the smallest upper bound or the greatest element in the set. Suppose \( 2 \cdot \beta(1) > \sup_{v_i(2)} \{ \beta(v_i(2)) \} \).
then for any bidder $i$, it is a best response to deviate from his equilibrium strategy by underbidding the pooling bid slightly with his single-unit bid (and thus raising his profit in the dual-winner equilibrium). This cannot be optimal and therefore we obtain the equilibrium requirement of $2 \cdot \beta(1) = \sup_{v_i(2)} \{ \beta(v_i(2)) \}$. As $\beta(\cdot)$ is a strictly monotonically increasing function in $v_i(2)$, we obtain condition (4.).

To derive condition (5.), apply the following reasoning: The dual-winner equilibrium requires mutually best responses of both bidders as support. Assume opponent $j$ sticks to his dual-winner equilibrium supporting strategy for all possible package values $v_j(1) \in V(1)$ and $v_j(2) \in V(2)$. Then we demonstrate that under conditions (1.) to (5.), $\beta_i = [\beta(1), \beta_i(2)]$ is a dual-winner equilibrium supporting strategy in relation to any other deviating bidding strategy $\hat{\beta}_i = [\hat{\beta}_i(1), \hat{\beta}_i(2)]$. Note that the profit in a dual-winner equilibrium for any bidder $i$ is given by $\pi_i(1) = v_i(1) - \beta_i(1)$ for all $v_i(1) \in V(1)$. Let us refer to this as equilibrium profit. Now we consider three different possible cases that can result from bidder $i$ playing any deviating strategy $\hat{\beta}_i = [\hat{\beta}_i(1), \hat{\beta}_i(2)]$ instead of the equilibrium strategy $\beta_i = [\beta(1), \beta_i(2)]$:

- **Case 1**: $\beta(1) + \hat{\beta}_i(1) > \hat{\beta}_i(2)$

Bidder $i$ deviates to a different dual-winner outcome and strictly avoids any single-winner outcome. However, bidder $i$ would never receive a dual-winner award if $\beta(1) + \hat{\beta}_i(1) < \beta(v_j(2))$ for all $v_j(2) \in V(2)$, i.e., $\beta(1) + \hat{\beta}_i(1) < \beta(v(2))$. Thus, it is a necessary condition for $\hat{\beta}_i(1)$ to exceed $\beta(1)$ to satisfy Case 1. Due to condition (4.), raising $\hat{\beta}_i(1)$ above $\beta(1)$ does not increase the probability of winning, which is already equal to one in the dual-winner equilibrium, but strictly lowers profits. Therefore, the rationalizable range for the deviating bid is defined by $\hat{\beta}_i(1) \in [\beta(v(2)) - \beta(1), \beta(1)]$.

The expected profit, $\hat{\pi}_i(1)$, of submitting a $\hat{\beta}_i(1)$ from this range for any single-unit value $v_i(1)$ is given by equation (1):
\[
\hat{\pi}_i(1) = (v_i(1) - \hat{\beta}_i(1)) \cdot P(\beta(v_j(2)) \leq \beta(1) + \hat{\beta}_i(1))
\] (I)

Now, (I) can be simplified as follows: By continuity of \(\beta(\cdot)\), for any \(\hat{\beta}_i(1) \in [\beta(v(2)) - \beta(1), \beta(1)]\), a unique proxy valuation for the package of two units \(\hat{v}_i(2) \in V(2)\) can be defined, so that \(\hat{\beta}_i(1) = \beta(\hat{v}_i(2)) - \beta(1)\). Using this expression for \(\hat{\beta}_i(1)\) to rewrite equation (I) we obtain equation (II):

\[
\hat{\pi}_i(1) = (v_i(1) - \beta(\hat{v}_i(2)) + \beta(1)) \cdot F_2(\hat{v}_i(2))
\] (II)

Deviating single-unit bids of the form \(\hat{\beta}_i(1) \in [\beta(v(2)) - \beta(1), \beta(1)]\) imply a focus on a deviation weakly below the pooling equilibrium price of \(\beta(1)\). In addition, any bidder \(i\) with single-unit value of \(v_i(1) = v(1)\) earns least in a dual-winner outcome and therefore has the highest incentive to deviate to a lower bid on one unit in equilibrium. To cancel this incentive, his dual-winner equilibrium profit of \(\pi_i(1) = v(1) - \beta_i(1)\) has to exceed his profit from deviating, \(\hat{\pi}_i(1)\), which is ensured in inequality (III):

\[
v(1) - \beta_i(1) \geq (v(1) - \beta(\hat{v}_i(2)) + \beta(1)) \cdot F_2(\hat{v}_i(2))
\] (III)

Rearranging and adding \(v_i(1)\), we obtain condition (5.) for all \(v_i(1) \in V(1)\) and \(\hat{v}_i(2) \in V(2)\), because \(v_i(1) \geq v(1)\):

\[
\beta(\hat{v}_i(2)) \geq \beta(1) + \frac{\beta(1) - v(1) \cdot (1 - F_2(\hat{v}_i(2)))}{F_2(\hat{v}_i(2))} \equiv G(\hat{v}_i(2), \beta(1))
\]

If bidder \(i\) with the lowest value for one unit has no incentive to deviate, then in fact no player with a higher value can have an incentive to deviate independent of the valuation for two units. The proposed dual-winner equilibrium is preferred to a Case 1 deviation by any bidder \(i\) as long as all deviating bids for two units are bounded from below by \(G(v_i(2), \beta(1))\). This is true for all valuations \(v_i(1) \in V(1)\) and \(v_i(2) \in V(2)\).

- Case 2: \(\beta(1) + \hat{\beta}_i(1) < \hat{\beta}_i(2)\)
Bidder $i$ deviates in a way that the auctioneer never selects any dual-winner outcome. He definitely does not win both units either if $\hat{\beta}_i(2) < \beta(v_i(2))$, which then defines the lower bound of his deviating bid for the double-unit package. If $\hat{\beta}_i(2) > \beta(\overline{v}(2))$, the bidder wins both units, but lowering the respective bid until $\hat{\beta}_i(2) = \beta(\overline{v}(2))$ strictly dominates in profit without changing the probability of winning. Thus, we obtain a rationalizable range for bidder $i$’s deviating double-unit bid of $\hat{\beta}_i(2) \in [\beta(v(2)), \beta(\overline{v}(2))]$ with an expected deviating profit of $\hat{\pi}_i(2) = (v_i(2) - \hat{\beta}_i(2)) \cdot P(\beta(v_j(2)) \leq \hat{\beta}_i(2))$ for all $v_i(2) \in V(2)$.

Let us again use the proxy notation $\hat{\beta}_i(2) = \beta(\hat{v}_i(2))$ to rewrite above profit as $\hat{\pi}_i(2) = (v_i(2) - \hat{\beta}_i(2)) \cdot F_2(\hat{v}_i(2))$. Bidder $i$ prefers the *dual-winner equilibrium* profit $\pi_i(1)$ to the deviating profit of $\hat{\pi}_i(2)$ if the following inequality (IV) holds:

$$0 \geq F_2(\hat{v}_i(2)) \cdot \left[ \frac{\beta(1) - v_i(1)}{F_2(\hat{v}_i(2))} + v_i(2) - \hat{\beta}_i(2) \right] \tag{IV}$$

The above inequality is true for all valuations $v_i(1) \in V(1)$, $v_i(2) \in V(2)$ and $\hat{v}_i(2) \in V(2)$ if the term in squared brackets is weakly negative. The respective term is in fact weakly negative for all $v_i(1) \in V(1)$, $v_i(2) \in V(2)$ and $\hat{v}(2) \in V(2)$ because the following inequality (V) is true:

$$\frac{\beta(1) - v_i(1)}{F_2(\hat{v}_i(2))} + v_i(2) \leq G(\hat{v}_i(2), \beta(1)) \tag{V}$$

Using the definition of $G(\hat{v}_i(2), \beta(1))$ and rearranging, we obtain (VI):

$$\underline{v}(1) - v_i(1) \leq F_2(\hat{v}_i(2)) \cdot [\beta(1) + \underline{v}(1) - v_i(2)] \tag{VI}$$

The LHS of the above inequality is weakly negative. Now we have to distinguish two cases regarding the RHS of inequality (VI): If $\beta(1) + \underline{v}(1) - v_i(2) \geq 0$, the inequality always holds. If $\beta(1) + \underline{v}(1) - v_i(2) < 0$, we have to show that $\beta(1) + \underline{v}(1) - v_i(2) \leq \underline{v}(1) - v_i(1)$. This is true for all $v_i(1) \in V(1)$ and $v_i(2) \in V(2)$ given the lower bound of condition (3.). As inequality (VI) holds, inequality (V) must
also be true. Remember from Case 1 that $\beta(\hat{v}_i(2)) \geq G(\hat{v}_i(2), \beta(1))$ must be given, which implies that inequality (IV) is satisfied. Therefore, any deviation considered in Case 2 is not profitable.

- Case 3: $\beta(1) + \hat{\beta}_i(1) = \hat{\beta}_i(2)$

In this case, bidder $i$ deviates as if he were indifferent between the dual-winner and single-winner outcome. Remember from condition (3.) that any bidder $i$ with valuations of $v_i = [v_i(1), v_i(2)]$ is indifferent between the dual-winner equilibrium and any single-winner outcome at the pooling price. Hence, the deviating behavior in Case 3 does in fact define his equilibrium strategy. Rewrite as the deviation in Case 3 to $\beta(1) = \hat{\beta}_i(2) - \hat{\beta}_i(1)$ and bear in mind the lower bound from condition (3.): $\beta(1) \geq \overline{v}(2) - v(1)$. Combining these two equations by substituting for $\beta(1)$ and rearranging, we obtain inequality (VII):

$$v(1) - \hat{\beta}_i (1) \geq \overline{v}(2) - \hat{\beta}_i (2) \quad \text{(VII)}$$

Now, consider player $i$ with values of $v_i = [v_i(1), v_i(2)]$, where $v_i(1) > v(1)$ and $v_i(2) \leq \overline{v}(2)$. For any such bidder, (VII) holds with strict inequality and he strictly prefers any deviating dual-winner award (LHS) to any single-winner award, which contradicts Case 3. Note that for bidder $i$ with valuations of $v_i = [v_i(1), v_i(2)]$, where $v_i(1) \geq v(1)$ and $v_i(2) < \overline{v}(2)$, the same reasoning holds.

Note in particular that by strictly decreasing the deviating bid on two units from $\hat{\beta}_i(2)$ to $\hat{\beta}_i(2)'$, so that the deviation from Case 3 becomes $\beta(1) + \hat{\beta}_i(1) > \hat{\beta}_i(2)'$, the bidder changes from a Case 3 deviation to a Case 1 deviation. As the latter always leads to some dual-winner outcome, it dominates Case 3 for all $v_i = [v_i(1), v_i(2)]$ with $v_i(1) > v(1)$ and $v_i(2) \leq \overline{v}(2)$, and for all $v_i = [v_i(1), v_i(2)]$, where $v_i(1) \geq v(1)$ and $v_i(2) < \overline{v}(2)$. Finally, as a Case 1 deviation is not beneficial, a Case 3 deviation cannot possibly be either. QED.
Theorem 4: Any bidder \( i \)'s vector of bids \( \beta_i \) is a symmetric single-winner equilibrium in the \( 2 \times 2 \) first-price sealed-bid package auction, if the following conditions hold:

\[
(1.) \beta_i(2) = v_i(2) - F_2(v_i(2))^{-1} \cdot \int_{v_i(2)}^{v_i(2)} F_2(v_j(2)) \cdot dv_j(2)
\]

\[
(2.) \beta_i(1) \in [0, v_i(2) - \overline{v}(1))
\]

Proof: In a single-winner equilibrium, any bidder \( i \) solely aims for the package of two units for all package valuations of \( v_i(1) \in V(1) \) and \( v_i(2) \in V(2) \). This scenario is strategically equivalent to the well-known first-price sealed-bid auction for a single package, in which two units are sold as the unique bundle. In this standard auction format, the equilibrium strategy of any bidder \( i \) takes the form of \( \beta_i(2) \) condition (1).

Note that the single-winner equilibrium requires any bidder \( i \) to possess ultimate “veto” power on the dual-winner outcome to make it unprofitable for his opponent to deviate from equilibrium. Suppose opponent \( j \) submits a very low “veto” bid \( \beta_j(1) \) on one unit, such as \( \beta_j(1) = 0 \), for example. Then bidder \( i \) would have to submit a deviating single-unit bid, \( \hat{\beta}_i(1) \), to retain the chance of winning the dual-winner outcome, where \( \hat{\beta}_i(1) \) is defined by the next inequality (I):

\[
\hat{\beta}_i(1) > \beta_i(v_i(2)) = v_i(2) \quad (I)
\]

In inequality (I), \( \beta_i(v_i(2)) \) is the optimal bid on the double-unit package of bidder \( i \) with lowest valuation for two units. Add the valuation for one unit \( v_i(1) \) on both sides of inequality (I) and rearrange to obtain inequality (I'):

\[
v_i(1) - \overline{v}(2) > v_i(1) - \hat{\beta}(1) \quad (I')
\]

The LHS of inequality (I') is strictly negative if \( \overline{v}(2) > v_i(1) \) for all single-unit valuations \( v_i(1) \in V(1) \), i.e., if \( \overline{v}(2) > \overline{v}(1) \) is true. The last inequality holds by assumption. As the LHS of (I') is strictly negative, the RHS of inequality (I') must be strictly negative. Note that the RHS corresponds to bidder \( i \)'s
profit in the forced deviating dual-winner outcome. Thus, if opponent \( j \) submits a "veto" bid in form of condition (2.), any deviating single-unit bid \( \beta_i(1) \) of bidder \( i \) to enforce the dual-winner outcome results in strictly negative profit. As he receives weakly positive expected profit in the single-winner equilibrium, a deviating bid of \( \hat{\beta}_i(1) \) is strictly dominated by any single-unit bid that supports the single-winner equilibrium. By symmetry, only a bid of the form \( \beta_i(1) < v(2) - \overline{v}(1) \) supports the single winner equilibrium for all \( v_i(1) \in V(1) \) and \( v_i(2) \in V(2) \) with certainty. QED.

**Theorem 5:** Any principal \( i \) prefers the dual-winner equilibrium to the single-winner equilibrium in the \( 2 \times 2 \) first-price sealed-bid package auction if the expected value of two objects exceeds two times the equilibrium bid, \( 2 \cdot (\overline{v}(2) - v(1)) < E(v_j(2)) \).

**Proof:** Remember from **Theorem 3** that any principal \( i \) submits the payoff-maximizing pooling bid of \( \beta_i(1) = \overline{v}(2) - v(1) \) in the dual-winner equilibrium and obtains respective equilibrium profit of \( v_i(1) - \overline{v}(2) + v(1) \) with certainty. The principal’s expected equilibrium profit in the single-winner equilibrium is

\[
\int_{v(2)}^{v_i(2)} F_2(v_j(2)) \cdot dv_j(2),
\]

as in the standard first-price sealed-bid auction, in which two units are sold as the sole package to two bidders. For principal \( i \) let us define the difference between expected profits in the dual-winner and single-winner equilibrium as a function \( \Delta[v_i(1), v_i(2)] \): \( V \to \mathbb{R} \) on the compact set \( V \subset \mathbb{R}^2 \), with

\[
\Delta[v_i(1), v_i(2)] = v_i(1) - \overline{v}(2) + v(1) - \int_{v(2)}^{v_i(2)} F_2(v_j(2)) \cdot dv_j(2)
\]

The above function is continuous, due to the differentiability of its constituents. It follows that it possesses a global maximum and a global minimum on \( V \). Moreover, \( \Delta[v_i(1), v_i(2)] \) is strictly increasing in its first argument and strictly decreasing in its second argument. Consequently, the function does not have a critical point in the interior of its domain, but on the boundary. Its maximum occurs at \((\overline{v}(1), v(2))\)
and the minimum at \((v(1), v(2))\), with values of \(\Delta[v(1), v(2)] = \overline{v}(1) - \overline{v}(2) + v(1)\) and \(\Delta[v(1), v(2)] = v(1) - \overline{v}(2) + v(1) - \int_{v(2)}^{\overline{v}(2)} \mathcal{F}(v) \cdot dv\), respectively. Remember that dual-winner efficiency is defined by \(\overline{v}(2) < 2v(1)\). This implies the maximum \(\Delta[v(1), v(2)]\) is always strictly positive and the minimum \(\Delta[v(1), v(2)]\) is strictly positive for all package valuations \(v_i(1) \in V(1)\) and \(v_i(2) \in V(2)\) if inequality (I) holds:

\[
\overline{v}(1) - \overline{v}(2) + v(1) - \int_{v(2)}^{\overline{v}(2)} \mathcal{F}(v) \cdot dv \geq 0 \quad (I)
\]

Using integration by parts, inequality (I) can be rewritten to (II):

\[
2 \cdot \overline{v}(1) - 2 \cdot \overline{v}(2) + \int_{v(2)}^{\overline{v}(2)} f(v) \cdot dv \geq 0 \quad (II)
\]

Finally, it follows that \(\Delta[v_i(1), v_i(2)]\) is strictly positive for all package valuations \(v_i(1) \in V(1)\) and \(v_i(2) \in V(2)\) if \(E(v_j(2)) > 2 \cdot \beta(1)\). QED.

**Theorem 6:** In a symmetric ex-post dual-winner equilibrium of the 2 × 2 ascending package auction, no bidder \(i\) bids on the double-unit package with dual-winner efficiency. Any bidder \(i\) only bids on one unit until the respective price reaches his valuation of \(v_i(1)\).

**Proof:** Let both bidders not bid on two units and solely start bidding on the single-unit package. Then there is no over-demand, and the auction immediately stops at a price of zero for the package of one unit. Any bidder \(i\) receives equilibrium profit of \(v_i(1)\) with certainty. In the remaining proof, we assume opponent \(j\) follows this proposed equilibrium strategy.

First, suppose principal \(i\) tries to win one unit, but does not immediately drop out from bidding on the double-unit package. Now the sum of both players’ prices for one unit has to exceed bidder \(i\)’s price for
two units. This cannot be optimal because any player \( i \) has to pay a positive price for one unit. His profit is decreased strictly below equilibrium profit of \( v_i(1) \).

Second, assume principal \( i \) tries to win two units and suppose he does not bid on the single unit. If player \( i \) wins two units at a price of \( v_j(1) \), he obtains a profit of \( v_i(2) - v_j(1) \). Remember, dual-winner efficiency implies \( v_i(1) + v_j(1) > v_i(2) \) for all possible valuations \( v_i(1), v_j(1) \in \mathcal{V}(1) \) and \( v_i(2) \in \mathcal{V}(2) \). Thus, for any principal \( i \), equilibrium profit of \( v_i(1) \) strictly exceeds \( v_i(2) - v_j(1) \). Now, let bidder \( i \) also bid on the package of one unit. This cannot possibly raise \( i \)'s profit on two units compared to not bidding on one unit. Finally, note that the proposed equilibrium strategy is independent of any bidder’s package valuations and therefore constitutes a symmetric ex-post equilibrium. QED.

**Theorem 7:** In a symmetric ex-post single-winner equilibrium of the 2 \( \times \) 2 ascending package auction, any bidder \( i \) remains active on the single-unit package as long as its price is strictly below \( \overline{v}(2) - \overline{v}(1) \) and continues to bid on the package of two units until the respective price reaches his valuation of \( v_i(2) \).

**Proof:** Suppose both bidders do not begin to bid on one unit, but start bidding on two units and continue to be active until their respective values are reached. Any bidder \( i \) obtains an expected equilibrium profit of \( F_2(v_i(2)) \cdot (v_i(2) - v_j(2)) \). Bidder \( i \) has the highest double-unit package value with probability of \( F_2(v_i(2)) \). In this case, he wins two units at a price of the second highest value, \( v_j(2) \), and receives a profit of \( v_i(2) - v_j(2) \). From now on, assume opponent \( j \) follows the proposed equilibrium strategy from Theorem 7.

Bidder \( i \) has no chance to profitably enforce the dual-winner outcome, because opponent \( j \) does in fact use his “veto” bid, given \( \underline{v}(2) > \overline{v}(1) \) is true. The reasoning is analogue to the proof of Theorem 4 and therefore omitted. Bidder \( i \) is indifferent between submitting any single-unit bid of \( b_i(1) \in [0, v_i(1)] \) as long as he continues to be active on the large bundle until the price reaches his corresponding value of
\( v_i(2) \). He obtains an expected profit of \( F_2(v_i(2)) \cdot (v_i(2) - v_j(2)) \). However, if bidder \( i \) decides to drop out on two units before the price reaches his value of \( v_i(2) \), he strictly lowers his probability of winning. This strictly decreases his expected profit and cannot be optimal. In the single-winner equilibrium, by symmetry, both bidders quit bidding on one unit before its price reaches \( v(2) - v(1) \) and remain active on the package of two units until its price reaches their respective valuations. The proposed equilibrium strategies are independent of the bidders’ actual package valuations and therefore are rationalizable ex post. QED.

**Corollary 1:** Any bidder \( i \) always strictly prefers the dual-winner equilibrium to the single-winner equilibrium in the \( 2 \times 2 \) ascending package auction in dual-winner efficiency.

**Proof:** In the dual-winner equilibrium from Theorem 6, the bidder with the lowest value for one unit obtains the lowest profit of \( v(1) \) with certainty. According to Theorem 7, the highest possible profit achievable in the single-winner equilibrium is \( v(2) - v(2) \). Using the definition of dual-winner efficiency and the fact that \( v(2) > v(1) \), the lowest obtainable profit in the dual-winner equilibrium strictly exceeds the highest possible profit in the single-winner equilibrium. Therefore, the profit in the dual-winner equilibrium is strictly greater than in the single-winner equilibrium for all possible bidder’s valuations \( v_i(1) \in V(1) \) and \( v_i(2) \in V(2) \). QED.

**Lemma 3:** A principal \( i \) can direct his agent on winning one unit together with his opponent with certainty in the second stage of the principal-agent \( 2 \times 2 \) first-price sealed-bid package auction model by assigning him package-dependent budget constraints of the form \( \alpha_i = [\alpha_i(1), \hat{\alpha_i}(2)] \), with \( \hat{\alpha_i}(2) = \alpha(v_i(2)) \) that satisfy inequalities \( \alpha(v_i(1)) + \alpha(v(1)) \geq \alpha(v_i(2)) \) and \( w(v_i(1)) \geq w(v_i(2)) \cdot F_1(\hat{\alpha_i}(2)). \)
Proof: Assume both agents follow their equilibrium strategy of submitting a positive bid for the single-unit package. Any agent $i$ can be coordinated on winning one unit together with his opponent with certainty. The sum of both single-unit bids must exceed each agent’s double-unit bid. Remember from Lemma 1 that any agent always spends his entire double-unit budget constraint. And according to Lemma 2, if an agent submits a non-zero bid on one unit, he must bid his entire single-unit budget constraint. Therefore, both principals have to implement a budget constraint scheme so that the sum of both single-unit budget constraints exceeds each agent’s double-unit budget constraint in equilibrium.

In this equilibrium, any agent $i$ pays his one-unit budget constraint for sure. Thus, his principal provides him with an optimal single-unit budget constraint of $\alpha_i(1) = \alpha(v_i(1))$. In addition, the principal choses a weakly reduced double-unit budget constraint $\alpha_i(2) = \alpha(\hat{v}_i(2))$ of the form $\alpha(\hat{v}_i(2)) \leq \alpha(v_i(2))$ with proxy valuation $\hat{v}_i(2) \leq v_i(2)$, so that condition $\alpha(v_i(1)) + \alpha(v(1)) \geq \alpha(\hat{v}_i(2))$ is satisfied. Principal $i$ does not know firm $j$’s package value for one unit. Thus, he has to make sure his choice of the weakly reduced two-unit budget constraint $\alpha(\hat{v}_i(2))$ is below the sum of both single-unit budget constraints. This has to be true for all possible single-unit budget constraints of his opponent, especially the smallest budget constraint of $\alpha(v(1))$. By symmetry, opponent $j$ follows the same strategy and condition $\alpha(v_i(1)) + \alpha(v(1)) \geq \alpha(\hat{v}_i(2))$ does in fact guarantee each agent will obtain the small package with certainty.

For agent $i$ to in fact submit a positive single-unit bid, his certain utility from bidding on this package must exceed his expected utility from winning two units. This is ensured in condition $w(v_i(1)) \geq w(v_i(2)) \cdot F_1(\hat{v}_i(2))$. On the RHS, agent $i$ does not bid on one unit, but can win two units instead. Here, $F_1(\hat{v}_i(2))$ is the probability with which his weakly reduced double-unit budget constraint $\alpha(\hat{v}_i(2))$ exceeds opponent $j$’s single-unit budget constraint $\alpha(v_j(1))$. The proxy value $\hat{v}_i(2)$ is chosen so that $F_1(\hat{v}_i(2))$ is low enough for the RHS to be lower than the LHS. Thus, agent $i$ prefers to bid on one unit.
and win with certainty. Note that we do not have to take into account the probability of $\alpha(\hat{v_i}(2))$ exceeding $\alpha(\hat{v_j}(2))$, because this weakly reduces the RHS further. QED.

**Theorem 8:** There is no vector of budget constraints $\alpha_i = [\alpha_i(1), \hat{\alpha_i}(2)]$, which constitutes a Perfect Bayesian Equilibrium for the principal-agent $2 \times 2$ first-price sealed-bid package auction model under dual-winner efficiency, in which the principal implements his dual-winner equilibrium strategies if inequality $2 \cdot (\overline{v}(2) - v(1)) > \overline{v}(1)$ is true.

**Proof:** In Theorem 3, every principal chooses the same pooling price of $\beta(1)$ in the dual-winner equilibrium. Thus, according to Theorem 5, any principal $i$ has to provide his agent with the same single-unit budget constraint. This budget constraint must be in the amount of the pooling bid: $\alpha(v_i(1)) = \beta(1) \forall i \in I$. Moreover, any principal $i$ selects a weakly reduced double-unit budget constraint that corresponds to his equilibrium bid on two units: $\alpha(\hat{v_i}(2)) = \beta(v_i(2))$. Due to Lemma 1, every agent always truthfully bids his entire weakly reduced double-unit budget constraint for two units $\alpha(\hat{v_i}(2))$. The following three conditions (I) to (III) then have to be satisfied:

\begin{align*}
2 \cdot \beta(1) & \geq \alpha(\hat{v_i}(2)) \quad (\text{I}) \\
\alpha(\hat{v_i}(2)) & \geq G(v_i(2), \beta(1)) \quad (\text{II}) \\
w(v_i(1)) & \geq w(v_i(2)) \cdot F_1(\hat{v_i}(2)) \quad (\text{III})
\end{align*}

Condition (I) ensures that the auctioneer will choose the dual-winner award in equilibrium. The last two conditions (II) and (III) ensure that no deviation from obtaining the single-unit package will be profitable for principal and agent in equilibrium. They correspond to conditions (5.) from Theorem 2 and (2.) from Theorem 5 respectively. Let us now check whether the above three conditions can be satisfied for the focal point pooling bid of $\beta(1) = \overline{v}(2) - v(1)$. Suppose firm $i$ has the highest possible value for
the package of two units \( v_i(2) = \overline{v}(2) \). In this case, condition (II) becomes (II’) by definition of \( G(v_i(2), \beta(1)) \):

\[
\alpha(\hat{v}_i(2)) \geq 2 \cdot (\overline{v}(2) - v(1)) \tag{II’}
\]

Given Theorem 2 (4.), for a package value of \( v_i(2) = \overline{v}(2) \) it must be true that \( \beta_i(\overline{v}(2)) = 2 \cdot (\overline{v}(2) - v(1)) \). Agent \( i \)’s bid on two units equals twice the pooling bid. Remember, principal \( i \) chooses a double-unit package budget constraint in of the amount of his respective equilibrium bid: \( \alpha(\hat{v}_i(2)) = \beta_i(\overline{v}(2)) \). This implies \( \alpha(\hat{v}_i(2)) = 2 \cdot (\overline{v}(2) - v(1)) \), which satisfies condition (I) and forces condition (II’) to hold with equality. Moreover, note that condition (III) implies \( \hat{v}_i(2) \leq \overline{v}(1) \), because \( \overline{v}(1) \) is the highest possible value in the support of \( F_1(\cdot) \). Applying this insight to the assumption of \( \alpha(\hat{v}_i(2)) \leq \hat{v}_i(2) \) we obtain condition (III’):

\[
\alpha(\hat{v}_i(2)) \leq \overline{v}(1) \tag{III’}
\]

Finally, combining conditions (II’) and (III’) is not possible if the condition from Theorem 6 is true. In this case, any firm \( i \) with package value of \( v_i(2) = \overline{v}(2) \) cannot implement budget constraints that satisfy two restrictions: They correspond to its principal’s equilibrium strategy and at the same time direct its agent to bid truthfully on both packages. Hence, the dual-winner equilibrium cannot be supported as a Perfect Bayesian equilibrium of the principal-agent \( 2 \times 2 \) first-price sealed-bid package auction model. QED.

**Theorem 9:** The vector of budget constraints \( \alpha_i = [\alpha_i(1), \hat{\alpha}_i(2)] \), with \( \alpha_i(1) = v_i(1) \) and \( \hat{\alpha}_i(2) = 0 \), constitutes a Perfect Bayesian equilibrium for the principal-agent \( 2 \times 2 \) ascending package auction model, in which any principal \( i \) implements the dual-winner equilibrium.
**Proof:** In Theorem 6, any principal $i$ does not bid on the package of two units. He remains active on the package of one unit at most until the price reaches his corresponding valuation of $v_i(1)$. To implement the principal’s dual-winner equilibrium strategy for his agent, the principal provides a zero budget constraint on the large package. This eliminates the agent’s possibility to win the double-unit package. Thus, he is in fact competing in a package auction that consists solely of one unit. As a consequence of Lemma 1, the agent then truthfully bids up to his budget constraint for the single-unit package. Therefore, the principal can simply provide his agent with a budget constraint in the amount of his valuation for one unit. QED.

**Theorem 10:** If the principal pays a bonus of $M(1) = w(\hat{v}(2)) - w(v(1))$ and $M(2) = 0$ in the case of winning the dual-winner outcome, then an agent will always implement the dual winner equilibrium in the first-price sealed-bid auction.

**Proof:** Note first, that any firm $i$’s optimal contract menu must involve budget constraints of the form $\alpha_i(1) = d$ and $\alpha_i(2) = 2 \cdot d$, i.e., $\alpha_i(2) = 2 \cdot \alpha_i(1)$. Next, assume opponent $j$ truthfully submits bids $\alpha_j(1) = d$ and $\alpha_j(2) = 2 \cdot d$. Now, suppose agent $i$ chooses $\hat{a}_i(1) > \alpha_i(1)$, which implies $\hat{a}_i(2) = 2 \cdot \hat{\beta}_i(1)$. Note that $2 \cdot \hat{\beta}_i(1) > \hat{\beta}_i(1) + \beta(1)$ and $2 \cdot \hat{\beta}_i(1) > 2 \cdot \beta(1)$. Thus, agent $i$ wins the double-unit package. For this deviation to be unprofitable, the following incentive-compatibility constraints (IC1) needs to be satisfied:

$$w(v(1)) + m_i(1) \geq w(v(2)) + m_i(2) \quad \text{(IC1)}$$

Any profit-maximizing principal chooses $m_i(2) = 0$, and thus IC1 can be rewritten as: $m_i(1) \geq w(v_i(2)) - w(v_i(1))$. Principal $i$ could now offer a menu of payments $m_i(1) = w(\hat{v}(2)) - w(v(1))$ with corresponding budget constraints of $\alpha_i(1) = d$ and $\alpha_i(2) = 2 \cdot d$. However, as $d$ could be the result of various $\hat{v}(2)$ and $\hat{v}(1)$ pairings agent $i$ has an incentive to choose the pairing that offers the highest
payment and still satisfies $d = \hat{v}(2) - \hat{v}(1)$. Hence, incentive-compatibility constraint IC2 needs to be satisfied, too:

$$w(v(1)) + m_i(1) \geq w(v(1)) + \hat{m}_i(1) \tag{IC2}$$

As the principal wants to minimize payments a constant payment of $m_i(1) = \hat{m}_i(1)$ is required for all possible reports of $\hat{v}(2)$ and $\hat{v}(1)$. Moreover, the payment must be high enough to induce the agent with the greatest incentive to deviate to the double-unit package not to do so. Agent $i$ with package valuations of $v_i = (\overline{v}(2), v(1))$ has the highest incentive and thus, the optimal payment is $m_i(1) = w(\overline{v}(2)) - w(\overline{v}(1))$.

Finally, suppose agent $i$ chooses $\hat{\alpha}_i(1) < \alpha_i(1)$, which implies $\hat{\alpha}_i(2) = 2 \cdot \hat{\beta}(1)$. Note that $2 \cdot \hat{\beta}(1) < \hat{\beta}(1) + \beta(1)$ and furthermore, $2 \cdot \hat{\beta}(1) < 2 \cdot \beta(1)$. Here, agent $j$ wins the double-unit package and agent $i$ wins nothing. According to IC1, this deviation cannot be optimal for agent $i$. QED.

**Appendix II: Multi-Unit Auctions Without Package Bidding**

The SMRA is often used in spectrum auctions, and it is interesting to compare SMRA in our model with budget constraints with package auctions. We use a uniform price multi-unit auction as an abstraction for a multi-unit SMRA, similar to Ausubel et al. (2014). The price starts at zero and only stops when demand no longer exceeds supply. Such a multi-unit auction is different from a package auction because it forces the agents to reveal their budget constraint for one unit to some extent, so that in dual-winner efficiency, the efficient allocation emerges.

**Theorem 11:** In the second stage of the principal-agent $2 \times 2$, ascending uniform-price multi-unit auction in which any agent $i$ faces package-dependent budget constraints $\alpha_i = [\alpha_i(1), \alpha_i(2)]$ of the form $\alpha_i(1) \leq$
\(\alpha_i(2)\), where \(\alpha_i(l) = \alpha(v_i(l))\), straightforward bidding constitutes a symmetric ex-post dual-winner equilibrium for agents.

**Proof:** At the beginning of the auction, both agents are active and the uniform unit price starts to increase. Suppose both agents engage in straightforward bidding as defined in **Definition 1** and remain active on both units until the price reaches half their budget constraints for the package of two units: \(\alpha(v_j(2))/2\). Any agent \(j\) distributes his double-unit budget constraint evenly, because he needs to beat agent \(i\) on both single units to obtain two units. Hence, an uneven distribution of the large budget constraint cannot be optimal. Without loss of generality, assume agent \(i\) to have the lower double-unit budget constraint.

As soon as \(i\) is overbid, he cannot win two units anymore and starts to bid up to his single-unit budget constraint on one unit. Thus, agent \(i\) always outbids opponent \(j\) on one single unit with his respective single-unit budget constraint, because \(\alpha(v_i(1)) > \alpha(v_j(2))/2\) is always true in dual-winner efficiency. However, as soon as \(j\) is outbid with half his double-unit budget constraint on one unit, he can no longer win two units. Therefore, he remains active on the other unit until his single-unit budget constraint is reached. As both bidders always end up in the dual-winner outcome, they are indifferent between bidding on one or on two units. Moreover, the above reasoning is independent of the agents’ actual value draws and therefore constitutes an ex-post equilibrium. QED.

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2 In a number of spectrum auctions, analysts argue that bidders inflated their demand and bid aggressively on large packages. Some comments even suggest that bidders bid beyond the marginal value of an additional license to win a larger package. This is difficult to explain in particular in uniform-price multi-unit auctions where demand reduction (Ausubel et al. 2014) is typically the main concern. We provide more extensive examples and references for price wars in the conclusions of this article.

3 Bulow et al. (2009) writes that “Prior to the AWS auction, analyst estimates of auction revenue ranged from $7 to $15 billion. For the recent 700 MHz auction, they varied over an enormous range – from $10 to $30 billion.” This is by no means an exception and estimates of investment banks and other external observers can quite different from the actual revenue of the auction. Prior to the German spectrum auction in 2010, most analysts expected low revenue (Berenberg Bank estimated €1.67 bn., the LBBW bank estimated €2.1 bn.). The actual revenue from the auction was €5 bn.

4 Burkett (2015) has to be given credit for introducing principal-agent relationships in a single-object auction context. He showed how the fact that budget constraints are endogenously set by the principal to mitigate the agency
problem affects the standard revenue comparisons between FPA and SPA. Later, Burkett (2016) studies a principal’s optimal choice of the budget constraint for an agent participating in an auction. Principal and agent are assumed to be equity holders in the firm, interested in maximizing the firm’s expected return at the auction, but the bidder receives an additional private payoff when the firm wins the good. In contrast, we model a complementary private-values environment with multiple units and focus on hidden information about the valuations. In our model, agents are no equity holders and have precise information about the valuations of the goods, but principals do not. The types of manipulation possible for agents in such multi-unit markets are quite different from single-object auctions. We show that the information asymmetry and the different preferences result in an agency dilemma that is difficult to resolve.

Many spectrum auctions are for homogeneous goods, i.e., multiple licenses of 5 MHz spectrum in a particular band. While the strategic problems discussed can also be found with heterogeneous goods, markets with homogeneous goods require less notational burden.

For example, France (2011) and Norway (2013) used a first-price sealed-bid package auction, whereas Romania (2012) used an ascending combinatorial clock auction.

We refer to risk aversion as is implied by a concave utility function \( w(\cdot) \) over possible outcomes of the auction (lottery) \( \{0, v_i(1), v_i(2)\} \) for some agent \( i \). Regarding Theorem 1 and Theorem 2 one could expect a risk averse agent to prefer the certain dual-winner outcome with utility of \( w(v_i(1)) \) over the lottery of winning the single-winner outcome with utility of \( w(v_i(2)) \cdot F_2(v_i(2)) \) and not winning at all.

See [https://en.wikipedia.org/wiki/Spectrum_auction#Germany](https://en.wikipedia.org/wiki/Spectrum_auction#Germany). Ausubel et al. (2014) argue that this was seen as a mistake on the part of Deutsche Telekom and that the company behaved differently in the subsequent auction in the much smaller Austrian market.


[https://www.rtr.at/en/tk/multibandauktion](https://www.rtr.at/en/tk/multibandauktion)