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Ascending Combinatorial Auctions with Allocation Constraints: On Game-Theoretical and Computational Properties of Generic Pricing Rules

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Combinatorial auctions are used in a variety of application domains such as transportation or industrial procurement using a variety of bidding languages and different allocation constraints. This flexibility in the bidding languages and the allocation constraints is essential in these domains, but has not been considered in the theoretical literature so far. In this paper, we analyze different pricing rules for ascending combinatorial auctions which allow for such flexibility: winning levels, and deadness levels. We determine the computational complexity of these pricing rules and show that deadness levels actually satisfy an ex-post equilibrium, while winning levels do not allow for a strong game-theoretical solution concept. We investigate the relationship of deadness levels and the simple price update rules used in efficient ascending combinatorial auction formats. We show that ascending combinatorial auctions with deadness level pricing rules maintain a strong game theoretical solution concept and reduce the number of bids and rounds required at the expense of higher computational effort. The calculation of exact deadness levels is a Π_2^P-complete problem. Nevertheless, numerical experiments show that for mid-sized auctions this is a feasible approach. The paper provides a foundation for allocation constraints in combinatorial auctions and a theoretical framework for recent Information Systems contributions in this field.

Key words: Electronic markets and auctions, Economics of IS, Electronic commerce, Decision support systems

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1. Introduction

Combinatorial auctions (CAs) allow selling or buying a set of heterogeneous items to or from multiple bidders. Bidders can specify package bids, i.e., a bid price is defined for a subset of the items for auction (Cramton et al. 2006). The price is only valid for the entire package and the package is indivisible. For example, in a CA a bidder might want to buy items x, y, and z for a package price of \$100, which might be more than the total of the prices for the items sold individually. CAs can be seen as generic mechanisms for multi-object markets as they allow expressing complex preferences such as items being substitutes or complements for bidders. They have found application in a variety of domains such as the auctioning of spectrum licenses (Cramton

2009), truck load transportation (Caplice 2007), bus routes (Cantillon and Pesendorfer 2006), or industrial procurement (Bichler et al. 2006).

The design of efficient CAs has drawn considerable attention, as they raise fundamental questions on pricing and efficiency in multi-object markets. If bidders revealed their preferences truthfully, the auctioneer only had to solve an optimization problem to find the efficient allocation. Typically bidders have incentives to speculate and deviate from truthful bidding. The Vickrey-Clarke-Groves (VCG) mechanism has therefore been a significant contribution in auction theory. Green and Laffont (1977) proved that the sealed-bid VCG mechanism is the unique auction mechanism in which truthful bidding is a dominant strategy. In spite of this powerful result, this mechanism has a number of practical problems when used in multi-object markets (Ausubel and Milgrom 2006b, Rothkopf 2007). Also, many auctioneers want to have a transparent open-cry bidding process rather than a sealed-bid format. For example, almost all spectrum auctions organized throughout the world are iterative auctions with multiple rounds.

It is interesting to understand, whether there are also iterative CA formats, which also satisfy strong game-theoretical solution concepts, which limit incentives for speculation. Preference elicitation in iterative auctions can invalidate dominant strategy equilibria existing in a single-step version of a mechanism (Conitzer and Sandholm 2002), but ex post equilibria can be achieved which also do not require agents to speculate about other bidders' valuations (Shoham and Leyton-Brown 2009). Much progress has been made on this question in the theoretical literature in the recent years. *iBundle* (Parkes and Ungar 2000), the Ascending Proxy Auction (APA) (Ausubel and Milgrom 2002), and dVSV (de Vries et al. 2007) lead to an efficient allocation, if bidders bid straightforward[1], which is an ex post equilibrium as long as the buyer submodularity condition holds for all valuations. Buyer submodularity is a restriction on the valuations of bidders, which defines that bidders are more valuable when added to a smaller coalition. It is interesting that such a strong solution concept is possible in iterative CAs, even though the result does not hold for general valuations. This line of work is heavily based on duality theory in linear programming.

All these auction formats use non-linear and personalized ask prices and increase these ask price for losing bids by a minimum increment, which causes a large number of auction rounds (Schneider et al. 2010).[2] The APA uses proxy bidders in order to cope with the large number of auction rounds, and to make sure that bidders follow the straightforward strategy, which turns the

[1] Bidders follow the straightforward strategy if they bid only on the surplus maximizing package(s) in every round

[2] We will refer to *ask prices* as the prices set by the auctioneer during the auction, and *bid prices* as the prices specified by bidders, if it is necessary to distinguish prices. In this paper, we assume that bidders always bid on the ask price.

mechanism into a sealed bid auction. Apart from these theoretical advances, a number of linear or item-price auction formats have been developed. These include versions of the combinatorial clock auction (Porter et al. 2003, Bichler et al. 2011), where item-level ask prices rise, whenever there is overdemand on an item, but also the family of auction formats with pseudo-dual linear ask prices (Kwasnica et al. 2005, Bichler et al. 2009). As of yet, there is little theory on equilibrium bidding strategies in such auctions, although lab experiments yielded high levels of allocative efficiency.

Adomavicius and Gupta (2005) introduce different pricing rules for CAs. *Pricing rules* refer to functions by which the auctioneer determines ask prices based on bids submitted in the auction. Winning levels (WLs) describe the lowest possible bid which would win if no other bids are submitted, whereas deadness levels (DLs) are a lower bound to bids, which still can become winning in the course of the auction. DLs and WLs describe natural bounds and interesting feedback for a losing bidder. While there is no rationale to bid below the DL, bidding at the WL could be rational in some situations, where a bidder is the only one able to outbid a winning coalition of bidders. WLs are equivalent to the minimal winning bids described by Rothkopf et al. (1998). These pricing rules are independent of the allocation rules, and laboratory experiments with respective auction formats yielded high levels of efficiency (Adomavicius et al. 2012). They are very generic and can be considered a fundamental contribution in the emerging Information Systems literature on decision support in smart markets (Xia et al. 2004, Bapna et al. 2007, Guo et al. 2007, Bichler et al. 2009, Scheffel et al. 2011, Bichler et al. 2010) and in the general literature on CAs.

The analysis in Adomavicius and Gupta (2005) is focused on a pure OR bidding language and assumes no substitutes valuations. OR bids can represent only bids that do not have any substitutabilities, i.e., purely additive and superadditive valuations (Nisan 2006). Also, this initial work did not analyze equilibrium strategies in such auctions. However, DLs and WLs introduce a very generic concept, which can be applied to any bidding language. It is natural to ask, how these generic pricing rules can be defined for auctions with XOR bidding languages and other types of allocation constraints as well, and which game-theoretical solution concepts can be satisfied. Auctions with strong game-theoretical solution concepts are strategically easier for bidders because there is no need to speculate about other bidders' valuations. This makes bidding strategically easy and respective auction formats are more likely to lead to high efficiency.

In this paper, we introduce WLs and DLs for general CAs, which allow for XOR bids and other constraints. Auctioneers use various types of allocation constraints to limit the number of winners or the number of items allocated to one bidder or to a group of bidders. Such constraints are actually the rule rather than the exception in application domains such as industrial procurement (Bichler et al. 2006, Sandholm and Suri 2006) or transportation (Caplice 2007). But they are typically not considered in the literature on iterative CAs. The beauty of DLs and WLs is that their definition is independent of the type of allocation constraints used in the winner determination. The resulting theory is applicable to a much broader set of real-world applications, where allocation constraints play a considerable role. We will refer to ascending CAs allowing for different types of allocation constraints as *flexible combinatorial auction (FCA)* in this text. Such constraints might be added by the auctioneer and the bidders, which allows for considerably more flexibility in the specification of the preferences of market participants.

While generic pricing rules for different CAs with allocation constraints would also have significant practical importance, we show that these pricing rules bare significant theoretical challenges. The main goal of this paper is not to introduce a new auction format, but *to define and analyze computational and game-theoretical properties of* WLs *and* DLs for FCAs. In particular, we want to understand which pricing rules allow for ex post equilibria and how these pricing rules relate to the theoretical framework on efficient and ascending CAs (Parkes and Ungar 2000, de Vries et al. 2007). This connects also recent IS contributions to the game-theoretical literature in this field.

The paper is structured as follows: In Section 2, we briefly discuss allocation constraints in CAs. We define winning and deadness levels and describe respective algorithms in Section 3. In Section 4 we determine the computational complexity for these pricing rules. In Section 5 we analyze the impact of allocation constraints on efficient CAs and perform an equilibrium analysis of ascending CAs using either winning or deadness levels as ask prices in Section 6. In Section 7 we report on our computational experiments, before we provide conclusions in Section 8.

2. Allocation Constraints in Combinatorial Auctions

Allocation constraints specify limits on the allocation of the available items to the bidders without explicitly limiting bid prices or revenue of a bidder or a group of bidders. On the other hand *price constraints* set price limits on items, packages, a bidder's budget or auctioneer revenue. It has been shown that incentive-compatible auctions are impossible in general if there are private budget limits (Dobzinski et al. 2008), and also reserve prices by the auctioneer increase expected revenue at the expense of efficiency (Myerson 1981). Table 1 provides an overview of allocaton and price constraints in CAs, subsumed by the term *side constraints*. We focus on efficient auctions, and therefore will limit ourselves to allocation constraints. One can also divide side constraints into bidder specific ones (bidder level) and such constraints, which concern more than a single bidder (group level). The latter are typically specified by the auctioneer, while bidder specific constraints could be imposed by the bidder or the auctioneer, especially when the OR bidding language is used.

Allocation constraints are important in many domains. Spectrum auctions, which have been the driving application for much research in this area, regularly face spectrum caps (max # items/bidder) (Seifert and Ehrhart 2005). In industrial procurement or the auction for transportation services allocation constraints are the rule rather than the exception (Caplice 2007, Bichler et al. 2006, Sandholm 2003, Sandholm and Suri 2006). Buyers need to specify lower or upper bounds on the number of suppliers overall or per group (min/max # winning bidders): lower bounds in order to hedge the risk that some suppliers fail to deliver and upper bounds in order to avoid administrative expenses. Market share constraints are defined on a group of bidders. For example, due to corporate requirements at least one minority supplier must be included in the set of winners on a particular set of items. Auctioneers also need to impose lower or upper bounds on the number of items (min/max # items/bidder), which can be awarded to a particular bidder or a group of bidders. Exclusive disjunction describes constraints such that a bidder is allowed to win one set of items or another one, but not both. For example, an auctioneer allows a bidder to win one of two different regions A and B, but not both. Such constraints can also be specified on a group of bidders. The XOR language can be seen as a separate bidder specific allocation constraint.

While the XOR language already captures many further allocation constraints, experimental analysis has been shown that OR bidding languages are often more efficient than XOR languages due to the reduced number of package bids that need to be submitted (Brunner et al. 2010). OR bids can represent only bids that do not have any substitutabilities, i.e., purely additive and superadditive valuations (Nisan 2006). If a bidder uses an OR bidding language, it might also be useful to specify constraints on the number of items (min/max # items/bidder) or budget awarded, in order to avoid the exposure problem, which can occur with substitutes valuations (e.g., a bidder is winning packages AB and CD, but only wants two lots at a maximum) or to express his capacity constraints in case he is a supplier in a procurement auction. Constraints of exclusive disjunction are relevant to the auctioneer in case of an OR bidding language, when a bidder is allowed to win one set of items or another one, but not both. Carriers in a transportation auction use such disjunctive constraints to communicate the message "give me this set of lanes, or this set of lanes, but not both" (Caplice 2007).

Having flexibility in the bidding language and the allocation constraints used by the auctioneer and the bidders allows for a much broader applicability of CAs, and this can be considered a prerequisite for most applications in transportation and industrial procurement.

side constraint	allocation constraint	price constraint
bidder level	min/max # items/bidder	budget
	exclusive disjunction	
group level	min/max # winning bidders	reserve prices
	market share	budget
	exclusive disjunction	

Table 1 Side constraints

3. Pricing Rules

Before we provide a formal definition, we will first provide an example to informally introduce the pricing rules DL and WL. We will contrast these with pricing rules as they are used in *iBundle* or in auction formats with pseudo-dual linear ask prices, such as RAD (Kwasnica et al. 2005) to highlight the connections among these different pricing rules.

packages	AB	BC	AC	B	C
bids	22^*_1,16_2	24_3	20_4	7_5	8^*_6
RAD	22	24	14	16	8
iBundle	22_1, 17_2	25_3	21_4	8_5	8_6
DL	22	24	20	7	8
WL	22	30	23	10	8

Table 2 Example with six bids and different ask prices.

The upper part of Table 2 describes six bids from different bidders $B1$ to $B6$, submitted on subsets of three items A, B, and C, while the lower part shows the resulting prices in various auction formats. Subscripts indicate bidders, i.e., 22_1 indicates a bid of \$22 from bidder $B1$. Prices have subscripts only if they are personalized. Asterisks denote the provisional winning bids.

The calculation of RAD prices requires solving a number of linear programs. Note that these linear prices can be lower than a losing bid (see the bid on AC of bidder $B4$). Another problem with pseudo-dual ask prices is the fact that they can also decrease if the competition shifts, and that the calculation does not take into account allocation constraints explicitly. This makes it difficult to define an equilibrium analysis and as of now there is little theory. Bichler et al. (2009) analyze the problems in defining pseudo-dual linear ask prices.

For the rest of the paper, we will focus on the family of efficient and ascending auctions (*iBundle*, APA, and dVSV). Apart from the use of proxy agents, APA and *iBundle* are equivalent and follow a subgradient algorithm, whereas dVSV uses a primal-dual approach (de Vries et al. 2007). Line 4 in our example shows a new set of ask prices in *iBundle*. The auction format increases the bids of all losing bidders by a minimum bid increment (\$1 in this example) (Parkes and Ungar 2000). DLs and WLs describe generic pricing rules, i.e., they are independent of the allocation constraints used by the auctioneer. DLs describe deadness level ask prices, which are simply the bids of all bidders in the last round in this example without any constraints. In this case, DLs do not need to be personalized. Losing bidders need to bid higher than this by a minimum bid increment. With a minimum bid increment of one, the DL would be at the *iBundle* ask price for the losing bidders $B3$, $B4$, and $B5$. In the presence of allocation constraints, DL ask prices can be much higher than the ones of *iBundle*, as we will see later.

The WL describes the lowest bid prices, at which a single bid would become winning without a new complementary bid of another bidder. Bidders $B4$ and $B5$ could become winning at a lower price, if they would form a coalition. A known problem with package bidding is the threshold problem, in which bidders seeking larger packages may be favoured, because small bidders do not have the incentive or capability to top the tentative winning bids of the large bidder. In such a problem, the WL for a small bidder might be way too high to outbid a winning bidder unilaterally, and with only WL ask prices, it will become difficult for bidders to coordinate. The spread between DLs and WLs can often be quite large in realistic value models.

Of course, one can also think of personalized and non-linear ask prices inbetween DLs and WLs. The coalition of $B4$ and $B5$ in our example would need to increase their bids by a combined \$3 plus increment in order to become winning. Both bidders would become winning, if bidder $B4$ bid \$21.5 and bidder $B5$ bid \$8.5, for example We refer to such prices as coalitional winning levels (CWL). A CWL could provide an alternative to linear-price auction formats, and help mitigate threshold problems and coordinate bidders. However, computing $CWLs$ turns out to be challenging. First, the number of losing coalitions can grow very fast with the number of items and bids in the auction. Second, and more importantly, there are a many ways how the costs to outbid the winning coalition can be shared among the bidders in a losing coalition. One can think about cost sharing which is proportional to the bids in the previous round or a cost sharing that satisfies certain fairness criteria. Independent of how $CWLs$ are set, there will be incentives to free-ride on other bidders in a losing coalition. Due to the ambiguities in the definition of $CWLs$, we will focus on DLs and WLs in this paper, which describe natural upper and lower bounds for bids in an ascending auction, and leave the analysis of $CWLs$ for future research.

3.1. Winning Levels

We first introduce the necessary terminology. Let $\mathcal{K}$ denote the set of items and $\mathcal{I}$ the set of bidders. A subauction on itemset $S \subseteq \mathcal{K}$ refers to an auction where only items $l \in S$ are auctioned. Auction state $k \in \mathbb{N}$ refers to the auction after the first k bids are submitted. A bid is a tuple $b = (S, v, k, i)$ where S denotes the package the bid refers to, v the amount bid, k the auction state after the bid submission[3] and i the bidder. Given bid b, we use the notation $S(b)$, $v(b)$, $k(b)$ and $i(b)$ to refer to its respective elements. A bid $b=(S,v,k,j)$ is foreign w.r.t. bidder i if $j\neq i$. $B_{S,k}=\{(T,v,k',i)|T\subseteq S,k'\leq k\}$ is the set of all bids in S and $B_{S,k,-i}=\{b\in B_{S,k}|i(b)\neq i\}$ is the set of all bids in S which are foreign to bidder i. An allocation C of packages to bidders is a set of nonoverlapping bids, i.e. for every $b,b'\in C, b\neq b'\Rightarrow S(b)\cap S'(b)=\emptyset$. An *allocation constraint* is a function $C\rightarrow\{0,1\}$ where the value 1 indicates fulfillment of the constraint. Our analysis applies to every possible allocation constraint in this form which can be verified in polynomial time[4]. An allocation C is feasible only if it satisfies every allocation constraint, in which case we write $feas(C)=1$. The set of all feasible allocations is denoted by $\mathbb{C}_k=\{C\subseteq B_{\mathcal{K},k}|b,b'\in C, b\neq b'\Rightarrow S(b)\cap S'(b)=\emptyset, feas(C)=1\}$. The combinatorial allocation problem (CAP), also known as winner determination problem (WDP), solves $max_{C\in\mathbb{C}_k}\sum_{b\in C}v(b)$. $CAP^k(\mathcal{K})$ denotes the maximum value of this optimization problem (henceforth we will refer to it simply as value) and $WIN_k(\mathcal{K})\in\mathbb{C}_k$ the value-maximizing allocation at state k.[5] $CAP(S)$ is the value of subauction S.

The *winning level* of a package S for bidder i at auction state k, $WL_k(S,i)$, is the minimal price i must bid to win S at auction state $k+1$.

Definition 1. $WL_k(S,i)=\underset{v}{argmin}:(S,v,k+1,i)\in WIN_{k+1}(\mathcal{K})$.

Adomavicius and Gupta (2005) define the anonymous WL of package S at auction state k by[6]

$$WL_k^{OR}(S)=CAP^k(\mathcal{K})-CAP^k(\mathcal{K}\backslash S) \tag{1}$$

Intuitively a bid on S can only win, if the bid price together with the value of the complementary set of items $\mathcal{K}\backslash S$ exceeds the actual value of the whole auction. Implicitly, the following assumptions are made: (i) OR bidding language and (ii) absence of allocation constraints. When these assumptions are relaxed, the calculation of WLs as in equation (1) is inappropriate. A first reason is that allocation constraints, which are not bidder specific, cannot be globally validated when solving subauction $CAP(\mathcal{K}\backslash S)$. In addition, WLs must be personalized as the following example demonstrates.

Example 1. Consider an auction with items A, B, C, bidders B1, B2 and B3 and the constraint that each bidder cannot win more than two items. Bidder $B1$ bids at $k=1$ \$5 on AB, B2 at $k=2$ \$1 on AB and at $k=3$ \$2 on C and $B3$ bids at $k=4$ \$3 on AB . $WL(C)$ is for B1 \$4 whereas for B2 it is only \$2.

[3] For example, a bid with state $k=3$ is the third bid submitted in chronological order.

[4] To the best our knowledge no allocation constraint has been proposed which does not fulfill this property.

[5] Depending on the context we may omit index k.

[6] The problem of finding this minimal price is introduced by Rothkopf et al. (1998) and referred to as the minimal winning bid problem.

We introduce the following formula to calculate personalized WLs that takes XOR bidding and allocation constraints into account:

PROPOSITION 1.

$$WL_k(S,i) = CAP^k(\mathcal{K}) - CAP^k(\mathcal{K}, S_i) \tag{2}$$

The proof is relegated to Appendix A. $CAP^k(\mathcal{K}, S_i)$ denotes the value of the whole auction provided that bidder i wins package S for free. Thus the auction value $CAP^k(\mathcal{K}, S_i)$ is raised from the items in $\mathcal{K}\backslash S$, as it is the case in $CAP(\mathcal{K}\backslash S)$. If the OR bidding language is used and no allocation constraints exist, the computation in (2) yields the same WLs as (1).

3.2. Deadness Levels

The *deadness level* of package S at auction state k, $DL_k(S,i)$, is the minimal price a bidder must bid to have a chance to win S in any future auction state.

DEFINITION 2. $DL_k(S,i) = \underset{v}{argmin} : \exists k' > k : (S, v, k+1, i) \in WIN_{k'}(\mathcal{K})$.

DLs constitute lower bounds on bid prices and all future bids below are "dead", i.e., they cannot win any more no matter which bids are submitted in future auction states $k' > k$. This implies that $DL_k(S,i)$ is monotonically increasing through the progress of any iterative auction.[7]

PROPOSITION 2.

$$a) DL_k(S,i) \le DL_{k+1}(S,i) \qquad b) DL_k(S,i) \le WL_k(S,i) \qquad c) DL_k(\mathcal{K},i) = WL_k(\mathcal{K},i)$$

The proofs are based on the definitions of the metrics and are given in Appendix A.

Adomavicius and Gupta (2005) define the anonymous DL of a package S by

$$DL_k^{OR}(S) = CAP^k(S) \tag{3}$$

In words, a bid on S cannot be part of the winning allocation if it is below the value of subauction S (i.e. $CAP(S)$). For example if $S = AB$ and there are already bids $A = \$10$ and $B = \$15$, then any bid on AB below \$25 is dead.

The DL in equation (3) is not valid if there are allocation constraints or any of the bidders uses an XOR bidding language. In these cases DLs must be personalized. We also need to understand what influence the additional allocation constraints might have in future auction states. For instance, consider again Example 1. $B3$ loses AB at current state $k = 4$. But if in the future state $k = 5$ $B1$

[7] Through our analysis, we assume no bid revocability, otherwise DLs would be always zero.

bids \$8 on C, then his previous winning bid on AB loses due to the allocation constraint[8]. Thus $B3$ can win AB for \$3 and his personal DL at $k=4$ is \$3. We say that the bid of $B1$ on AB has been "blocked" at $k=5$ due to the constraint.

A bid on package S loses in an auction with allocation constraints if at least one of the following conditions is met: (i) There exists a higher bid or bid combination in subauction S that wins in the whole auction (without violating allocation constraints). (ii) S is not part of the value maximizing allocation. (iii) The bid in interaction with other bids that win in the whole auction violates an allocation constraint.

The first two conditions are common for every CA, with or without constraints. The third one leads us to the definition of *blocked bids*, which we use later on; a bid b is blocked if it does not win due to allocation constraints. In this case, one or more winning bids in the complementary subauction, which we denote by B', prevent the blocked bid to win. If the bids in B' had not existed, b would have won. The winning bids after the removal of a bid set B' are denoted by $WIN_k^{-B'}(\mathcal{K})$. In the context of Example 1 extended by a bid of \$8 of bidder $B1$ on item C, and of \$8 of $B1$ on additional item D, the bid set B' comprises these two bids. The removal of both bids in B' would turn the bid b of $B1$ on AB into winning. Since $b \in WIN_k^{-B'}(\mathcal{K})$, b is blocked by B' and does not win due to allocation constraints.

Definition 3. Bid $b=(S,v,k^-,i)$ is *blocked* at state $\bar{k} \geq k^-$ if $b \notin WIN_{\bar{k}}(\mathcal{K})$ and $\exists B' \subseteq B_{\mathcal{K}\setminus S,\bar{k}}$ with $b \in WIN_{\bar{k}}^{-B'}(\mathcal{K})$.

Computing $DL_k(S,i)$ requires knowing which currently winning bids in S can be blocked in future auction states. We call these bids blockable.

Definition 4. Bid $b=(S,v,k^-,i)$ is *blockable* at state k if $b \in WIN_k(\mathcal{K})$ with $k \geq k^-$ and $\exists \bar{k} > k$ so that b is blocked at state $\bar{k}$.

Removing a blockable foreign to i bid in S causes another lower foreign bid (or bids) in S to win. If a distinct future state $\bar{k}$ can be reached where both the second and the first bid are blocked, then the $DL_k(S,i)$ becomes even lower, since i does not have to overbid any of them. We will say that the two bids are simultaneously blockable at current state k and simultaneously blocked at $\bar{k} > k$. To see this happen, consider again Example 1 with an additional item D and seek for $DL(AB,B2)$. We have already argued that the highest bid on AB by $B1$ is blockable. Removing it causes the bid of $B3$ to win. This second bid is also blockable due to a similar reason as the first blockable bid ($B3$ may bid high on D). More importantly, these two bids are simultaneously blockable since

[8] A bid of \$5 would produce the same effect. With \$8 we indicate here and in the sequel an arbitrarily high future bid, submitted due to increased competition.

there is a future state with both bids blocked (see Table 3). Had item D not existed, this would not have been possible. Only one of the bids would have been blockable but the two of them would not have been simultaneously blockable.

<table>
<tr><th></th><th>A</th><th>B</th><th>C</th><th>D</th></tr>
<tr><td>$B1$</td><td colspan="2">$\not{5}^{4}$</td><td>8^{5}</td><td>0</td></tr>
<tr><td>$B2$</td><td colspan="2">1^{2}</td><td>2^{3}</td><td>0</td></tr>
<tr><td>$B3$</td><td colspan="2">$\not{3}^{4}$</td><td>0</td><td>8^{6}</td></tr>
</table>

Table 3 Bids in bold block crossed out bids and $DL(AB, B2) = 1$, k is denoted as superskript

We just highlighted the central role of simultaneously blockable bid sets in computing $DL_k(S,i)$. These are bids by rivals of i, thus foreign to him, submitted in S. We denote a *simultaneously blockable bid set* in subauction S, at state k, comprising bids foreign to bidder i, as $B^{block}_{S,k,-i}$. Undoubtedly, foreign bids on packages which overlap with S but are not entirely in S are also preventing i to win S but these bids are not relevant at the most favorable future state for i that $DL(S,i)$ represents. The reason is that in this most favorable state the value of subauction $\mathcal{K}\backslash S$ is high enough such that S is part of the winning allocation. Before providing the formal definition for $B^{block}_{S,k,-i}$, it is meaningful to sketch the algorithm which utilizes these sets to compute $DL(S,i)$.

The algorithm identifies all simultaneously blockable bid sets. In our running example there was only one set when item D was present, comprising the bids of $B1$ and $B3$ on AB and two single-item sets when D was not present, which were the two bids separately. The algorithm removes each identified set in turn and computes the value of $CAP(S)$. The lowest value is $DL(S,i)$, because $DL(S,i)$ represents the most favorable future state for i to win S.

Having outlined the algorithm, we can state the conditions a set must fulfill in order to be declared as simultaneously blockable at current state k, with $\bar{k}$ being the future state where the set will be actually blocked. We begin with the indispensable conditions and continue with conditions which only accelerate the sketched algorithm. Their enumeration and motivation precedes the formal descrpition given in 5. We have already discussed that going from k to $\bar{k}$ in order to block bids in S, we add only bids in $\mathcal{K}\backslash S$ since if they would overlap with S, they would prevent i from winning S. Their set is $B' = B_{\mathcal{K},\bar{k}} \backslash B_{\mathcal{K},k}$, i.e. all bids between k and $\bar{k}$. Bids in B' must be winning else they cannot block any bids, or generally impact the allocation or prices. The first condition states that after the submission of B' it must be feasible that i wins S (condition i). Secondly, inherent to the definition of a blocked bid is the condition that every bid in $B^{block}_{S,k,-i}$ must be losing at $\bar{k}$ (condition ii).

Moving to conditions which only accelerate the sketched algorithm, we observe that if all bids in the simultaneosuly blockable set are losing at k, their removal will not lower the price i has to pay for S, hence at least one of them must be winning at k (condition iii). In Table 3 removing only bid 4 has no impact. Furthermore, it is not meaningful that a blockable bid set contains a bid that never wins, even after the removal of the rest bids of the set. As we begin to block some bids, previously losing bids turn to winning and these are the ones that lower the price of S if blocked. Hence we demand $b \in B^{block}_{S,k,-i} \Rightarrow b \in WIN_k^{-(B^{block}_{S,k,-i} \setminus b)}(\mathcal{K})$ (condition iv). Furthermore, we observe that it is redundant to block a foreign bid on $T \subseteq S$ if it is lower than a bid already submitted by i on $T' \subseteq T$, since i can never pay less than his own bid. The set of all these bids, which we call non-i-dominated, is denoted by $B^{ndom}_{S,\bar{k},-i}$ and the corresponding condition states that all bids in $B^{blocked}_{S,\bar{k},-i}$ are non-i-dominated (condition v). The concluding condition specifies that simultaneously blockable bid sets must be maximal (condition vi) since blocking and removing an extra bid can only lower the value of $CAP(S)$. A set is defined as maximal if it satisfies a property (in our case the property is the satisfaction of conditions i to v) and there exist no strict superset of it that also satisfies the property. Hence adding an item in a maximal set causes violation of the property. In Table 3 bid 1 is not identified as simultanoeously blockable only because it is not maximal.

Definition 5. Bid set $B^{block}_{S,k,-i}$ is *simultaneously blockable* at state k if:
$\exists B' = (B_{\mathcal{K},\bar{k}} \setminus B_{\mathcal{K},k}) \subseteq (B_{\mathcal{K}\setminus S,\bar{k}} \cap WIN_{\bar{k}}(\mathcal{K}))$:

$i)$ $feas(WIN_{\bar{k}}(\mathcal{K} \setminus S) \cup b) = 1$, where $S(b) = S, i(b) = i$

$ii)$ $B^{block}_{S,k,-i} \cap WIN_{\bar{k}}(\mathcal{K}) = \emptyset$

$iii)$ $B^{block}_{S,k,-i} \cap WIN_k(\mathcal{K}) \neq \emptyset$

$iv)$ $b \in B^{block}_{S,k,i} \Rightarrow b \in WIN_k^{-(B^{block}_{S,k,-i} \setminus b)}(\mathcal{K})$

$v)$ $B^{block}_{S,k,-i} \subseteq B^{ndom}_{S,k,-i}$

$vi)$ $\nexists B'' \supset B^{block}_{S,k,-i}$ with B'' satisfying conditions $i)$ to $v)$.

With this definition we can introduce the general method to compute $DL_k(S,i)$ for arbitrary allocation constraints.

General Method to Compute $DL_k(S,i)$

In the first phase, the method takes as input all bids which are candidates to form simultaneously blockable bid sets. These are the bids in $B^{ndom}_{S,k,-i}$. The output of the first phase is the set of all simultaneously blockable bid sets in S. We can represent the first phase as a function f with argument $B^{ndom}_{S,k,-i}$ and value the output of this phase, i.e. $f : B^{ndom}_{S,k,-i} \to \{B^{block}_{S,k,-i}\}$. We consider here f as a black box since for the general description of the method only f's argument and value matters.

In the second phase each of the identified sets $B^{block}_{S,k,-i} \in f(B^{ndom}_{S,k,-i})$ is removed consecutively and the value of subauction S, denoted as $CAP^k(S, B_{S,k} \backslash B^{block}_{S,k,-i})$ is computed. The lowest of these values equals to $DL(S,i)$. It is not always necessary to compute all these values. If we encounter a value not greater than the value of the bids of i in S, we do not need to continue since i must overbid the own bids. Equation 4 summarizes the general two-phase method.

Proposition 3.

$$DL_k(S,i) = \min_{B^{block}_{S,k,-i}} \left\{ CAP^k(S, B_{S,k} \backslash B^{block}_{S,k,-i}) : B^{block}_{S,k,-i} \in f(B^{ndom}_{S,k,-i}) \right\} \tag{4}$$

Proof: The proof draws on the definition of $DL_k(S,i)$ and of simultaneously blockable bid sets. Let c be the right side of equation 4 and B^{block}_* the argument of the minimum. First we show that for $b = (S, c, k^-, i)$ $\exists k^* > k^- : b \in WIN_{k^*}(\mathcal{K})$. We define k^* as the state at which: a) the bids in B^{block}_* are actually blocked and do not win due to condition i, b) a bid set B' in $\mathcal{K} \backslash S$ is submitted so that $B' \subseteq WIN_{k^*}(\mathcal{K})$ and c) $\forall b$ with $k(b) \in (k, k^*], i(b) \neq i, S(b) \subseteq \mathcal{K} \backslash S$ (i.e. no new foreign bids overlapping with S are submitted). We can ensure that the bids in B^{block}_* lose at k^* without the need of submission of overlapping bids at states in $(k, k^*]$ due to condition ii. Thus the bid b^* with value $c = CAP^k(S, B_{S,k} \backslash B^{block}_*)$ is part of the winning allocation (due to condition i and that the winning allocation maximizes CAP over all feasible allocations). Similarly it can be argued that for $b' = (S, c - \epsilon, k, i)$ $\nexists k' > k : b \in WIN'_k(\mathcal{K})$: After removing each simultaneously blockable bid set $B^{block}_{S,k,-i}$, there is a feasible allocation which includes the bids which maximize $CAP^k(S, B_{S,k} \backslash B^{block}_{S,k,-i})$ and have a value at least c. Thus b' cannot be part of the winning allocation. Collectively, we showed that $c = \underset{v}{argmin} : \exists k' > k : (S, v, k+1, i) \in WIN_{k'}(\mathcal{K})$. This is the definition of $DL_k(S,i)$. Q.E.D.

With an OR bidding language and without allocation constraints (4) reduces to (3), since without these constraints, there are no blockable bids and thus $DL_k(S,i) = CAP^k(S, B_{S,k} \backslash B^{block}_{S,k,-i}) = CAP^k(S, B_{S,k} \backslash \emptyset) = CAP^k(S, B_{S,k}) = CAP^k(S)$.

Example 2. Consider an auction with items A, B, C, D, bidders $B1$ to $B5$ and the XOR bidding language. The bids in AB are: $(AB, \$9, 1, B1)$, $(A, \$5, 2, B2)$, $(B, \$8, 3, B2)$, $(A, \$10, 4, B3)$, $(AB, \$15, 5, B4)$ and $(AB, \$19, 6, B5)$. We compute $DL_6(AB, B1)$ using the above formula. Function f's argument $B^{ndom}_{AB,6,-B1}$ contains all bids except the bid of $B1$ and f returns all simultaneously blockable sets $B^{block}_{AB,6,-B1}$ and

$$f(B^{ndom}_{AB,6,-B1}) = \{\{2,3,6\}, \{4,6\}, \{5,6\}\}$$

Bids are referred by their state. Bids {2,3,6} are blockable due to XOR bidding if $B2$ and $B5$ win C and D respectively, bids {4,6} if $B3$ and $B5$ win C and D respectively and bids {5,6} if $B4$ and

<table>
<tr><th></th><th>A</th><th>B</th><th>C</th><th>D</th></tr>
<tr><td>$B1$</td><td colspan="2">9^1</td><td colspan="2">0</td></tr>
<tr><td>$B2$</td><td>5^2</td><td>8^3</td><td colspan="2">0</td></tr>
<tr><td>$B3$</td><td>10^4</td><td colspan="3">0</td></tr>
<tr><td>$B4$</td><td colspan="2">15^5</td><td colspan="2">0</td></tr>
<tr><td>$B5$</td><td colspan="2">19^{6*}</td><td colspan="2">0</td></tr>
</table>

Table 4 **Bids of five bidders on items A,B,C and D. The superskript denotes k and * currently winning. What is the DL of the package AB for bidder B1?**

$B5$ win C and D respectively. Note that {2,5,6} is not listed since even if bids 5,6 are removed, bid 2 does not win and violates condition *iv* of definition 5. Sets without bid 6, like {2,3,4}, {2,3,5}, {4,5} violate condition iii since none of their bids currently wins. The sets {3,6} and {2,6} are not maximal, since their superset {2,3,6} is simultaneously blockable. Subsequently we remove each simultaneously blockable set from subauction AB and compute CAP. After removing the first set $CAP =$ \$15, after the second one $CAP =$ \$15 and after the third one \$18. The minimum CAP is \$15, thus $DL_6(AB, B1) =$ \$15. $B1$ could win AB for \$15 in a future state in which $B2$ wins C and $B5$ wins D.

Function f is specific to allocation constraints and the bidding language, which determine whether a bid set is simultaneously blockable or not. In Appendix B we provide an algorithm to calculate DLs for the XOR bidding language without additional allocation constraints. The XOR bidding language is fully expressive (Nisan and Segal 2006), and it is used in high-stakes applications such as in spectrum auctions across Europe nowadays.

4. Computational Complexity

Computational complexity has turned out to be a practically relevant topic for the winner determination problem in CAs. We show that computational complexity is also a problem when computing pricing rules and that the computation of DLs is even one of the rare examples of a Π_2^P-complete problem.

The CAP is a well-known NP-complete problem in its decision version. The NP-completeness is proven for a variety of cases and bidding languages. Lehmann et al. (2006) prove the NP-completeness of CAP for the OR and XOR bidding language by reducing from the independent set problem (Garey and Johnson 1972) using intersection graphs. Sandholm and Suri (2001) prove the completeness for numerous cases dealing with side constraints such as bounds on the maximal number of winners. It is easy to see that the computation of WLs must be at least as hard as CAP.

Proposition 4. *Deciding $WL(S, i)$ is $NP-complete$.*

Proof: Every instance of CAP can be viewed as an instance of WL for $S = \mathcal{K}$. For $S = \mathcal{K}$, equation (2) yields $WL(K,i) = CAP(\mathcal{K}) - CAP(\mathcal{K}, \mathcal{K}_i) = CAP(\mathcal{K})$. Thus WL contains CAP as a special case which is NP-complete. Q.E.D.

Interestingly, the DL problem (equation (4)) is even harder to solve. It requires to solve one CAP for every simultaneously blockable bid set. The number of bids in S is $O(2^{|S|}|\mathcal{I}|)$. Hence the number of bid sets is $O(2^{2^{|S|}|\mathcal{I}|})$. Due to the set maximality property many of these double exponential bid sets need not to be evaluated. That means in the worst case only $\binom{2^{|S|}|\mathcal{I}|}{\lfloor (2^{|S|}|\mathcal{I}|)/2 \rfloor}$ $CAPs$ need to be solved (Engel 1997), which is still super exponential to the number of items in S. The actual number of blockable bid sets depends strongly on the allocation constraints.

Obviously the problem belongs to a higher complexity class than NP. This class is Π_2^P if the decision version is formulated as $DL(S,i) > c$ and the Σ_2^P if $DL(S,i) \leq c$. Completeness in these classes informs of what can (or cannot) be done in polynomial time with access to an NP oracle[9], i.e. an oracle which is able to decide the decision version of the winner determination problem in a single operation. Every CA employs a method to solve CAP. If we assume the presence of such an efficient method, completeness in these classes plays exactly the role of NP-completeness for "ordinary" optimization problems - it distinguishes the intractable from the efficiently solvable (Umans 2000). $\Pi_2^P = coNP^{NP}$ and $\Sigma_2^P = NP^{NP}$.

We first show $DL(S,i) \in \Pi_2^P$ with the decision problem $DL(S,i) > c$.

Lemma 1. *$DL(S,i)$ is in Π_2^P.*

Proof: We show that there exists a polynomially balanced, polynomial-time decidable 3-ary relation R such that $DL = \{x : \forall y_1 \exists y_2 \text{ such that } (x, y_1, y_2 \in R)\}$. x represents the graph containing all bids in S. Each node in x represents a \$1 bid and nodes are connected if and only if the bids they represent are compatible (e.g. do not overlap). y_1 represents a subgraph of x which is induced according to function f (i.e. each subgraph represents a bid set after removing a simultaneously blockable bid set). y_2 is an independent set of y_1. Note that its cardinality equals the value of the auction. Relation R decides in polynomial time that y_1 is a subgraph of x, y_2 an independent set of y_1 and that y_2 contains more than c nodes. Thus R is polynomially decidable. Furthermore R is polynomially balanced (since the lengths of y_1 and y_2 are bounded by a polynomial in the length of x). Q.E.D.

[9] The exponent NP denotes that the non-deterministic Turing machine accepting Σ_2^P and the complement of Π_2^P uses a NP oracle. We refer to Papadimitriou (1993) for formal definitions.

Intuitively the DL is greater than c if all bid combinations prescribed by f result in a subauction value greater than c. In graph terms, all of the induced subgraphs must have an independent set of cardinality greater than c. Remember that DL is a minimization problem and if it exists one subgraph with value not greater than c then the answer to the question becomes negative. That explains the necessity of the first quantifier $\forall$ and is central for the complexity of the problem, which is a min-max optimization problem (Ko and Lin 1995).

We now prove the completeness of $DL(S,i)$ in Π_2^P. We make use of the structures of a fairly restrictive case of the problem, as it is common ground in such proofs (Lehmann et al. 2006), to reduce from the minmax-Clique problem defined in Ko and Lin (1995).

Definition 6. minmax-Clique: Given is a graph $G=(V,E)$ with its vertices V partitioned into subsets $V_{i,j}$, $1 \le i \le I$, $1 \le j \le J$. For any function $t: \{1,...,I\} \to \{1,...,J\}$, G_t denotes the induced subgraph of G on the vertex set $V_t = \bigcup_{i=1}^{I} V_{i,t(i)}$. Find $f_{Clique}(G) = \min_t \max_Q \{|Q|: Q \subset V \text{ is a clique in } G_t\}$.
Intuitively the graph represents a network with I components, with each component V_i having J subcomponents $V_{i,1},...,V_{i,J}$. At any time t only one subcomponent $V_{i,t(i)}$ of each V_i is active and the problem is to find the maximum clique size of all possible active subgraphs G_t. The minmax-Clique problem is Π_2^P-complete by reduction from SAT_2. The completeness is shown for $J=2$ and subsets $V_{i,j}$ of same cardinality.

Theorem 1. *Deciding $DL(S,i)$ is Π_2^P-complete.*

Proof: We reduce from minmax-Clique. For each subcomponent $V_{i,j}$ we create a bidder who participates in subauction S. Since $J=2$, each component corresponds to a pair of bidders. We introduce the allocation constraint that no two bidders belonging to the same pair are allowed to win together in $\mathcal{K}\backslash S$. For each node we create a \$1 bid on T with $T \subseteq S$ submitted from the bidder associated to the subcomponent the node belongs to. Whenever two nodes in G are connected, their associated bids are compatible.[10] It can be observed now that $f_{Clique}(G) = DL(S,i)$. There is a one-to-one correspondence between the $2^{|I|}$ possible active subgraphs G_t and the bidder (bid) sets that remain after removing each of the $2^{|I|}$ simultaneously blockable bidder (bid) sets of cardinality $|I|$, due to the constraint in $\mathcal{K}\backslash S$. Furthermore, the max-Clique problem on the complementary intersection graphs is equivalent to the maximum independent set problem on the actual intersection graph and thus equivalent to CAP. Q.E.D.

[10] The graph we work on is complementary to the intersections graphs described by Lehmann et al. (2006).

An interpretation of the constraint we introduced to prove completeness is that the auctioneer is not willing to deal with more than a single winner from a region, assuming that bidders are paired according to their region. We conjecture that the DL problem without such constraints cannot be easier since the number of the simultaneously blockable bidders and thus the number of $CAPs$ which must be solved becomes much greater. To see this, consider an example with 8 foreign bidders in S and $|\mathcal{K}\backslash S| = 4$. Without the constraint the bidder sets amount to $\binom{8}{4}$ whereas with the constraint only 2^4 bidder sets need to be evaluated.

5. Allocation Constraints and their Impact on Equilibrium Strategies in Efficient Auction Designs

In what follows, we want to understand equilibrium strategies in CAs with allocation constraints. In order to analyze such CAs with respect to efficiency and incentive compatibility we first need to understand the impact of allocation constraints on those CA formats, which are known to be efficient with a strong game-theoretical solution concept. First, we analyze the VCG auction, which is known to be the unique CA format that is strategy proof, efficient and individually rational (Green and Laffont 1977). Second, we focus on ascending CAs as *iBundle*, the APA, and dVSV in which straightforward bidding is an ex post equilibrium for buyer submodular valuations. The analysis integrates the auction mechanisms in Adomavicius and Gupta (2005) in this game-theoretical framework and provides conditions, when this auction format leads to an ex post Nash equilibrium.

Definition 7. A strategy profile $s^* = (s_1^*, ..., s_n^*)$ is an *ex-post Nash equilibrium* iff the utility functions u_i satisfy

$$u_i(s^*, t) \geq u_i(s_i^{'}, s_{-i}^*, t) \; \forall s_i^{'}, t, i$$

In other words, truthful bidding in every round of an auction is an ex-post (Nash) equilibrium, if for every bidder $i \in \mathcal{I}$, if all other bidders follow the truthful bidding strategy, then bidder i maximizes his payoff in the auction by following the truthful bidding strategy independent of the type t of other bidders (Mishra and Parkes 2007). Ex post equilibria avoid speculation about other bidders' valuations or types and could therefore reduce the strategic complexity for bidders considerably, leading to higher efficiency, and also an increased adoption of ascending CAs. Note that this is weaker than a dominant strategy equilibrium, where bidders do not have to speculate about other bidders' valuations and strategies. In contrast to dominant strategy and ex post equilibria, Bayes-Nash equilibria do always exist, but they require bidders to speculate on both, the type and the strategy of others. We refer to dominant and ex post equilibria as *strong solution concepts*. For ascending auctions we focus on ex post equilibria, as preference elicitation in an indirect mechanism typically does not allow for dominant strategy equilibria (Conitzer and Sandholm 2002).

Let us first introduce two definitions to describe bidder valuations, before we discuss individual auction formats.

DEFINITION 8. A *coalitional value function* V maps a set of bidders J to a real number $V(J)$, equal to the total value created from trade among these bidders and the auctioneer.

CAP implements a coalitional value function in the context of CAs. Bidder submodularity describes a property of the coalitional value function, which allows for strong solution concepts in ascending CAs and core outcomes in the VCG auction, as we will see below.

DEFINITION 9. (bidder submodular (BSM) condition) A coalitional value function V is *bidder submodular* if bidders are more valuable when added to smaller coalitions: for all $i \in \mathcal{I}$ and all coalitions J and J' satisfying $J \subset J'$, $V(J \cup \{i\}) - V(J) \geq V(J' \cup \{i\}) - V(J')$.
Since allocation constraints may alter (lower) $V(J)$, imposing them may turn a coalitional value function from not BSM to BSM or vice versa[11]. The simplest example to see this, is to impose the allocation constraint "max one winner". Any function will then turn into BSM. The constraint "min three winners" turns any function into not BSM. In the following, when we refer to $V(J)$ or BSM, these are computed by taking into account present allocation constraints.

5.1. The VCG Mechanism

The VCG outcome serves as a baseline for all other efficient auction formats. For example, under BSM valuations APA, *iBundle*, and dVSV terminate with VCG prices which are in the core, eliminating incentives for speculation. Core prices have the property that no coalition of bidders can renegotiate the outcome with the auctioneer in order to increase everyones payoff in this coalition.

The VCG auction is a sealed bid auction allowing for package bids on all combinations of items. Bidders place sealed XOR bids on their desired packages without getting any feedback by the auctioneer or knowing bids of other bidders. The auctioneer calculates a feasible allocation X^* that maximizes the sum of bid prices. Bidders payments are calculated in a second step. Winning bidders pay their bid prices $b_i(S)$ reduced by a discount which is equal to their marginal contribution to the whole economy. $p_i(S) = b_i(S) - (V(\mathcal{I}) - V(\mathcal{I} \backslash i)) \quad \forall S \in X^*$ and zero otherwise.

We concentrate on allocation constraints in this paper. While Ausubel and Milgrom (2006b) show that budget constraints can lead to inefficiency in the VCG mechanism, allocation constraints do not affect its properties. However, the calculation of the VCG prices and in particular of the

[11] Also the implication by Ausubel and Milgrom (2002) (Goods Are Subtitutes) $\Rightarrow$ BSM does not hold in the presence of allocation constraints.

coalitional value from $V(J)$ with $J \subset \mathcal{I}$ has to consider the allocation constraints, as otherwise the auctioneer could suffer a negative payoff and participation would not be individually rational.

Definition 10. (Shoham and Leyton-Brown (2009)) An environment exhibits the *no-single-agent effect* if $\forall i, \forall v_{-i}, \forall X$ there exists an allocation X' that is feasible without i and $\sum_{j \neq i} v_j(X') \geq \sum_{j \neq i} v_j(X)$.

A mechanism is weakly budget balanced when it will not lose money, this means if the mechanism is not weakly budget balanced the auctioneer might be confronted with a negative payoff which would contradict individual rationality of the mechanism.

Theorem 2. *(Shoham and Leyton-Brown (2009)) The VCG mechanism is weakly budget balanced when the no single agent effect property holds.*

CAs without allocation constraints are always weakly budget balanced since the no single agent effect property always holds (Shoham and Leyton-Brown 2009). The theorem extends to VCG auctions with allocation constraints but in this case the no single agent effect property may not hold. Hence these auctions are not always weakly budget balanced.

Corollary 1. *The VCG mechanism with allocation constraints is not always weakly budget balanced.*

Proof: It may happen that $V(\mathcal{I} \backslash i)$ in the VCG payment computation is zero because of allocation constraints. Consider an example where the auctioneer requires 2 winning bidders and only 2 bidders participate, such that the no-single-agent effect does not hold. Bidder $B1$ values item A at \$10 and $B2$ values B at \$10. The Vickrey payments are $p_1(A) = p_2(B) = 10 - (20 - 0) = -10 \leq 0$ and the auctioneer loses money. Q.E.D.

If the no-single-agent effect does not hold, the auctioneer might want to consider bids by bidders $i \notin J$ to assure a feasible allocation, while maximizing $V(J)$.

5.2. Efficient Ascending CAs

The recent game-theoretical research has led to a coherent theoretical framework and a family of ascending CAs (*iBundle*, APA, dVSV) which satisfy an ex post equilibrium under BSM. These efficient ascending CAs use personalized and non-linear prices. They calculate a provisional value maximizing allocation at the end of every round and increase the prices for a certain group of bidders. The different approaches can be interpreted as implementations of primal-dual algorithms (dVSV) or subgradient algorithms (*iBundle*, APA) to solve an underlying linear programming problem (de Vries et al. 2007). This linear program (CAP_3) always yields integral solutions and the dual variables have a natural interpretation as non-linear and personalized ask prices (Bikhchandani and Ostroy 2002).

We want to understand, whether additional allocation constraints have an impact on equilibrium strategies and efficiency in these auction formats. For this reason, we analyze the impact of allocation constraints on CAP_3. The original CAP_3 formulation changes with additional allocation constraints. An arbitrary allocation constraint can make certain allocations infeasible. Rather than modeling specific allocation constraints, we keep our analysis general and partition the set of all allocations in two subsets: the feasible allocations $\mathbb{C}$ and the infeasible ones $\mathbb{C}_u$, which turn infeasible due to the violation of certain allocation constraints (e.g. the maximum number of winners). This extends CAP_3 by constraint set (LP4):

$$
\begin{array}{rlll}
\max\limits_{x_i(S)} \sum\limits_{S \subseteq \mathcal{K}} \sum\limits_{i \in \mathcal{I}} x_i(S) v_i(S) & & & \\
s.t. & & & \\
\sum\limits_{S \subseteq \mathcal{K}} x_i(S) \leq 1 & \forall i & (\pi_i) & (LP1) \\
x_i(S) \leq \sum\limits_{X \in \mathbb{C} \cup \mathbb{C}_u : S_i \in X} y(X) & \forall i, S & (p_i(S)) & (LP2) \\
\sum\limits_{X \in \mathbb{C} \cup \mathbb{C}_u} y(X) \leq 1 & & (\pi_s) & (LP3) \\
y(X) \leq 0 & \forall X \in \mathbb{C}_u & (t(X)) & (LP4) \\
x_i(S), y(X) \geq 0 & \forall i, S, X \in \mathbb{C} \cup \mathbb{C}_u & &
\end{array}
\tag{5}
$$

The dual to the extended CAP_3 in (5) is:

$$
\begin{array}{rlll}
\min\limits_{\pi_i, \pi_s} \sum\limits_{i \in \mathcal{I}} \pi_i + \pi_s & & & \\
s.t. & & & \\
\pi_i + p_i(S) \geq v_i(S) & \forall i, S & (x_i(S)) & (DLP1) \\
\pi_s - \sum\limits_{S_i \in X} p_i(S) \geq 0 & \forall X \in \mathbb{C} & (y(X)) & (DLP2a) \\
\pi_s + t(X) - \sum\limits_{S_i \in X} p_i(S) \geq 0 & \forall X \in \mathbb{C}_u & (y(X)) & (DLP2b) \\
\pi_i, \pi_s, p_i(S) \geq 0 & \forall i, S & &
\end{array}
\tag{6}
$$

The decision variables of the primal are: $x_i(S)$ denoting whether bidder i wins package S and $y(X)$ denoting whether allocation X is realized or not. In the dual, π_s and π_i denote the auctioneers' and bidder i's profit respectively whereas $p_i(S)$ denotes the ask price for the package S and bidder i. Prices are summarized by p. Theorem 3.1 in Bikhchandani and Ostroy (2002) shows that allocation X and prices p form a competitive equilibrium (CE), if X is an optimal solution to the primal CAP_3 and (p, π_i, π_S), where π_i, π_S are payoffs resulting from X and prices p are an optimal solution to the corresponding dual linear program. Their proof is based on the resulting complementary slackness conditions. We show that additional allocation constraints causing additional infeasible solutions do not impact the theorem and the equivalence between competitive equilibrium and optimal solution to (5) is still given. Let us first enumerate the complementary slackness (CS) conditions:

$$\left(\sum_S x_i(S)-1\right)\pi_i = 0\ \forall i \quad (CS1) \qquad (\pi_i + p_i(S) - v_i(S))\,x_i(S) = 0\ \forall i,S \quad (CS5)$$
$$\left(x_i(S) - \sum_{S_i\in k} y(X)\right)p_i(S) = 0\ \forall i,S \quad (CS2) \qquad \left(\pi_s - \sum_{S_i\in X} p_i(S)\right)y(X) = 0\ \forall X\in\mathbb{C} \quad (CS6)$$
$$\left(\sum_X y(X)-1\right)\pi_s = 0 \quad (CS3) \qquad \left(\pi_s + t(X) - \sum_{S_i\in X} p_i(S)\right)y(X) = 0\ \forall X\in\mathbb{C}_u \quad (CS7)$$
$$y(X)t(X) = 0\ \forall X\in\mathbb{C}_u \quad (CS4)$$

The competitive equilibrium (CE) conditions are:

$$\pi_i = \max_S (v_i(S) - p_i(S))\ \forall i\ (CE1) \qquad \pi_s = \max_{X\in\mathbb{C}} \sum_{S_i\in X} p_i(S) \quad (CE2)$$

Lemma 2. *(X^*, p^*) is a CE if and only if the integral solution dictated by X^* is an optimal solution to primal CAP_3 (5) and (p^*, π_i^*, π_S^*) is an optimal solution to dual CAP_3 (6).*

Proof: We follow the proof of Theorem 3.1 by Bikhchandani and Ostroy (2002), and show that the additional infeasible allocations due to additional allocation constraints do not violate the equivalence of competitive equilibrium and optimality of the winner determination problem. The due to the allocation constraints additional (CS4) and (CS7) do always hold, as $y(X) = 0$ for the infeasible allocations (cf. (LP4) and $y(X) \geq 0$). Denote with S_i^* the package bidder i is assigned under allocation X^*.

Sufficiency: Suppose the LP (5) has an integral solution X^* with $x_i(S) = 1$ iff $S = S_i^*$ and $y(X) = 1$ iff $X = X^*$. Let $(\pi_s^*, \pi_i^*, p^*, t(X^*))$ be an optimal solution of the DLP (6). $t(X^*) \geq \sum_{S_i\in X} p_i(S)$ because it does not appear anywhere else than in (DLP2b) and the program minimizes π_s. (CS5) and (DLP1) imply the first CE condition (CE1). (DLP2a) and (DLP2b) imply

$$\pi_s \geq \max\left\{\max_{X\in\mathbb{C}} \sum_{S_i\in X} p_i(S), \max_{X\in\mathbb{C}_u} \sum_{S_i\in X} (p_i(S) - t(X))\right\}.$$

Due to $t(X^*) \geq \sum_{S_i\in X} p_i(S)$ the last term is always smaller or equal to zero, while the first term is always greater or equal to zero. Due to (CS6) the above inequality implies the second CE condition (CE2). Hence (X^*, p^*) is a CE.

Necessity: Let (X^*, p^*) be a CE. Therefore by definition:

$$\pi_i^* \equiv v_i(S^*) - p_i(S_i^*) = \max_S (v_i(S) - p_i(S))\ \forall i\ (CE1)$$
$$\pi_s^* \equiv \sum_{S_i\in X^*} p_i(S) = \max_{X\in\mathbb{C}} \sum_{S_i\in X} p_i(S) \quad (CE2)$$

Let $x_i(S) = 1$ iff $S = S_i^*$ and $y(X) = 1$ iff $X = X^*$, else 0. X^* is a feasible solution to LP (5) since the allocation is supported in a CE equilibrium. Similarly $(\pi_s^*, \pi_i^*, p^*, t(X^*))$ is feasible to DLP (6). The dual variable $t(X)$ does not impact this equivalence. The remaining proof showing that the integral solution we just constructed is optimal, is identical to the proof of Theorem 3.1 in Bikhchandani and Ostroy (2002). Q.E.D.

DEFINITION 11. A straightforward bidder bids only for those packages that maximize his payoff given the current ask prices.

COROLLARY 2. *The CAP_3 formulation with allocation constraints (5) yields integral solutions and thus the efficient ascending CAs (iBundle, APA, dVSV) terminate at a CE even if allocation constraints are present and bidders follow the straightforward bidding strategy.*

This follows directly from Lemma 2 and the original proofs of the efficiency of *iBundle* (Parkes and Ungar 2000) and dVSV (de Vries et al. 2007). Parkes and Ungar (2000) show that all complementary slackness conditions except (CS1) are satisfied in each round of the *iBundle* auction. (CS1) states that every bidder with a positive utility for some packages at the current prices must receive a package in the allocation. Only in the last round this condition is satisfied for all bidders. The new complementary slackness conditions (CS4) and (CS7) due to allocation constraints are trivially satisfied, because $y(X)$ is null, and do not impact the proof. While the price updates in dVSV follow a primal-dual algorithm, *iBundle* and APA can be considered subgradient algorithms (de Vries et al. 2007).

Ausubel and Milgrom (2006a) show that the APA (and therefore *iBundle*) terminates with an efficient solution and straightforward bidding is an ex post Nash equilibrium strategy when the BSM condition holds. The proof is defined on some coalitional value function, which might be implemented by CAP_3 but also a CAP_3 with additional allocation constraints, and it is therefore not affected by allocation constraints. In summary, allocation constraints neither have an impact on the efficiency of the family of efficient ascending CAs, nor on the incentive properties. While *iBundle*, the APA, and dVSV allow for allocation constraints, they do not explicitly take them into account in the pricing rule, but implement simple price increments for subsets of bidders.

6. Efficiency and Equilibrium Analysis of FCAs

In what follows, we want to understand economical characteristics of FCAs and whether pricing rules such as WL, and DL can also achieve 100% efficiency with a strong solution concept, and how they relate to other efficient ascending CAs such as *iBundle*, APA, and dVSV introduced earlier. Except from the ask price calculation (i.e., pricing rules) the following auctions (FCA_{WL} and FCA_{DL}) are equivalent to APA and *iBundle*. As in all other efficient ascending CAs we will first assume a straightforward bidding strategy where the bidders only have to reveal their demand set in each round. We will show that while FCA_{WL} does not even lead to an efficient solution with this bidding strategy, FCA_{DL} leads to an efficient outcome and straightforward bidding is an ex-post equilibrium with buyer submodular valuations. Throughout we will assume an XOR bidding language.

6.1. FCA$_{WL}$

In the FCA$_{WL}$ auction losing bidders in a round get an ask price of $WL(S,i)+\epsilon$ for a package S. In each round WLs for losing bids of losing bidders have to be calculated. This causes the FCA$_{WL}$ to be an ascending CA, although generally, WLs are not monotonically increasing as DLs are (cf. Section 3, Proposition 2).

Proposition 5. *FCA$_{WL}$ is an ascending CA.*

Proof: It is more convenient to speak of auction rounds r than of states here. $WL_r(S,i)$ is only updated right after the submission of a bid $(S,v,r-1,i)$ and therefore $WL_r(S,i)$ can never be lower than the bid's value v. But v is equal to the $WL_{r'}(S,i)$ presented to the bidder before the submission, hence for any $r'<r$ it holds $WL_{r'}(S,i) \leq WL_r(S,i)$ (the round r' is either the previous round the bidder had bid on S or the first round). Q.E.D.

The efficiency of a FCA$_{WL}$ can be as low as 0% if the bidders bid straightforward and valuations are *demand masking.*

Definition 12. A demand masking set of bidder valuations is given if the following properties are fulfilled. For each item, there is one bidder. Each l-th bidder values the big package which contains all items with V_b and the l-th single item with V_s. All other package valuations are zero. We set $mV_s > V_b > V_s$ so that at the efficient allocation every bidder wins a single item.

	$\mathcal{K}$	item l	item $l' \neq l$
l-th bidder	V_b	V_s	0

Table 5 Demand masking set of bidder valuations

Let $m=|\mathcal{K}|$ be the number of items. We will first provide an example with $m=4$, $V_s=\$2$ and $V_b=\$5$, where FCA_{WL} is inefficient.

Example 3. There are four bidders, $B1$ to $B4$ and four items A to D. Table 6 indicates the auction progress. Prices are initialized to \$0. At the beginning, all bidders bid on the big package. When its price increases to \$3, then the losing bidders bid also on the single items, since their payoff is \$2, i.e. equals the payoff of the big package. Their bids on the single items are unsuccessful and the prices are updated to \$4. These updated prices exceed their valuations $V_s=\$2$, therefore they never bid again on the single items and the auction fails to reach the efficient solution.

Theorem 3. *If bidder valuations are demand masking, the efficiency of FCA$_{WL}$ with straightforward bidding converges to $2/m$ in the worst case with $m>1$.*

	FCA_{WL}					
packages	A	B	C	D	ABCD	$\emptyset$
valuations	2_1	2_2	2_3	2_4	$5_1, 5_2, 5_3, 5_4$	
round 1					0_1^*, 0_2, 0_3, 0_4	
round 2					0_1, 1_2^*, 1_3, 1_4	
round 3					2_1^*, 1_2, 2_3, 2_4	
round 4		0_2	0_3	0_4	2_1, 3_2^*, 3_3, 3_4	
round 5	0_1				3_2, 4_3^*, 4_4	
round 6					5_1^*, 5_2, 4_3, 5_4	0_1, 0_2^*, 0_4^*
round 7					5_1^*, 5_2, 5_3, 5_4	0_1, 0_2^*, 0_3^*, 0_4^*
	Termination					

Table 6 **FCA_{WL} process**

Proof: We distinguish two cases:

$$V_b \geq 2V_s \tag{7}$$

If inequality (7) holds, FCA_{WL} is inefficient. At the beginning, all bidders bid on the big package $\mathcal{K}$. They do so until the round r at which its winning level exceeds $V_b - V_s$ and thus its payoff falls below the payoff of a single item. Denote with w the winner of the big package at this round. All other bidders lose. Their payoffs are higher for the single items than the big package and therefore they bid on them at the current price 0 (payoff(item l) $= V_s - 0 >$ payoff($\mathcal{K}$) $= V_b - (V_b - V_s + \delta) = V_s - \delta$, whereby δ denotes a small constant). These bids of zero value are unsuccessful and the single item prices (i.e. winning levels) for the next round amount to the winning bid of w which was $V_b - V_s$. Their payoff(item l) $= V_s - V_b + V_s = 2V_s - V_b$ and payoff($\mathcal{K}$) $= V_b - V_b + V_s = V_s >$ payoff(item l) since $V_b > V_s$. Thus they bid again on the big package. Furthermore, due to (7) their payoffs for the single items (equal to $2V_s - V_b$) are negative and they will never bid on them again. The auction ends by assigning the big package to an arbitrary bidder. The efficiency is $V_b/(mV_s)$ and for $V_b = 2V_s + \delta$, it becomes $(2 + \delta')/m$. Thus for a large m, it converges to 0%. On the contrary, if (7) does not hold, the bidders will bid again on the small items at price $V_b - V_s$ when the payoff of the big package falls below payoff(item l) $= 2V_s - V_b$. They win the single items and the auction ends with the efficient outcome. Note also that for $m = 2$ the reverse inequality (7) cannot be satisfied due to requirement $mV_s > V_b$ and hence the auction is efficient for $m = 2$ and inefficient for $m > 2$. Q.E.D.

While there might also be other bidder valuations leading to low efficiency, it is sufficient for our purposes to show that the efficiency of FCA_{WL} can actually be as low.

6.2. FCA_{DL}

Contrary to the negative results on FCA_{WL}, we show that FCA_{DL} leads to full efficiency with straightforward bidding, but it requires less rounds and less bids than *iBundle* and the APA.

The only difference between FCA$_{DL}$ and FCA$_{WL}$ is that in FCA$_{DL}$, ask prices are updated to $DL(S,i)+\epsilon$.

LEMMA 3. *DL ask prices are always higher or equal to iBundle prices given the same bids.*

Proof: Let $p_i^k(S)$ be the *iBundle* price after k bids are submitted. *iBundle* uses an XOR bidding language, and let $CAP_k(S,i)$ denote the highest bid of i in S (bids of one bidder cannot be combined due to XOR): $CAP_k(S,i) = max\{v|(T,v,k',i), T \subseteq S, k' \leq k\}$. Equation (4) implies $DL_k(S,i) \geq CAP_k(S,i)$. The price update rules in *iBundle* ensure that in each round $p_i^k(S) = max\{v|(T,v,k',i),\, T \subseteq S, k' \leq k\} + \epsilon$ (Parkes and Ungar 2000)[12]. Thus $CAP_k(S,i) + \epsilon = p_i^k(S)$ and $DL_k(S,i) + \epsilon \geq p_i^k(S)$. Q.E.D.

To prove the efficiency of FCA$_{DL}$, we draw on the proof of optimality by Parkes and Ungar (2000) and show that optimality is not affected by the requirement to bid $DLs + \epsilon$ instead of only an ϵ above the last losing bid. Their proof works on a primal and a dual version of CAP, due to Bikhchandani and Ostroy (2002). This version is known as CAP_2 and is very similar to CAP_3. The main difference is that its prices are anonymous and it corresponds to the auction *iBundle*(2). To prove the efficiency of *iBundle*(2), it is assumed that no single bidder bids on two non-overlapping (safety condition. The efficiency of *iBundle*(3) or simply *iBundle* follows then directly from *iBundle*(2) (Parkes and Ungar 2000) and it can be dispensed with the safety condition.

LEMMA 4. *FCA$_{DL}$ terminates with the efficient solution and with CE prices if bidders bid straightforward.*

Proof: The only modification of FCA$_{DL}$, i.e. to quote DLs instead of simple price updates, only affects the proof with respect to the complementary slackness condition CS-6. We only need to show that CS-6, which states that "the allocation must maximize the auctioneer's profit at prices $p(S)$, over all possible allocations and irrespective of bids received by agents", is satisfied by FCA$_{DL}$ too. Replace $p(S)$ by $DL(S)$. From the DL computation follows that there is always a bidder or group of bidders willing to pay $DL(S)$ for every package in the value-maximizing allocation X^*_{DL} that is computed based on the prices (DLs) and irrespective of the bids. For this, observe that the highest possible $DL(S)$, which is the case when no bids are blockable, is equal to $CAP(S)$ considering all submitted bids. Hence there is always a bidder or a group of bidders willing to pay $DL(S)$. Therefore, allocation X^*_{DL} with auctioneer's profit $\sum_{S^* \in X^*_{DL}} DL(S^*)$ can

[12] Due to the free disposal assumption implying that packages are priced at least as high as the greatest price of any package they contain, i.e. $p_i^k(S) \geq p_i^k(T)$ for $S \supseteq T$.

be realized by assigning each S^* to a subset of bidders $J(S^*), J(S^*) \subseteq \mathcal{I}$ with $\bigcap J(S^*) = \emptyset$. Every bidder receives at most one package and hence the XOR constraint is not violated. The reason is that packages S^* form a feasible allocation and are obviously non-overlapping and no single bidder bids on non-overlapping packages due to the bid safety condition. In summary, we showed that it is always possible for the auctioneer to realize the profit-maximizing allocation at prices $DL(S)$ irrespective of bids received, since the computation of DLs ensures there are always bidders willing to take these prices[13]. Q.E.D.

THEOREM 4. *FCA_{DL} is efficient if bidders follow the straightforward bidding strategy. This strategy is an ex-post Nash equilibrium if the BSM condition holds.*

Theorem 4 follows directly from Lemma 4 and Ausubel and Milgrom (2006a).

FCA_{DL} can reduce the number of auction rounds, which is a considerable problem of *iBundle* as shown by Scheffel et al. (2011) and Schneider et al. (2010). The reason is that dead bids in *iBundle*, which will never be part of the winning allocation, are skipped and prices increase faster. We provide a simple example that FCA_{DL} can terminate with strictly less rounds than *iBundle*.

EXAMPLE 4. Consider items A,B,C are auctioned among bidders $B1$ to $B4$ in *iBundle* and FCA_{DL} using an increment of $\epsilon = 1$. Bidders bid straightforward and are single minded which means they value only one package positively and all others with zero. The exact valuations of each bidder and the auction rounds are described in Table 7. Ties are broken in favor of more winners.

	iBundle					FCA_{DL}			
packages	A	B	C	ABC		A	B	C	ABC
valuations	5_1	5_2	5_3	8_4		5_1	5_2	5_3	8_4
round 1	1_1^*	1_2^*	1_3^*	1_4		1_1^*	1_2^*	1_3^*	1_4
round 2	1_1^*	1_2^*	1_3^*	2_4		1_1	1_2	1_3	4_4^*
round 3	1_1^*	1_2^*	1_3^*	3_4		2_1^*	2_2^*	2_3^*	4_4
round 4	1_1	1_2	1_3	4_4^*		2_1	2_2	2_3	7_4^*
round 5	2_1^*	2_2^*	2_3^*	4_4		3_1^*	3_2^*	3_3^*	7_4
round 6	2_1^*	2_2^*	2_3^*	5_4		3_1^*	3_2^*	3_3^*	$\emptyset_4^*$
round 7	2_1^*	2_2^*	2_3^*	6_4		Termination			
round 8	2_1	2_2	2_3	7_4^*					
round 9	3_1^*	3_2^*	3_3^*	7_4					
round 10	3_1^*	3_2^*	3_3^*	$8_4, \emptyset_4^*$					
	Termination								

Table 7 *iBundle* **and FCA_{DL} process**

[13] To see why this is not the case by WLs, consider only one bid \$10 on AB. Then $WL(A) = \$10$ but no bidder is willing to pay the price. $DL(A)$ is instead \$0.

The example illustrated in Table 7 shows that FCA_{DL} reduces the number of auction rounds, the communication effort (since dead bids are not submitted) and also the computational effort. In general the reduction of auction rounds and communication effort comes at the price of higher computational effort as the Π_2^P-hard DL determination problem has to be solved several times.

In what follows, we introduce two economically motivated value models to demonstrate the benefits of FCA_{DL} concerning the number of auction rounds and the communication effort. Let $RRR = \frac{rounds\_iBundle - rounds\_FCA_{DL}}{rounds\_iBundle}$ denote the round reduction rate. Let C denote the communication effort measured as the number of all bids and ask prices exchanged. $CRR = \frac{C\_iBundle - C\_FCA_{DL}}{C\_iBundle}$ is the corresponding reduction rate.

Definition 13. Value model $VM1$ comprises m single minded regional bidders who value pairwise non-overlapping packages to at least $\$\epsilon$ and one global bidder who values package $\mathcal{K}$ of all items to at least the sum of the valuations of the regional bidders.

Definition 14. Value model $VM2$ comprises m single minded bidders with identical valuations for a specific package S, with $|S| > m - |\mathcal{K}| + 1$, i.e. the complementary to S subauction is not too large and ensures the existence of blockable bids and thus high DL.

Theorem 5. *In $VM1$ $RRR = \frac{m-1}{m+1}$ and $CRR = \frac{1}{2} - \frac{1}{2m}$. In $VM2$ $RRR = CRR = \frac{1}{m}$.*

The proofs are in Appendix A. It follows that in $VM1$ $RRR \to 100\%$ and $CRR \to 50\%$ for $m \to \infty$. Surprisingly, in a realistic value model with regional and global bidders, which resembls the setting of FCC sprectum auctions, FCA_{DL} can substantially decrease the number of rounds and communication effort. In more generic $VM2$, which is the case when many bidders have homogeneous valuations, $RRR \to 50\%$ and $CRR \to 50\%$ for $m = 2$.

7. Numerical Experiments

The computational complexity results for exact DLs suggest that the computational costs outweigh the benefits. We supplement the theoretical analysis with an experimental comparison of FCA_{DL} and *iBundle*. Since there are hardly any real-world CA data sets available, we have adopted value models of the Combinatorial Auctions Test Suite (CATS) (Leyton-Brown et al. 2000). In addition to CATS value models, we have used an extended version of the Pairwise Synergy value model from An et al. (2005). A more detailed description of the value models is provided in Appendix C, while the computation of DLs is described in Appendix B. For each value model we created 30 auction instances with different valuations, and ran each of them with both auction formats. All auctions used a bid increment of 1. Table 8 depicts the results for small and medium sized auctions. We used the Symphony MIP solver (http://www.coin-or.org/SYMPHONY/) and computers with an Intel

Core 2 CPU with 2.67GHz and 4GB main memory. Efficiency and final pay prices of both auction formats were the same (in accordance with Section 6.2). Hence, we report only on the reduction in rounds RRR and communication effort CRR and the computation times.

	iBundle(3)		*DL*		*iBundle*(3)		*DL*
Real Estate		RRR_{max}	7.547%	**Real Estate**		RRR_{max}	1.190%
		CRR_{max}	49.222%			CRR_{max}	49.029%
2x2 items	∅ R = 43.8	∅ RRR	3.395%	3x3 items	∅ R = 96.3	∅ RRR	0.513%
10 bidders	∅ C = 3943.1	∅ CRR	44.995%	12 bidders	∅ C = 341261.6	∅ CRR	45.976%
		∅ RF	1.425			∅ RF	30.756
		∅ PCT	3.402ms			∅ PCT	169.377ms
Transportation		RRR_{max}	17.021%	**Transportation**		RRR_{max}	0.000%
		CRR_{max}	13.584%			CRR_{max}	19.790%
4 items	∅ R = 108.4	∅ RRR	5.333%	9 items	∅ R = 104.7	∅ RRR	0.000%
10 bidders	∅ C = 842.2	∅ CRR	5.635%	12 bidders	∅ C = 1448.4	∅ CRR	4.519%
		∅ RF	0.986			∅ RF	1.094
		∅ PCT	1.252ms			∅ PCT	1.596ms
Pairwise Synergy+		RRR_{max}	47.945%	**Pairwise Synergy+**		RRR_{max}	29.132%
		CRR_{max}	39.447%			CRR_{max}	24.314%
3 items	∅ R = 110.9	∅ RRR	36.625%	8 items	∅ R = 302.5	∅ RRR	5.001%
9 bidders	∅ C = 1055.9	∅ CRR	30.028%	12 bidders	∅ C = 115947.0	∅ CRR	21.022%
		∅ RF	0.714			∅ RF	5.467
		∅ PCT	1.025ms			∅ PCT	8.862ms

Table 8 Comparison of FCA_{DL} to *iBundle*. Left part for small size and right part for medium size auctions.
RRR_{max} = Max Round Reduction Rate, CRR_{max} = Max Communication Reduction Rate,
∅ RRR = Average Round Reduction Rate, ∅ CRR = Average Communication Reduction Rate,
∅ RF = Average Runtime Factor, ∅ PCT = Average Price Calculation Time

In small size auctions (see Table 8) we observe a considerable reduction of the auction rounds across all value models and particularly in Pairwise Synergy+, where the maximum over the 30 auctions (RRR_{max}) is 47.945%. This is due to the existence of two bidder segments, regional and global, which leads to high RRR. The average CRR is over 30% in all value models except the Transportation value model, where bidders are only interested in a very limited set of packages. RF is the ratio of total run time of FCA_{DL} to *iBundle*. Surprisingly, despite of the high computational complexity of DLs, the run times of FCA_{DL} is sometimes even lower than that of *iBundle* (see 8 where $RF < 1$). The reason is the lower number of bids submitted, which shortens the winner determination in each round. The average computation time of a DL, denoted as PCT, ranges from 1.0 to 3.4 milliseconds only.

The mid-sized auctions with up to 9 items led to higher computation times than *iBundle* since many more DLs had to be computed. Also in these experiments computing a DL took between

1ms and 170ms only. The communication between auctioneer and bidders (CRR) was reduced substantially in all value models, but the total runtime RF increased.

We conducted additional experiments with 10 items and 9 bidders in different value models, where bidders submit bids on every possible package. Also in these larger instances, the computation lasted only 1.48s on average. Obviously, the computation time depends on many parameters such as the package size, the size of the complementary subauction, the number of bidders and their bids. However, our results indicate that computing DLs in ascending auctions might well be used in practical applications.

8. Conclusion

Designing efficient combinatorial auctions turned out to be a challenging task. A few recent papers have described efficient and ascending combinatorial auctions which satisfy strong game-theoretical solution concepts. In many applications the consideration of additional allocation constraints and flexibility in the choice of the bidding language are essential. These requirements have not been considered in the design of price feedback in the theoretical literature so far. It is important to extend the theory respectively. This could increase the applicability of ascending combinatorial auctions in domains such as transportation or industrial procurement considerably and bare significant practical potential.

We consider ascending combinatorial auctions allowing for side constraints and OR as well as XOR bidding languages. We draw on the work by Adomavicius and Gupta (2005) and define winning and deadness levels (WLs and DLs) as a general pricing rule for ascending combinatorial auctions, which allow for different bidding languages and allocation constraints. This extension leads to a number of theoretical challenges. We show that straightforward bidding is an ex post equilibrium in ascending combinatorial auctions with DLs, and how this pricing rule can be integrated in the theoretical framework of efficient and ascending combinatorial auctions.

While both, *iBundle* and the FCA_{DL} allow for allocation constraints, DLs take allocation constraints into account and actually lead to a lower number of auction rounds and bids that need to be submitted. The high number of auction rounds turned out to be one of the main obstacles for efficient ascending combinatorial auctions such as *iBundle*, the Ascending Proxy Auction, and dVSV. DLs come at a computational cost, however. The computation is a Π_2^P-complete problem. We show, however, that such ask prices can be calculated for up to 10 items and 9 bidders with realistic value models in less than 1.5 seconds in experiments, which suggests that these approaches might well be used in applications. Approximations to the exact computation of DLs could potentially be an area of future research.

These results provide a theoretical foundation for practical auction design. Such designs can leverage different pricing rules. Experimental research is required to gain insights on bidding behavior and efficiency in complex markets with allocation constraints.

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Appendix A: Proofs

Proposition 1: $WL_k(S,i) = CAP_k(\mathcal{K}) - CAP_k(\mathcal{K}, S_i)$

Proof: We give a self-contained proof that is based on the proof of Theorem 2 of Adomavicius and Gupta (2005). We introduce the symbol $\mathbb{C}_k^E(S) = \{C' \in C \cup E | C \in \mathbb{C}_k, E \in \bigcup_{i=1}^{I}(S,0,0,i) \cup \emptyset\}$[14] to denote the set of feasible allocations that can also include bids of zero value on S[15]. We define a binary relation $\prec$ on bid allocations to compare the values of two allocations: $C' \prec C'' \Rightarrow v(C') < v(C'')$ where $v(C) = \sum_{b \in C} v(b)$ is

[14] In the referenced proof $\mathbb{C}_k(S)$ denotes the set of feasible allocations for subauction S. In our setting, note that $B_1 \in \mathbb{C}_k(S)$ and $B_2 \in \mathbb{C}_k(\mathcal{K} \backslash S)$ does not imply that $B_1 \cup B_2 \in \mathbb{C}_k(\mathcal{K})$ due to allocation constraints. Thus, generally $\mathbb{C}_k(S)$ where $S \subset K$ is not defined.

[15] We need this extension since otherwise allocations where a bidder wins a package for free would not be feasible. These allocations are considered in $CAP_k(\mathcal{K}, S_i)$.

the value of the allocation C. $S(C) = \bigcup_{b \in C} S(b)$ denotes the items covered in allocation C. $WIN_k^E(\mathcal{K}, S, i) = max_{\prec}\{C \in \mathbb{C}_k^E(\mathcal{K}) | (S, 0, 0, i) \in C\}$ represents the winning allocation of the whole auction at state k subject to the condition that bidder i wins S for free. We consider a new bid b_{k+1} of bidder i on package S at state $k+1$. Let $C_1 = \{C \in \mathbb{C}_{k+1}(\mathcal{K}) | b_{k+1} \in C\}$ and $C_2 = \{C \in \mathbb{C}_{k+1}(\mathcal{K}) | b_{k+1} \notin C\}$ be the set of all allocations with and without b_{k+1} respectively. It holds $C_1 \cap C_2 = \emptyset$ since they cannot share a common allocation (every allocation in C_1 contains b_{k+1} and every allocation in C_2 does not) and $C_1 \cup C_2 = \mathbb{C}_{k+1}(\mathcal{K})$. Therefore:

$$WIN_{k+1}(\mathcal{K}) = max_{\prec}\{C \in \mathbb{C}_{k+1}(\mathcal{K})\} = max_{\prec}\{C_1 \cup C_2\} = max_{\prec}\{max_{\prec} C_1, max_{\prec} C_2\} \tag{8}$$

$$\text{and since } b_{k+1} \notin C \ \forall C \in C_2, \text{ it follows } C_2 = \mathbb{C}_k(\mathcal{K}) \text{ and } max_{\prec} C_2 = max_{\prec} \mathbb{C}_k(K) = WIN_k(\mathcal{K}) \tag{9}$$

Furthermore:

$$\begin{aligned} max_{\prec} C_1 &= max_{\prec}\{C \in \mathbb{C}_{k+1}(K) | b_{k+1} \in C\} \\ &= \{b_{k+1}\} \cup \max_{\prec}\{C \backslash \{b_{k+1}\} | C \in \mathbb{C}_{k+1}, b_{k+1} \in C\} \\ &= \{b_{k+1}\} \cup \max_{\prec}\{C | C \in \mathbb{C}_k(K), S(C) \cap S(b_{k+1}) = \emptyset\} \\ &= \{b_{k+1}\} \cup \max_{\prec}\{C \in \mathbb{C}_k^E(\mathcal{K}), (S(b_{k+1}), 0, 0, i(b_{k+1})) \in C\} \\ &= \{b_{k+1}\} \cup WIN_k^E(\mathcal{K}, S(b_{k+1}), i(b_{k+1})) \end{aligned}$$

The last equation together with (8) and (9) imply:

$$WIN_{k+1}(\mathcal{K}) = max_{\prec}\{WIN_k(\mathcal{K}), \{b_{k+1}\} \cup WIN_k^E(\mathcal{K}, S(b_{k+1}), i(b_{k+1}))\}$$

$$\text{and } b_{k+1} \in WIN_{k+1} \iff v(WIN_k) < v(b_{k+1}) + v(WIN_k^E(\mathcal{K}, S(b_{k+1}), i(b_{k+1})))$$

Thus, for a new bid b_{k+1} to win, its value $v(b_{k+1})$ together with $v(WIN_k^E(\mathcal{K}, S(b_{k+1}), i(b_{k+1})))$ which is the value of CAP subject to the constraint that the bidder $i(b_{k+1})$ wins $S(b_{k+1})$ for free (we denoted this CAP as $CAP^k(\mathcal{K}, S_i)$), must exceed $v(WIN_k)$ which is the current value $CAP^k(\mathcal{K})$. This completes the proof. Q.E.D.

Proposition 2:

$$a) DL_k(S,i) \le DL_{k+1}(S,i) \qquad b) DL_k(S,i) \le WL_k(S,i) \qquad c) DL_k(\mathcal{K}, i) = WL_k(\mathcal{K}, i)$$

Proof: [16] a)Assume $DL_{k+1}(S,i) < DL_k(S,i)$. Denote with ε a very small positive number. Then $DL_k(S,i)$ implies that $\nexists k' > k : (S, DL_k(S,i) - \varepsilon, k', i) \in WIN_{k'}(\mathcal{K}) \Rightarrow \nexists k' > k : (S, DL_{k+1}(S,i), k', i) \in WIN_{k'}(\mathcal{K})$ (since we assumed $DL_{k+1}(S,i) < DL_k(S,i)$). But $DL_{k+1}(S,i)$ implies that $\exists k' > k+1 : (S, DL_{k+1}(S,i), k', i) \in WIN_{k'}(\mathcal{K})$. Contradiction.

In words, all bids below $DL_k(S,i)$ are destined to lose whatever happens in future auction states. But in the future state $k+1$ a bid amounting to $DL_{k+1}(S,i)$ and thus below DL_k has a chance to win in a state greater than $k+1$. Therefore $DL_k(S,i)$ is not minimal and by definition not a DL.

[16] We provide a definition-based proof without using a formula which calculates DLs. We have not derived such a formula yet.

b)Assume $DL_k(S,i) > WL_k(S,i)$. The WL definition implies that the bid $(S, WL_k(S,i), k+1, i) \in WIN_{k+1}(\mathcal{K})$. The DL definition together with the assumption $DL_k(S,i) > WL_k(S,i)$ implies that $\nexists k' > k$: $(S, WL_k(S,i), k', i) \in WIN_{k'}(\mathcal{K})$. Contradiction.

In words, the DL is the minimal price to win the item in a possible future auction state. The WL is the minimal price to win it at the next state, thus it cannot be lower.

c)$(\mathcal{K}, v, k+1, i) \in WIN_{k+1}(\mathcal{K}) \Rightarrow v \geq CAP_k(\mathcal{K})$. Thus the minimum v, i.e. the WL, is $WL_k(\mathcal{K}) = CAP_k(\mathcal{K})$. Furthermore if $v < CAP_k(\mathcal{K}) \Rightarrow \nexists k' > k : (S, v, k+1, i) \in WIN_{k'}(\mathcal{K})$ since by CAP definition $k' > k \Rightarrow CAP_{k'}(\mathcal{K}) \geq CAP_k(\mathcal{K})$. Thus $DL_k(\mathcal{K}, i) = CAP_k(\mathcal{K}) = WL_k(\mathcal{K}, i)$.

In words, to win in the next auction state all auctioned items, a bidder must bid at least the current auction value. Every lower bid will not be winning in any future auction state since the auction value will not sink. Q.E.D.

THEOREM 5: In $VM1$ $RRR = \frac{m-1}{m+1}$ and $CRR = \frac{1}{2} - \frac{1}{2m}$. In $VM2$ $RRR = CRR = \frac{1}{m}$.

Proof: $VM1$: In each round, either the coalition of the global bidder alone wins or the coalition of all regional bidders. Denote a round as G if the global bidder wins, else as S. We consider the sequence of rounds which comprises of two consecutive winning rounds for the global bidder. In FCA_{DL} the coalitions win alternately, thus the sequence is GSG. After a G, each regional bidder increases his bid by ϵ and after a S, the global bidder by $m\epsilon$ since his DL increases by this amount. In FCA_{DL} the sequence is GSG while in *iBundle* $G\overbrace{S\cdots S}^{m}G$. RRR and CRR equal to the reductions rates of this cyclical sequence, excluding the last G. Thus $RRR = \frac{m+1-2}{m+1} = \frac{m-1}{m+1}$ and $CRR = \frac{2m-(m+1)}{2m} = \frac{1}{2} - \frac{1}{2m}$

$VM2$: Every bidder bids on the same package. Let p_r denote its highest price over all bidders at round r. In FCA_{DL} p_r increases by ϵ after each round. In *iBundle* it can be easily seen that in every m consecutive rounds, there is one round where p_r remains unchanged since all losing bidders just level the price of the previously winning bid, thus on average the price increase is $(1 - \frac{1}{m})\epsilon$. The number of rounds is equal to the final price p^T divided by the average price increase, thus $RRR = \frac{(1-\frac{1}{m})\epsilon - \epsilon}{(1-\frac{1}{m})\epsilon} = \frac{m-1}{m+1}$. Regarding CRR, in *iBundle* each bidder submit $\frac{p^{final}}{\epsilon} + 1$ bids. In FCA_{DL}, w.l.o.g. we assume that all but two bidders always lose (tie breaking). These two bidders alternately increase their bids by 2ϵ and the computational effort for these two is equal to the effort for one of the always losing bidders. Thus $CRR = \frac{m(\frac{p^T}{\epsilon}+1) - (m-1)(\frac{p^T}{\epsilon}+1)}{m(\frac{p^T}{\epsilon}+1)} = \frac{1}{m}$. Q.E.D.

Appendix B: Computation of *DLs* for an XOR bid language

We proceed according to the two-phase method and firstly seek for simultaneously blockable bid sets. We observe that a winning bid of bidder j on a single item $l \in \mathcal{K} \backslash S$ suffices to simultaneously block all his bids in S. Consequently the number of foreign bidders, whose bids in S are simultaneously blockable, amounts to $|\mathcal{K} \backslash S|$. If this number is greater or equal than the number of all foreign bidders in S who have a least one non-i-dominated bid in S, denoted as n_S, we are done. All of them can be blocked and DL is equal to the highest bid of i in S, i.e he must overbid only his own bids. Otherwise we proceed to phase 2 and determine which bidder set leads, if removed from subauction S, to the minimum CAP value in S.[17] If

[17] Here it is more convenient to speak about removal of bidder sets instead of bid sets. The removal of a bidder set from subauction S corresponds to the removal of all the bids in S submitted from bidders in the bidder set.

Input: package S, bidder i
set of bids $B^{ndom}_{S,k,-i}$
Output: $DL(S,i)$

$lowerBound \leftarrow \max_b\{v(b) : i(b) = i, S(b) \subseteq S\}$
$DL(S,i) \leftarrow \infty$
$\mathcal{F} \leftarrow getForeignBiddersOnPackage(S,i)$
if $|\mathcal{F}| \leq |\mathcal{K}\backslash S|$ **then**
  $DL(S,i) \leftarrow lowerBound$
else
  for all $\mathcal{F}_i \subset \mathcal{F} : |\mathcal{F}_i| = |\mathcal{K}\backslash S|$ **do**
    $thisPrice \leftarrow CAP(S, B_{S,k}\backslash\{b' | b' \in B^{ndom}_{S,k,-i}, i(b') \in \mathcal{F}_i\})$
    if $thisPrice < DL(S,i)$
      $DL(S,i) \leftarrow thisPrice$
    end if
    if $DL(S,i) = lowerBound$ **then**
      break for
    end if
  end for
end if
return $DL(S,i)$

Algorithm 1: DL XOR algorithm

$|S| = 1$ or if all bids in S are overlapping, it is easy to find which bidders should be removed; the ones with the highest bids. But the general case is obviously combinatorial and requires solving the CAP for each of the $\binom{n_S}{n_S - |\mathcal{K}\backslash S|} = \binom{n_S}{|\mathcal{K}\backslash S|}$ bidder sets remaining after removing the blockable ones. The pseudocode is given in 1.

The outlined calculation of XOR DL can serve as a basis to calculate DL for a number of other constrained cases, surprisingly even for cases with the OR bidding language. We provide an example, as this cannot be claimed for every conceivable case.

Proposition 6. *The $DL(S,i)$ for the OR bidding language in presence of the constraint "max a winners" can be calculated as XOR DL.*

Proof: The most opportune case for bidder i to win S is when he places a huge bid on a item of the complementary subauction $\mathcal{K}\backslash S$ so that he is surely among the a winners[18] and additionally some bidders leading to the highest $CAP(S)$ are not among the a winners because other "low" bidders in S place huge bids in $\mathcal{K}\backslash S$. Together with bidder i, $min(a-1, |\mathcal{K}\backslash S| - 1)$ foreign bidders can win in $\mathcal{K}\backslash S$ and all other

[18] It is not known whether i truly desires items in $\mathcal{K}\backslash S$. We take it for granted since we want to minimize his price on S. Knowing additional information such maximal willingness to pay in $\mathcal{K}\backslash S$ leads to additional constraints in the minimization problem of DL and the optimal value increases. In our analysis we do not cope with this issue but with the standard definition of DLs of Adomavicius and Gupta (2005) which is $DL(S) = CAP(S)$. Thereby winning subauction S does not implies winning S. This especially will not happen if we incorporate the additional information that every bidder is interested in packages overlapping with S and there is little interest in $\mathcal{K}\backslash S$ so that S and $\mathcal{K}\backslash S$ will never be part of the efficient allocation. Although incorporating such information is beyond the scope of this work, we believe that our analysis and algorithms provide the necessary tools to guide the computation of DLs even in such cases. What changes is how the simultaneously blocked bids are computed and this has to be engineered for every concrete case.

$max(n_S - min(a-1, "|\mathcal{K}\backslash S|-1)), 0)$ foreign bidders in S can be simultaneously blocked. Knowing this, we can proceed to calculate $DL(S,i)$ as in the XOR DL case by solving one CAP for each bidder sets of the derived cardinality. Q.E.D.

Appendix C: Value Models

The ***Real Estate*** value model is based on the *Proximity in Space* model from CATS (Leyton-Brown et al. 2000). Items sold in the auction are the real estate lots l, which have valuations v_l drawn from the same normal distribution for each bidder. Adjacency relationships between two pieces of land m and n (e_{mn}) are created randomly for all bidders. There is a 90% probability of a vertical or horizontal edge, and an 80% probability of a diagonal edge. Edge weights w_{mn} are then generated randomly for each bidder (mean 0.4, deviation 0.2), and they are used to determine package valuations of adjacent pieces of land: $v(S) = (1 + \sum_{e_{mn}:m,n \in S} w_{mn}) \sum_{l \in S} v_l$.

The ***Transportation*** value model uses the *Paths in Space* model from CATS (Leyton-Brown et al. 2000). It models a nearly planar transportation graph in Cartesian coordinates, where each bidder is interested in securing a path between two randomly selected vertices (cities). The items traded are edges (routes) of the graph. Parameters for the Transportation value model are the number of items (edges) m and graph density ρ, which defines an average number of edges per city, and is used to calculate the number of vertices as $(m*2)/\rho$. The bidder's valuation for a path is defined by the Euclidean distance between two nodes multiplied by a random number, drawn from a uniform distribution. Consequently only a limited number of packages, which represent paths between both selected cities, are valuable for the bidder. This allows us to consider even larger transportation networks in a reasonable time. In our simulations we set the mean of ρ to 1.8 and 2.5 for the small and medium size auctions respectively. The standard deviation was set to 0.25.

The ***Pairwise Synergy*** value model in An et al. (2005) is defined by a set of valuations of individual items $\{v_l\}$ and a matrix of pairwise item synergies $\{syn_{k,l} : k, l \in \mathcal{K}, syn_{k,l} = syn_{l,k}, syn_{k,k} = 0\}$. The valuation of a package S is then calculated as $v(S) = \sum_{k=1}^{|S|} v_k + \frac{1}{|S|-1} \sum_{k=1}^{|S|} \sum_{l=k+1}^{|S|} syn_{k,l}(v_k + v_l)$. A synergy value of 0 corresponds to completely independent items, and the synergy value of 1 means that the package valuation is twice as high as the sum of the individual item valuations. The relevant parameters for the Pairwise Synergy value model are the interval for the randomly generated item valuations, set to $[0.0, 30.0]$, and the interval for the randomly generated synergy values, set to $[0.0, 2.0]$. In ***Pairwise Synergy+*** we specified additionally two bidder segments. In small size auctions, six bidders were interested in packages of cardinality 1 and three bidders of cardinality 3. In medium size auctions, eight bidders were interested in packages of cardinality in the interval $[1, 3]$ and four in the interval $[7, 8]$.